MATH 692 SPRING 2024
HOMEWORK 3
DUE JUNE 7, 2024.

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Solve any five of these problems. (Problems marked with stars are also available as mini-paper topics.)

(1) In class, we discussed Thurston's classification of mapping classes as periodic, reducible, or pseudo-Anosov. How does this work in the genus 1 case? That is, which elements of $SL(2,\mathbb{Z})$ correspond to periodic diffeomorphisms? To reducible ones? To Anosov ones? Be as explicit as you can.

(2) In class, we argued that (orientable) Σ_g-bundles over X for $g > 1$ correspond to homotopy classes of maps $X \to BMod_g$. Show that this is not the case for $g = 0$ by finding a nontrivial S^2 bundle over S^2.

(3) Let $p\colon X \to \mathbb{C}P^1$ be a (topological) Lefschetz fibration. Prove that the monodromy around a critical point is a Dehn twist. (Hint: I think this problem takes some work. Reduce this to a local computation for a model where the base is a small disk and the regular fiber is an annulus.)

(4) Show that any factorization of the identity map of Σ_g into positive Dehn twists induces a Lefschetz fibration over $\mathbb{C}P^1$ with generic fiber Σ_g and one critical point for each factor in the factorization.

(5) Let M be a (G,X)-manifold and $D\colon \widetilde{M} \to X$ the developing map of M, starting from some point $\widetilde{m}_0 \in \widetilde{M}$. Show that for $\gamma \in \pi_1(M, m_0)$ there is a unique $g_\gamma \in G$ so that

$$D(\gamma \cdot \widetilde{m}_0) = g_\gamma \cdot D(\widetilde{m}_0).$$

(On the left side, $\cdot$ denotes the action of deck transformations on the universal cover, while on the right it denotes the action of G on X.)

(6) Continuing the previous problem, show that D induces a homomorphism $\rho\colon \pi_1(M, m_0) \to G$, $\rho(\gamma) = g_\gamma$.

(7) Continuing the previous problem, show that different choices in the construction of D give conjugate representations.

(8) Suppose that X is a Riemannian manifold and G is a subgroup of the group of isometries of X which acts transitively on X. Let M be a (G,X)-manifold. Show that M inherits a Riemannian metric from X.

(9) With notation as in the previous problem, explain how the exponential map of M (plus some choices of basepoints) induces a map $\widetilde{M} \to X$. Show that this map agrees with the developing map. Deduce that M is complete in the sense of (G,X)-manifolds if and only if M is geodesically complete.

(10) We defined an affine structure on T^2 by viewing T^2 as a quotient of the sector

$$\{re^{i\theta} \mid 1 < r < r_0,\ 0 < \theta < \theta_0\}$$

by multiplication by r_0, $e^{i\theta_0}$. Give an explicit description of this affine structure, in terms of coordinate charts, and check that the transition functions are affine transformations.

(11) Choose a particular quadrilateral in $\mathbb{R}^2$ so that all four sides have different lengths. As we discussed in class, this induces an affine structure on T^2. Draw the images of, say, 10 fundamental domains under the corresponding developing map $\widetilde{T^2} = \mathbb{R}^2 \to \mathbb{R}^2$, either carefully by hand or using a computer.

(12) As in the previous problem, any quadrilateral in $\mathbb{R}^2$ induces an affine structure on T^2. For which quadrilaterals is the corresponding developing map a covering space of $\mathbb{R}^2$ (i.e., complete)?

(13) With notation as in the previous problem, for which quadrilaterals is the developing map a covering space of its image?

(14) Show that $\mathbb{C}P^2 \# \mathbb{C}P^2 \not\simeq \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, where $\overline{\mathbb{C}}P^2$ is $\mathbb{C}P^2$ with its orientation reversed. (Hint: consider the cup product, i.e., the intersection forms.)

(15) Given a closed, oriented 3-manifold Y, let $TH_1(Y)$ denote the torsion subgroup of $H_1(Y)$. Define a map $TH_1(Y) \otimes TH_1(Y) \to \mathbb{Q}/\mathbb{Z}$ as follows. Given $x \in TH_1(Y)$ there is some integer n so that $nx = 0 \in H_1(Y)$. Thus, $nx = \partial z$ for some $z \in C_2(Y)$. Given another $y \in TH_1(Y)$ let $\gamma = \mathrm{PD}(y) \in H^2(Y)$ and let

$$L(x, y) = \frac{1}{n}\gamma(z).$$

This is called the *linking form* of Y.

(a) Prove that the linking form is well-defined.

(b) Compute the linking form for the lens space $L(p, q)$, and deduce that $L(p, q) \simeq L(p, q')$ only if $qq' \equiv \pm n^2 \pmod{p}$ for some n. (Compare Hatcher, Exercise 3.E.2.)

(c) Use the linking form to show that $L(3, 1)$ has no orientation-reversing self-homeomorphism.

(d) Use the linking form to show that $L(3, 1)\#\overline{L(3, 1)} \not\simeq L(3, 1)\#L(3, 1)$.

(16) Prove that every torus $T^2 \subset S^3$ bounds a solid torus. (This should be fairly easy from results we proved in class.) Give an example of a (closed, orientable) 3-manifold Y and a compressible torus $T^2 \subset Y$ which does not bound a solid torus.

(17) (Hatcher's notes Exercise 1.5) Show that if $M^3 \subset \mathbb{R}^3$ is a compact submanifold with $H_1(M) = 0$, then $\pi_1(M) = 0$.

(18) Let $F = F_1 \amalg \cdots \amalg F_k \subset M$ be a collection of disjoint, incompressible surfaces. Prove that a surface $\Sigma \subset M \setminus F$ is incompressible in M if and only if Σ is incompressible in $M \setminus F$.

(19) We stated in class that, given a compact, orientable, irreducible M^3, there is a bound on the number of disjoint, incompressible surfaces so that no component of their complement is a product $\Sigma \times I$ of a closed surface and an interval. (See also Hatcher's notes.) Is the hypothesis that M be irreducible needed?

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