MATH 636 HOMEWORK 3
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(1) Hatcher 4.1.4 (p. 358).
(2) Hatcher 4.1.11 (pp. 358–359).
(3) (a) Suppose that \( M \) is a closed, connected \( m \)-manifold. Suppose further that \( M \) is triangulated, i.e., is an \( m \)-dimensional simplicial complex. Suppose \( p \in M \) is on a codimension-1 face (or facet) \( \partial \sigma \) of some simplex \( \sigma \). Show that \( p \) is on a codimension-1 face of exactly two simplices \( \sigma, \sigma' \).

(b) With notation as in the previous part, let \( \alpha = \sum \sigma_i \in C_m(M; \mathbb{F}_2) \) be the sum of the \( m \)-simplices in \( M \), viewed, via their characteristic maps, as maps \( \sigma_i: \Delta^m \to M \). Show that \( \alpha \) is a cycle.

(c) Show that for any point \( p \) in the interior of some \( m \)-simplex \( \sigma_i \), the image of \( \alpha \) in \( H_m(M, M \setminus \{p\}; \mathbb{F}_2) \) is a generator. Deduce that \( \alpha \) is the generator of \( H_m(M; \mathbb{F}_2) \cong \mathbb{F}_2 \) so \( \alpha \) is a (in fact, the) mod-2 fundamental class for \( M \).

(d) Now, suppose that \( M^m \subset N^n \) is a closed submanifold of a manifold \( N \), and that \( N \) is triangulated in such a way that \( M \subset N \) is a subcomplex. Show that \( i_*[M] \in H_m(N; \mathbb{F}_2) \), the homology class represented by \( M \), is the sum of the \( m \)-simplices in \( N \) which are contained in \( M \). (Hint: this is easy.)

Suggested review / qualifying exam practice (not to turn in):

(1) Hatcher 4.1.3, 4.1.5, 4.1.8, 4.1.12, 4.1.13, 4.1.14.
(2) In class, we showed that if \( f: Y \to Z \) is a weak homotopy equivalence then for any CW complex \( X \), \( f_*: [X,Y] \to [X,Z] \) is injective. We also sketched a proof that \( f_* \) is surjective. Fill in the details of that proof.
(3) With notation as in the previous problem, fill in the proof that the map \( f_*: [(X,x_0),(Y,y_0)] \to [(X,x_0),(Z,z_0)] \) of based homotopy classes of maps is bijective.

More problems to think about but not turn in:

(1) Hatcher 4.1.10, 4.1.18.
(2) Extend Problem 3 to the case that instead of the manifold \( M \) being triangulated, \( M \) is an \( n \)-dimensional CW complex and \( \alpha \in C_n^{\text{cell}}(M; \mathbb{F}_2) \) is the sum of the \( n \)-cells in \( M \). (You’ll have to follow \( \alpha \) through the isomorphism between cellular and singular homology.)
(3) Extend Problem 3 to \( \mathbb{Z} \)-coefficients.

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