MATH 636 SPRING 2024
HOMEWORK 8
DUE MAY 28, 2024 (TUESDAY)

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Required problems:

(1) Hatcher 4.B.2 (p. 428). Note: you solved part of this in a previous homework. You don't have to re-prove that part; just cite it.
(2) Prove that, for the category of based topological spaces, the reduced suspension functor is left adjoint to the based loop space functor.
(3) Hatcher 4.2.34 (p. 392).
(4) Hatcher 4.2.37 (p. 392). (You'll have to read the definitions of Whitehead products and H-spaces first.)

Optional problems:
Some good qual-level problems:

- Hatcher 4.2.14, 4.2.19, 4.2.30.

Some more problems to think about but not turn in:

- In required problem (2), you prove that $\mathrm{Hom}(\Sigma(X, x_0), (Y, y_0)) \cong \mathrm{Hom}((X, x_0), \Omega(Y, y_0))$, naturally. Here, there are two ways you can interpret $\cong$: a bijection of sets, which is easier and what the problem intended, or as a homeomorphism of topological spaces, where the mapping spaces have the compact-open topology. Prove the stronger result, for the category of based locally compact, Hausdorff spaces. In particular, this implies that $[\Sigma(X, x_0), (Y, y_0)] = [(X, x_0), \Omega(Y, y_0)]$ (for X, Y locally compact, Hausdorff).
- In this problem, we show that Hopf's original definition of the Hopf invariant is a homotopy invariant. The problem assumes a little more knowledge of transversality than we have covered in class.
  (1) Let $f\colon S^3 \to S^2$ be a smooth map and p, q, r regular values of f. Show that $\mathrm{lk}(f^{-1}(p), f^{-1}(q)) = \mathrm{lk}(f^{-1}(p), f^{-1}(r))$. (Hint: consider a path γ from q to r with the property that f is transverse to γ.)
  (2) Show that every continuous map $S^3 \to S^2$ is homotopic to a smooth map, and that if two smooth maps are homotopic then they are homotopic via a smooth map $S^3 \times [0, 1] \to S^2$.
  (3) Now, suppose that $f, g\colon S^3 \to S^2$ are homotopic smooth maps and p, q are regular values of both f and g. Show that $\mathrm{lk}(f^{-1}(p), f^{-1}(q)) = \mathrm{lk}(g^{-1}(p), g^{-1}(q))$.
  (4) Given a continuous map $f\colon S^3 \to S^2$, let $\bar{f}$ be a smooth map homotopic to f and p, q regular values of $\bar{f}$. Prove that

$$\mathrm{lk}(\bar{f}^{-1}(p), \bar{f}^{-1}(q))$$

  is an invariant of the homotopic class of f.
- Let $K_1, K_2 \subset S^3$ be knots (smoothly embedded circles) and $\Sigma_1, \Sigma_2 \subset B^4$ smoothly embedded, orientable surfaces with $\partial\Sigma_i = K_i$. Show that $\mathrm{lk}(K_1, K_2) = \pm\#\Sigma_1 \cap \Sigma_2$.

(Optionally, note that orientations of the K_i induce orientations of the Σ_i, and the sign in the formula becomes $+$.)

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