MATH 635  
OPTIONAL HOMEWORK ON INVERSE LIMITS.

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This is purely for your enjoyment, not to turn in.

Let \((I, \leq)\) be a directed set. An inverse system of abelian groups (or, more generally, \(R\)-modules) consists of:

- For each \(\alpha \in I\), an abelian group (\(R\)-module) \(G_\alpha\), and
- For each pair \(\alpha, \beta \in I\) with \(\alpha \leq \beta\), a homomorphism \(f_{\alpha, \beta} : G_\beta \to G_\alpha\),

such that:

- For any \(\alpha \in I\), \(f_{\alpha, \alpha} = \text{id}\).
- For any \(\alpha, \beta, \gamma \in I\) with \(\alpha \leq \beta \leq \gamma\), \(f_{\alpha, \gamma} = f_{\alpha, \beta} \circ f_{\beta, \gamma} : G_\gamma \to G_\alpha\).

Equivalently, if we view \((I, \leq)\) as a category \(\mathcal{I}\) then an inverse system is a contravariant functor from \(\mathcal{I}\) to the category of abelian groups (or \(R\)-modules).

Here are some examples of inverse systems:

(a) \(I = \mathbb{Z}\) with the usual order, \(G_\alpha = \mathbb{Z}/24\), and every map \(f_{m, n}\) is the zero map.

(b) \(I = \mathbb{Z}\) with the usual order, \(G_\alpha = \mathbb{Z}/24\), and every map \(f_{m, n}\) is the identity map.

(c) \(I = \mathbb{Z}_{\geq 0}\) the non-negative integers, \(G_\alpha = \mathbb{R}[x]/(x^{n+1})\) the set of polynomials of degree \(\leq n\), and \(f_{m, n} : G_n \to G_m\) the obvious quotient map, i.e., \(f_{m, n}(x^k) = x^k\) if \(k \leq m\) and \(0\) if \(k > m\).

(d) \(I = \mathbb{Z}_{> 0}\). Fix a prime \(p\), and let \(G_n = \mathbb{Z}/p^n\mathbb{Z}\), and \(f_{m, n} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}\) be the map which sends 1 to 1.

(e) \(X\) a topological space, \(I\) the set of compact subsets of \(X\) ordered by inclusion, and \(f_{K, L} : H_j(X, X \setminus L) \to H_j(X, X \setminus K)\) induced by the inclusion of pairs \((X, X \setminus L) \hookrightarrow (X, X \setminus K)\) whenever \(K \subset L\).

(1) Convince yourself that the inverse systems described above are, in fact, inverse systems.

(2) Given an inverse system \(\{G_\alpha\}_{\alpha \in I}, \{f_{\alpha, \beta}\}_{\alpha \leq \beta \in I}\), define

\[
\varprojlim G_\alpha = \{(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} G_\alpha \mid f_{\alpha, \beta}(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta\} \subset \prod_{\alpha \in I} G_\alpha.
\]

(Note: the TeX commands for direct and inverse limits are \texttt{\textbackslash varinjlim} and \texttt{\textbackslash varprojlim}, respectively.)

Show that there are homomorphisms \(p_\alpha : \varprojlim G_\alpha \to G_\alpha\) so that for all \(\alpha \leq \beta\), \(p_\alpha \circ f_{\alpha, \beta} = p_\beta\).

(3) Compute the inverse limits of the systems (a)–(c), i.e., identify these inverse limits with familiar objects.

(4) Prove that the inverse limit satisfies the following universal property: given an abelian group \(H\) and homomorphisms \(q_\alpha : H \to G_\alpha\) so that for all \(\alpha \leq \beta\), \(q_\alpha \circ f_{\alpha, \beta} = q_\beta\) there is a unique homomorphism \(g : H \to \varprojlim G_\alpha\) so that for all \(\alpha\), \(p_\alpha = q_\alpha \circ g\).
The inverse limit of abelian groups (or $R$-modules) is naturally a topological space: give each group $G_\alpha$ the discrete topology, give $\prod_{\alpha \in I} G_\alpha$ the product topology (not the box topology), and then give $\lim_{\leftarrow} G_\alpha$ the subspace topology from $\prod_{\alpha \in I} G_\alpha$.

Show that for example (d), the inverse limit is not discrete, though it is totally disconnected (every connected component consists of a single point). This inverse limit is denoted $\mathbb{Z}_p$, the $p$-adic integers. If you know another definition of $\mathbb{Z}_p$, prove it agrees with this one.

(6) Show that an inverse limit of chain complexes is a chain complex, in a natural way.

(7) The inverse limit functor does not preserve exact sequences so, in particular, does not commute with taking homology. To see this, consider the following inverse system of short exact sequences. The indexing set is $\mathbb{Z}_{>0}$. Let $A_i = \mathbb{Z}$, $B_i = \mathbb{Z}$, $C_i = \mathbb{Z}/p^i\mathbb{Z}$, $h_i: A_i \to B_i$, and maps given by

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A_i = \mathbb{Z} & \longrightarrow & B_i = \mathbb{Z} & \longrightarrow & C_i = \mathbb{Z}/p^i\mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A_{i-1} = \mathbb{Z} & \longrightarrow & B_{i-1} = \mathbb{Z} & \longrightarrow & C_{i-1} = \mathbb{Z}/p^{i-1}\mathbb{Z} & \longrightarrow & 0 \\
\end{array}
$$

So, the inverse system of $C_i$ is example (d). Show that the map from the inverse limit of the $B_i$ to the inverse limit of the $C_i$ is not surjective.

(There is a derived functor associated to inverse limits, called $\lim_1$, similar to how $\text{Tor}^1$ is associated to tensor products.)

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