MATH 635 HOMEWORK 7
DUE FEBRUARY 26, 2021.

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(1) Use the Künneth theorem to compute the homology groups of the following spaces. Then use the universal coefficient theorem to compute their cohomology groups:
  (a) $S^1 \times M_g$, where M_g is the closed, orientable surface of genus g.
  (b) $K \times K$, where K is the Klein bottle.
  (c) $M(\mathbb{Z}/p\mathbb{Z}, m) \times M(\mathbb{Z}/q\mathbb{Z}, n)$ (a product of two Moore spaces), where p and q are (not necessarily distinct) primes.
  (d) $M(\mathbb{Z}/4\mathbb{Z}, 2) \times M(\mathbb{Z}/6\mathbb{Z}, 3)$.

(2) Hatcher 3.2.15 (p. 230). (You can skip the "and the spaces in the preceding three exercises" part.)

(3) Hatcher 3.B.3 (p. 280).

(4) Let $F, G\colon C_*(X) \otimes C_*(Y) \to C_*(X) \otimes C_*(Y)$ be chain maps, for each pair of spaces X, Y, with the following properties:
  (a) F and G are natural, in the sense that given $f\colon X \to X'$, $g\colon Y \to Y'$,

$$F(f_*(\alpha) \otimes g_*(\beta)) = (f_* \otimes g_*)(F(\alpha \otimes \beta))$$
$$G(f_*(\alpha) \otimes g_*(\beta)) = (f_* \otimes g_*)(G(\alpha \otimes \beta)).$$

  (b) On $C_0(X) \otimes C_0(Y)$, F and G are the identity map.

  Prove that F is chain homotopic to G. (This was an omitted step in our proof of the Eilenberg-Zilber theorem.)

Suggested review / qualifying exam practice (not to turn in):

(1) Hatcher 3.2.16, 3.2.18, 3.B.1.

More problems to think about but not turn in:

(1) Hatcher 3.B.5 (p. 280).

(2) Read the remaining problems in Sections 3.2 and 3.B and solve any that seem challenging or interesting.

(3) By the Eilenberg-Zilber theorem, $C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$; let $E\colon C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ be a one of the chain homotopy equivalences constructed in the proof of the Eilenberg-Zilber theorem. There is an induced chain homotopy equivalence

$$E^T\colon C^*(X) \otimes C^*(Y) \to C^*(X \times Y).$$

  Now, define a map $\cup\colon C^i(X) \otimes C^j(X) \to C^{i+j}(X)$ to be the composition

$$C^i(X) \otimes C^j(X) \xrightarrow{E^T} C^{i+j}(X \times X) \xrightarrow{\Delta^*} C^{i+j}(X),$$

  where $\Delta\colon X \to X \times X$ is the diagonal map.
  (a) Show that $\cup$ induces a map $\cup\colon H^i(X) \otimes H^j(X) \to H^{i+j}(X)$.

(b) Show that the map $\cup$ on homology is independent of the choice of chain homotopy equivalence E.
(c) Show that the this cup product on cohomology is natural, unital, and associative. (Associativity probably takes a little work.)
(d) Show that this definition of the cup product agrees with the one given in Hatcher.

(4) Read the acyclic models theorem in, for instance, Spanier and use it to shorten our proof of the EIlenberg-Zilber theorem.
(5) Give a proof of the algebraic Künneth theorem similar to the proof of the universal coefficient theorem on Homework 5.

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