# MATH 635 HOMEWORK 5
# DUE FEBRUARY 12, 2021.

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(1) Hatcher 3.1.9 (p. 205).
(2) Hatcher 3.1.10 (p. 205), part about the universal coefficient theorem (you did the rest last week).
(3) Hatcher 3.1.11 (p. 205).
(4) Hatcher 3.A.1 (p. 267).

Suggested review / qualifying exam practice (not to turn in):

(1) Hatcher 3.1.1, 3.1.2, 3.1.3, 3.1.4.
(2) Hatcher 3.A.2, 3.A.3, 3.A.4, 3.A.5, 3.A.6.
(3) Here is a more abstract proof of the universal coefficient theorem; fill in the details.
   (a) Let $C_*$ be a chain complex of free abelian groups, and $H_*$ the homology of $C_*$, which we view as a chain complex with trivial homology. Assume that $C_i = 0$ for all sufficiently small $i$ (i.e., $C_*$ is *bounded below*). Show that there is a chain map $C_* \to H_*$ inducing an isomorphism on homology (i.e., a *quasi-isomorphism* from $C_*$ to $H_*$).
   (b) Given a bounded-below chain complex $C_*$ over a ring $R$, choose a chain complex $P_*$ of free $R$-modules which is quasi-isomorphic to $C_*$. For $M$ an $R$-module, define $\mathrm{Tor}^i(C_*, M)$ to be the $i^{\text{th}}$ homology group of $P_* \otimes M$. Prove that:
      (i) $\mathrm{Tor}^i(C_*, M)$ is independent of the choice of $P_*$, up to (canonical) isomorphism.
      (ii) If $C_*$ is quasi-isomorphic to $D_*$ then $\mathrm{Tor}^i(C_*, M) \cong \mathrm{Tor}^i(D_*, M)$.
   (c) If the differential on $C_*$ vanishes, then
$$\mathrm{Tor}^i(C_*, M) \cong (C_i \otimes M) \oplus \mathrm{Tor}^1(C_{i-1}, M) \oplus \mathrm{Tor}^2(C_{i-2}, M) \oplus \cdots .$$
   (d) By definition, if $C_*$ is a chain complex of free $R$-modules then $\mathrm{Tor}^i(C_*, M) \cong H_i(C_* \otimes M) =: H_i(C_*; M)$. So, deduce the universal coefficient theorem for homology.
(4) Prove the universal coefficient theorem with $\mathbb{Z}$ replaced by any PID.

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