MATH 635 HOMEWORK 9
DUE MARCH 12, 2021.

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(1) Hatcher 3.3.15 (p. 259).
(2) Hatcher 3.3.16 (p. 259).
(3) Hatcher 3.3.21 (p. 259). (You don't have to answer the parenthetical question.)
(4) Hatcher 3.3.30 (p. 260).
(5) Hatcher 3.3.33 (p. 260).

Suggested review / qualifying exam practice (not to turn in):

(1) Hatcher 3.3.20, 3.3.25, 3.3.26.

More problems to think about but not turn in:

(1) Hatcher 3.3.22, 3.3.23, 3.3.27–29.
(2) Let X be a topological space. A *locally-finite singular n-chain* in X is a (possibly infinite) formal linear combination $\sum k_\sigma \sigma \in \prod_{\sigma\colon \Delta^n \to X} \mathbb{Z}$ so that for each compact set $C \subset X$,
$$\{\sigma \mid k_\sigma \neq 0 \text{ and } \sigma^{-1}(C) \neq \emptyset\}$$
is finite. Let $C_n^{BM}(X)$ denote the abelian group of locally-finite singular n-chains in X.
   (a) Show that the "obvious" boundary operator $\partial\colon C_n^{BM}(X) \to C_{n-1}^{BM}(X)$,
   $$\partial \sum_\sigma k_\sigma \sigma = \sum_\sigma \sum_{i=0}^{n} (-1)^i k_\sigma \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$
   is well-defined and makes $C_n^{BM}(X)$ into a chain complex. (*Hint.* This would *not* be true without the "locally-finite" condition, so your proof had better use it!)
   (b) The homology of (C_*^{BM}, ∂) is called the *Borel-Moore homology of M*, and denoted $H_*^{BM}(X)$. Observe that if X is compact then $H_*^{BM}(X) \cong H_*(X)$.
   Compute $H_*^{BM}(\mathbb{R})$.
   (c) Show that if M is a connected, orientable n-manifold, not necessarily compact, then $H_n^{BM}(M) \cong \mathbb{Z}$. In particular, any oriented n-manifold M has a fundamental class $[M] \in H_n^{BM}(M)$.
   (d) Show that there is a well-defined cap product
   $$H_{i+j}^{BM}(X) \otimes H^i(X) \to H_j^{BM}(X).$$
   (e) Imitate Hatcher's proof of Poincaré duality to show that for any oriented n-manifold M,
   $$[M] \cap \cdot\colon H^i(X) \to H_{n-i}^{BM}(X)$$
   is an isomorphism. In particular, this gives an alternative proof of Poincaré duality for compact manifolds.

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