Minerva Mini-Course Lecture 1 - Overview

Supplement to the slides

Robert Lipshitz

September 8, 2025

Here are some supplementary comments to the slides, spelling out precisely some statements that they leave vague. It is organized slide-by-slide; not every slide has comments.

6: Intuition from curves I

Lots of definitions here:

- A map $f : X \to Y$ is *injective* if different points in X map to different points in Y; *surjective* if every point in Y is the image of some point in X; and *bijective* if both injective and surjective.
- If $f : X \to Y$ bijective, then it has a well-defined inverse function $g : Y \to X$, satisfying $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in X$ and $y \in Y$ (exercise). Often one writes $g = f^{-1}$, of course.
- Suppose A is a subset of $\mathbb{R}^n$ and B is a subset of $\mathbb{R}^m$. A function $f : A \to B$ is *continuous* if it sends nearby points in A to nearby points in B. More precisely, for every point $x \in A$ and every $\epsilon > 0$ there is a $\delta > 0$ so that if $y \in A$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
- A function $f : X \to Y$ is a *homeomorphism* if f is continuous and bijective and f^{-1} is also continuous. The spaces X and Y are *homeomorphic* if there exists a homeomorphism from X to Y.
- *Exercise.* The property of being homeomorphic is an equivalence relation (for spaces).
- A *curve* is a subset $C \subset \mathbb{R}^n$ so that for every point $c \in C$, there is an open neighborhood $U \subset C$ of c which is homeomorphic to $(-\epsilon, \epsilon)$ or $[0, \epsilon)$ for some $\epsilon > 0$.
- *Exercise*. Equivalently, U is homeomorphic to $\mathbb{R}$ or $[0, \infty)$.
- *Exercise*. Implicit in the definition of curve on the slide is the following: if $f : (-\epsilon, \epsilon) \to (-\delta, \delta)$ is a continuous, bijective map, then f^{-1} is also continuous. (So, in this case, I can drop the latter condition from the definition of homeomorphism.)
- During the lecture, I typically require subspaces to be closed and bounded subsets of $\mathbb{R}^n$, i.e., *compact*.

- Note that the word "closed" is used for two different things for curves: being a closed subspace (contains all its limit points) and being a closed curve (no boundary). The latter usage is old and, in particular, predates the (relatively recent) modern formulation of point-set topology. It does cause confusion, but is unlikely to go away. (I think perhaps all students get confused by this double use of the world "closed" at some point.)
- A curve is *smooth* if both the maps $(-\epsilon, \epsilon) \to U \subset C$ can be chosen so that they and their inverses $U \to (-\epsilon, \epsilon)$ are both C^∞.
- *Exercise*. It is not enough to just require that the maps $(-\epsilon, \epsilon) \to U$ are C^∞. For example, find a C^∞ function $\mathbb{R} \to \mathbb{R}^2$ whose image is a square.
- A curve $C \subset \mathbb{R}^n$ is *topologically locally flat* (or, for these lectures, just *locally flat*) if for each point $p \in C$ there is an open neighborhood U of p in $\mathbb{R}^n$ and a homeomorphism $f : U \to B^n$ so that $f(U \cap C) = \{(x, 0, 0, \dots, 0) \in B^n\}$. That is, each point has a standard neighborhood that sits inside $\mathbb{R}^n$ in a standard way.
- *Exercise*. Smooth curves are locally flat. (Hint: use the Implicit Function Theorem, or one of its cousins.)
- *Exercise*. Write down a curve which is not locally flat.
- A point p on a curve is a *boundary point* if p has a neighborhood homeomorphic to $[0, \epsilon)$ by a homeomorphism sending p to 0, and an *interior point* if p has a neighborhood homeomorphic to $(-\epsilon, \epsilon)$. Prove that p cannot be both a boundary point and an interior point.

