MATH 636 SPRING 2024
HOMEWORK 10
DUE JUNE 8, 2024

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Required problems:

(1) Hatcher 4.3.16 (p. 420).

(2) Up to homotopy equivalence, how many CW complexes X are there with $\pi_2(X) \cong \mathbb{Z}$, $\pi_3(X) \cong \mathbb{Z}/2\mathbb{Z}$, and all other homotopy groups trivial? Describe each of these spaces as explicitly as you can.

(3) Fix $n \geq 2$. Describe explicitly the first $2n$ stages of a Postnikov tower for $\mathbb{C}P^n$ consisting of principal fibrations (up through what Hatcher labels X_n), and compute the corresponding k-invariants $k_1, \dots, k_{2n-1}$. Show further that k_{2n} is nontrivial.

Optional problems:

Some good qual-level problems:

- Hatcher 4.3.11, 4.3.12, 4.3.15, 4.3.17, 4.3.20, 4.3.21, 4.3.22.

Some more problems to think about but not turn in:

- (1) Fix $n \geq 2$. Let X be a CW complex such that $\pi_i(X) = 0$ for $i < n$. Show that there is a map $X \to K(\pi_n(X), n)$ inducing an isomorphism on π_n.
  (2) Given a simply-connected space X and an integer n, by taking iterated homotopy fibers, build another space Y and a map $f\colon Y \to X$ so that $\pi_i(Y) = 0$ for $i < n$ and $f_*\colon \pi_i(Y) \to \pi_i(X)$ is an isomorphism for $i \geq n$. (Do not use the existence of Moore-Postnikov towers.)
- Hatcher 4.3.8, 4.3.9.
- Deduce the bundle homotopy lemma for fibrations as stated in class from Hatcher, Proposition 4.62.
- The point of this exercise is to illustrate that inverse limits can be somewhat bonkers, even for nice maps between nice spaces. Consider the inverse system

$$S^1 \leftarrow S^1 \leftarrow S^1 \leftarrow \cdots,$$

  where each map in the sequence is the double cover $z \mapsto z^2$. Let S^1_n denote the n^{th} term in this sequence and $p_n\colon S^1_{n+1} \to S^1_n$ the n^{th} map. Let $S^1_\infty = \varprojlim S^1_n$.
  (1) Let $(z_1, z_2, \dots) \in S^1_\infty$. Show that the universal covering maps $q_n\colon \mathbb{R} \to S^1_n$, $q_n(t) = z_n^{-1} e^{2\pi i t/2^n}$, induce a continuous map $q_\infty\colon \mathbb{R} \to S^1_\infty$.
  (2) Show that the image of q_∞ is a path component of S^1_∞.
  (3) Show that q_∞ is not surjective.
  (4) Show that the image of q_∞ is dense.
  (5) Conclude that S^1_∞ is not locally path connected.

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