MATH 636 HOMEWORK 9
DUE MAY 28, 2021.

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(1) Hatcher 4.3.12 (p. 420).
(2) Hatcher 4.3.16 (p. 420).
(3) (a) Suppose $\pi_i(X) = 0$ for $i < n$, for some $n \geq 2$. Show that there is a map $X \to K(\pi_n(X), n)$ inducing an isomorphism on π_n.
  (b) Given a simply-connected space X and an integer n, by taking iterated homotopy fibers, build another space Y and a map $f\colon Y \to X$ so that $\pi_i(Y) = 0$ for $i < n$ and $f_*\colon \pi_i(Y) \to \pi_i(X)$ is an isomorphism for $i \geq n$. (Do not use the existence of Moore-Postnikov towers.)
(4) Up to homotopy equivalence, how many spaces X are there with $\pi_2(X) \cong \mathbb{Z}$, $\pi_3(X) \cong \mathbb{Z}/2\mathbb{Z}$, and all other homotopy groups trivial? Describe each of these spaces as explicitly as you can.
(5) Fix $n \geq 2$. Describe explicitly the first n stages of a Postnikov tower for $\mathbb{C}P^n$ consisting of principal fibrations (up through what Hatcher labels X_n), and compute the corresponding k-invariants $k_1, \dots, k_{2n-1}$. Show further that k_{2n} is nontrivial.
(6) (a) Let V be a finite-dimensional vector space over a field k, and F a filtration on V. Prove that V is isomorphic (as vector spaces) to the associated graded vector space (with respect to this filtration).
  (b) Give a counterexample if, instead, V is a free abelian group.

Suggested review / qualifying exam practice (not to turn in):

(1) Hatcher 4.3.11, 4.3.15, 4.3.17, 4.3.20, 4.3.21, 4.3.22.

More problems to think about but not turn in:

(1) Deduce the bundle homotopy lemma for fibrations as stated in class from Hatcher, Proposition 4.62.
(2) The point of this exercise is to illustrate that inverse limits can be somewhat bonkers, even for nice maps between nice spaces. Consider the inverse system

$$S^1 \leftarrow S^1 \leftarrow S^1 \leftarrow \cdots,$$

where each map in the sequence is the double cover $z \mapsto z^2$. Let S^1_n denote the n^{th} term in this sequence and $p_n\colon S^1_{n+1} \to S^1_n$ the n^{th} map. Let $S^1_\infty = \varprojlim S^1_n$.
  (a) Let $(z_1, z_2, \dots) \in S^1_\infty$. Show that the universal covering maps $q_n\colon \mathbb{R} \to S^1_n$, $q_n(t) = z_n^{-1} e^{2\pi i t/2^n}$, induce a continuous map $q_\infty\colon \mathbb{R} \to S^1_\infty$.
  (b) Show that the image of q_∞ is a path component of S^1_∞.
  (c) Show that q_∞ is not surjective.
  (d) Show that the image of q_∞ is dense.
  (e) Conclude that S^1_∞ is not locally path connected.

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