MATH 636 HOMEWORK 6
DUE MAY 7, 2021.

INSTRUCTOR: ROBERT LIPSHITZ

(1) Hatcher 4.2.22 (p. 390).
(2) The case $n > 1$ of Hatcher 4.2.23 (p. 390). (You can view the $n = 1$ case as an optional exercise.)
(3) Hatcher 4.2.31 (p. 392). (You may assume the nullhomotopy of the inclusion $F \hookrightarrow E$ is constant on $\widetilde{x}_0$.)
(4) Hatcher 4.B.2 (p. 428). Note: you solved part of this in a previous homework. You don't have to re-prove that part; just cite it.

Suggested review / qualifying exam practice (not to turn in):

(1) 4.2.28, 4.2.29, 4.2.34, 4.B.1.
(2) Give a direct proof of exactness for the long exact sequence

$$\cdots \to \pi_n(F, \widetilde{x}_0) \to \pi_n(E, \widetilde{x}_0) \to \pi_n(B, x_0) \to \pi_{n-1}(F, \widetilde{x}_0) \to \cdots$$

associated to a fibration $F \to E \to B$. (That is, don't use exactness of the long exact sequence for a pair.)

More problems to think about but not turn in:

(1) In this problem, we show that Hopf's original definition of the Hopf invariant is a homotopy invariant. The problem assumes a little more knowledge of transversality than we have covered in class.
  (a) Let $f\colon S^3 \to S^2$ be a smooth map and p, q, r regular values of f. Show that $\mathrm{lk}(f^{-1}(p), f^{-1}(q)) = \mathrm{lk}(f^{-1}(p), f^{-1}(r))$. (Hint: consider a path γ from q to r with the property that f is transverse to γ.)
  (b) Show that every continuous map $S^3 \to S^2$ is homotopic to a smooth map, and that if two smooth maps are homotopic then they are homotopic via a smooth map $S^3 \times [0, 1] \to S^2$.
  (c) Now, suppose that $f, g\colon S^3 \to S^2$ are homotopic smooth maps and p, q are regular values of both f and g. Show that $\mathrm{lk}(f^{-1}(p), f^{-1}(q)) = \mathrm{lk}(g^{-1}(p), g^{-1}(q))$.
  (d) Given a continuous map $f\colon S^3 \to S^2$, let $\bar{f}$ be a smooth map homotopic to f and p, q regular values of $\bar{f}$. Prove that

$$\mathrm{lk}(\bar{f}^{-1}(p), \bar{f}^{-1}(q))$$

  is an invariant of the homotopic class of f.
(2) Let $K_1, K_2 \subset S^3$ be knots (smoothly embedded circles) and $\Sigma_1, \Sigma_2 \subset B^4$ smoothly embedded, orientable surfaces with $\partial\Sigma_i = K_i$. Show that $\mathrm{lk}(K_1, K_2) = \pm\#\Sigma_1 \cap \Sigma_2$. (Optionally, note that orientations of the K_i induce orientations of the Σ_i, and the sign in the formula becomes $+$.)

(3) Let $M^{n-1}, N^{n-1} \subset S^{2n-1}$ be closed, orientable submanifolds. Suppose that $M = \partial P$ for some n-dimensional manifold-with-boundary $P \subset S^{2n-1}$. Show that if $N \pitchfork P$ then $\mathrm{lk}(M, N) = \pm \# P \cap N$.

Email address: `lipshitz@uoregon.edu`