MATH 635
OPTIONAL HOMEWORK ON INVERSE LIMITS.

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This is purely for your enjoyment, not to turn in.

Let $(I, \leq)$ be a directed set. An *inverse system* of abelian groups (or, more generally, R-modules) consists of:

- For each $\alpha \in I$, an abelian group (R-module) G_α, and
- For each pair $\alpha, \beta \in I$ with $\alpha \leq \beta$, a homomorphism $f_{\alpha,\beta} \colon G_\beta \to G_\alpha$,

such that:

- For any $\alpha \in I$, $f_{\alpha,\alpha} = \mathbb{I}$.
- For any $\alpha, \beta, \gamma \in I$ with $\alpha \leq \beta \leq \gamma$, $f_{\alpha,\gamma} = f_{\alpha,\beta} \circ f_{\beta,\gamma} \colon G_\gamma \to G_\alpha$.

Equivalently, if we view $(I, \leq)$ as a category $\mathcal{I}$ then an inverse system is a contravariant functor from $\mathcal{I}$ to the category of abelian groups (or R-modules).

Here are some examples of inverse systems:

(a) $I = \mathbb{Z}$ with the usual order, $G_n = \mathbb{Z}/24$, and every map $f_{m,n}$ is the zero map.
(b) $I = \mathbb{Z}$ with the usual order, $G_n = \mathbb{Z}/24$, and every map $f_{m,n}$ is the identity map.
(c) $I = \mathbb{Z}_{\geq 0}$ the non-negative integers, $G_n = \mathbb{R}[x]/(x^{n+1})$ the set of polynomials of degree $\leq n$, and $f_{m,n} \colon G_n \to G_m$ the obvious quotient map, i.e., $f_{m,n}(x^k) = x^k$ if $k \leq m$ and 0 if $k > m$.
(d) $I = \mathbb{Z}_{>0}$. Fix a prime p, and let $G_n = \mathbb{Z}/p^n\mathbb{Z}$, and $f_{m,n} \colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ be the map which sends 1 to 1.
(e) X a topological space, I the set of compact subsets of X ordered by inclusion, and $f_{K,L} \colon H_j(X, X \setminus L) \to H_j(X, X \setminus K)$ induced by the inclusion of pairs $(X, X \setminus L) \hookrightarrow (X, X \setminus K)$ whenever $K \subset L$.

(1) Convince yourself that the inverse systems described above are, in fact, inverse systems.
(2) Given an inverse system $(\{G_\alpha\}_{\alpha \in I}, \{f_{\alpha,\beta}\}_{\alpha \leq \beta \in I})$, define

$$\varprojlim G_\alpha = \{(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} G_\alpha \mid f_{\alpha,\beta}(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta\} \subset \prod_{\alpha \in I} G_\alpha.$$

(Note: the TeX commands for direct and inverse limits are `\varinjlim` and `\varprojlim`, respectively.)

Show that there are homomorphisms $p_\alpha \colon \varprojlim G_\alpha \to G_\alpha$ so that for all $\alpha \leq \beta$, $p_\alpha \circ f_{\alpha,\beta} = p_\beta$.
(3) Compute the inverse limits of the systems (a)–(c), i.e., identify these inverse limits with familiar objects.
(4) Prove that the inverse limit satisfies the following universal property: given an abelian group H and homomorphisms $q_\alpha \colon H \to G_\alpha$ so that for all $\alpha \leq \beta$, $q_\alpha \circ f_{\alpha,\beta} = q_\beta$ there is a unique homomorphism $g \colon H \to \varprojlim G_\alpha$ so that for all α, $p_\alpha = q_\alpha \circ g$.

(5) The inverse limit of abelian groups (or R-modules) is naturally a topological space: give each group G_α the discrete topology, give $\prod_{\alpha\in I} G_\alpha$ the product topology (*not* the box topology), and then give $\varprojlim G_\alpha$ the subspace topology from $\prod_{\alpha\in I} G_\alpha$.

Show that for example (d), the inverse limit is not discrete, though it is totally disconnected (every connected component consists of a single point). This inverse limit is denoted $\mathbb{Z}_p$, the p-adic integers. If you know another definition of $\mathbb{Z}_p$, prove it agrees with this one.

(6) Show that an inverse limit of chain complexes is a chain complex, in a natural way.

(7) The inverse limit functor does not preserve exact sequences so, in particular, does not commute with taking homology. To see this, consider the following inverse system of short exact sequences. The indexing set is $\mathbb{Z}_{>0}$. Let $A_i = \mathbb{Z}$, $B_i = \mathbb{Z}$, $C_i = \mathbb{Z}/p^i\mathbb{Z}$, $h_i\colon A_i \to B_i$, and maps given by

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_i = \mathbb{Z} \arrow[r, "\cdot p^i"] \arrow[d, "\cdot p"] & B_i = \mathbb{Z} \arrow[r] \arrow[d, "Id"] & C_I = \mathbb{Z}/p^i\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i-1} = \mathbb{Z} \arrow[r, "\cdot p^{i-1}"] \arrow[d] & B_i = \mathbb{Z} \arrow[r] \arrow[d] & C_{i-1} = \mathbb{Z}/p^{i-1}\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

So, the inverse system of C_i is example (d). Show that the map from the inverse limit of the B_i to the inverse limit of the C_i is not surjective.

(There is a derived functor associated to inverse limits, called $\varprojlim^1$, similar to how Tor^1 is associated to tensor products.)

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