7: Intuition from curves II

For continuous curves, by *intrinsically the same* I mean homeomorphic (see above). For smooth curves, we require both the homeomorphism and its inverse to be smooth. That is:

- Given (closed) subsets $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$, a function $f : A \to B$ is *smooth* if f extends to a C^∞ function from an open neighborhood of A to $\mathbb{R}^n$.
- A function $f : A \to B$ is a *diffeomorphism* if f is bijective and smooth, and $f^{-1} : B \to A$ is also smooth. A and B are *diffeomorphic* if there exists a diffeomorphism from A to B.
- *Exercise*. Being diffeomorphic is an equivalence relation.
- For smooth curves, being diffeomorphic is what I mean by being intrinsically the same.

8: Intuition from curves III

The notion of isotopy from the slides is typically called *ambient isotopy*. That is, a *continuous ambient isotopy* of $\mathbb{R}^n$ is a continuous function $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ so that $F|_{\mathbb{R}^n \times \{0\}} = Id$ and $F|_{\mathbb{R}^n \times \{t\}}$ is a homeomorphism for every t. We say that subsets $A, B \subset \mathbb{R}^n$ are *continuously ambiently isotopic* (or, for this lecture, just *continuously isotopic*) if there is an ambient isotopy F of $\mathbb{R}^n$ so that $F(A \times \{1\}) = B$.

A *smooth ambient isotopy* is defined the same way, except F should be C^∞ and

$F|_{\mathbb{R}^n \times \{t\}}$ should be a diffeomorphism.

Exercise. Being continuously ambiently isotopic is an equivalence relation, as is being smoothly ambiently isotopic.

The two versions of ambient isotopy are our notions of extrinsically equivalent.

9: Surface basics

The basic definitions for curves carry over without real changes for surfaces:

- A *surface* is a subset $S \subset \mathbb{R}^n$ so that every point in S has a neighborhood homeomorphic to the open unit disks $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ or to the half-disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ y \geq 0\}$.
- A surface is *smooth* if these homeomorphisms can be chosen to be diffeomorphisms.
- We require surfaces to be *topologically locally flat*, meaning that for each point $p \in S$ there is an open neighborhood U of p in $\mathbb{R}^n$ and a homeomorphism $f : U \to B^n$ so that $f(U \cap C) = \{(x, y, 0, \dots, 0) \in B^n\}$.
- *Exercise*. Smooth surfaces are topologically locally flat.
- Surfaces are continuously/smoothly locally equivalent if they are homeomorphic/diffeomorphic.

11: Intrinsic equivalence

To reiterate, the reason the longitude and latitude on earth have to be badly behaved somewhere is that the sphere and the torus are not homeomorphic.

12: Orientability

There are many equivalent definitions of what it means for a space to be orientable, none of which are quick to state precisely. For a surface inside $\mathbb{R}^3$, orientability corresponds to having two sides, i.e., being able to construct a continuous normal vector field along the surface. This has the disadvantage of not looking intrinsic, and also doesn't generalize to surfaces in $\mathbb{R}^4$, so the slides mention a notion of "clockwise" and "counterclockwise". If you have a normal vector field to a surface in $\mathbb{R}^3$, you can get a notion of counterclockwise using the right-hand rule.

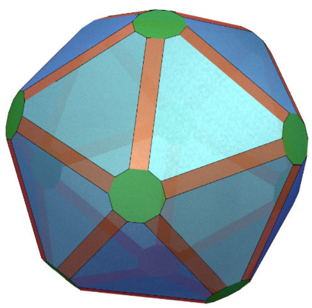
The usual generalization of having a well-defined notion of counterclockwise is in terms of having a preferred equivalence class of ordered bases for the tangent space. Another useful definition is being able to choose a notion of signed volume (or, for surfaces, area), as shows up in Stokes's Theorem.

13: Euler characteristic

The usual general setting for the Euler characteristic is *CW complexes* (also called *cell complexes*). See, for instance, Allen Hatcher's *Algebraic Topology* or John M. Lee's *Introduction to Topological Manifolds* for basic definitions.

Triangulated surfaces, or surfaces divided into polygons, are a special case.

