MATH 635 HOMEWORK 8
DUE MARCH 6, 2019.

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Problems to turn in:

(1) Hatcher 3.3.21 (p. 259).
(2) Hatcher 3.3.30 (p. 260).
(3) Hatcher 3.3.31 (p. 260).
(4) Hatcher 3.3.32 (p. 260).
(5) (a) Let Y be a closed, connected, orientable 3-manifold with $H_1(Y)$ finite and $\phi\colon Y \to Y$ a map of degree -1. Show that ϕ has a fixed point.
  (b) Construct an example of a closed, connected, orientable 3-manifold Y with $H_1(Y) \cong \mathbb{Z}$ and a map $\phi\colon Y \to Y$ of degree -1 with no fixed points.

Review / qualifying exam practice (not to turn in):

(1) Hatcher 3.3.20, 3.3.25, 3.3.26.

More problems to think about but not turn in:

(1) Hatcher 3.3.22, 3.3.23, 3.3.27–29.
(2) Let X be a topological space. A *locally-finite singular n-chain* in X is a (possibly infinite) formal linear combination $\sum k_\sigma \sigma \in \prod_{\sigma\colon \Delta^n \to X} \mathbb{Z}$ so that for each compact set $C \subset X$,
$$\{\sigma \mid k_\sigma \neq 0 \text{ and } \sigma^{-1}(C) \neq \emptyset\}$$
is finite. Let $C_n^{BM}(X)$ denote the abelian group of locally-finite singular n-chains in X.
  (a) Show that the "obvious" boundary operator $\partial\colon C_n^{BM}(X) \to C_{n-1}^{BM}(X)$,
$$\partial \sum_\sigma k_\sigma \sigma = \sum_\sigma \sum_{i=0}^n (-1)^i k_\sigma \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$
is well-defined and makes $C_n^{BM}(X)$ into a chain complex. (*Hint.* This would *not* be true without the "locally-finite" condition, so your proof had better use it!)
  (b) The homology of (C_*^{BM}, ∂) is called the *Borel-Moore homology of M*, and denoted $H_*^{BM}(X)$. Observe that if X is compact then $H_*^{BM}(X) \cong H_*(X)$.
  Compute $H_*^{BM}(\mathbb{R})$.
  (c) Show that if M is a connected, orientable n-manifold, not necessarily compact, then $H_n^{BM}(M) \cong \mathbb{Z}$. In particular, any oriented n-manifold M has a fundamental class $[M] \in H_n^{BM}(M)$.
  (d) Show that there is a well-defined cap product
$$H_{i+j}^{BM}(X) \otimes H^i(X) \to H_j^{BM}(X).$$
  (e) Imitate Hatcher's proof of Poincaré duality to show that for any oriented n-manifold M,
$$[M] \cap \cdot\colon H^i(X) \to H_{n-i}^{BM}(X)$$

is an isomorphism. In particular, this gives an alternative proof of Poincaré duality for compact manifolds.

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