The last example on the slide is a space built as a union of handles, instead of as a CW complex. An n-dimensional k-handle just means $D^k \times D^{n-k}$, attached along $S^{k-1} \times D^{n-k}$. So, if $n = 2$, 0-handles are attached along $\emptyset$, 1-handles are strips $D^1 \times D^1$ attached along pairs of arcs $S^0 \times D^1$, and 2-handles are disks $D^2 \times D^0$ attached along their entire boundaries. In the picture, the green 10-gons are 0-handles, the brown rectangles are 1-handles, and the blue hexagons are 2-handles. Surfaces divided into handles, as in the last example on the slide, induce cell complex structures, so one also gets an Euler characteristic as the alternating sum of the dimensions of the handles.



14: Classification of surfaces

The Classification of Surfaces theorem states:

Theorem. Let S and S' be compact surfaces, possibly with boundary. Assume that:

- S and S' have the same number of boundary components.
- Either S and S' are both orientable or they are both nonorientable.
- The Euler characteristics agree, $\chi(S) = \chi(S')$.

Then S and S' are homeomorphic. If S and S' are smooth surfaces, then they are diffeomorphic.

16 and 17: Weird phenomena in dimension 4

Two good graduate-level textbooks to read more about strange smooth phenomena in dimension 4 are *4-Manifolds and Kirby Calculus* by Robert Gompf and András Stipsicz and *The Wild World of 4-Manifolds* by Alexandru Scorpan.

19: Surfaces in the sea

Given a smooth function $f : S \to \mathbb{R}$, a *critical point* of f is a point p on S where the derivative of f vanishes. To make sense of this, we have to first parameterize S near p

by a smooth function $\mathbb{R}^2 \to \mathbb{R}^3$. That is, suppose $S \subset \mathbb{R}^3$. (The case of surfaces in $\mathbb{R}^4$ is similar.) Choose a smooth (C^∞) function

$$(x(u,v), y(u,v), z(u,v)) : \mathbb{R}^2 \to \mathbb{R}^3$$

so that $(x(0,0), y(0,0), z(0,0)) = p$ and

$$\begin{bmatrix} \frac{\partial x}{\partial u}(0,0) & \frac{\partial x}{\partial v}(0,0) \\ \frac{\partial y}{\partial u}(0,0) & \frac{\partial y}{\partial v}(0,0) \\ \frac{\partial z}{\partial u}(0,0) & \frac{\partial z}{\partial v}(0,0) \end{bmatrix}$$

has rank 2. (The latter guarantees that the function gives a reasonable set of coordinates near p.) Then the composition

$$g(u,v) := f(x(u,v), y(u,v), z(u,v))$$

is a function $\mathbb{R}^2 \to \mathbb{R}$. The function f is *smooth* if $g(u,v)$ is, and p is a critical point of f is $(0,0)$ is a critical point of $g(u,v)$ (i.e., $\frac{\partial g}{\partial u}(0,0) = \frac{\partial g}{\partial v}(0,0) = 0$).

Exercise. Prove that these definitions do not depend on which $(x(u,v), y(u,v), z(u,v))$ you choose.

If p is a critical point, with this choice of coordinates we get a second derivative matrix

$$Hess_p(f) \begin{bmatrix} \frac{\partial^2 g}{\partial y^2} & \frac{\partial^2 g}{\partial u \partial v} \\ \frac{\partial^2 g}{\partial v \partial u} & \frac{\partial^2 g}{\partial v^2} \end{bmatrix}.$$

I say *f passes the second derivative test at the critical point* p if $Hess_p(f)$ is invertible. (In that case, the second derivative test tells you if p is a local min, local max, or saddle. Otherwise, the second derivative test tells you nothing.)

Exercise. Prove that passing the second derivative test at p is independent of the choice of coordinates $(x(u,v), y(u,v), z(u,v))$ near p.

A *Morse function* is one that passes the second derivative test at all of its critical points.

Exercise. Prove that a Morse function on a (closed and bounded) surface S has only finitely many critical points.

21: Movies for surfaces in $\mathbb{R}^4$, II

The surface of revolution in the slide is given more precisely as follows. Fix an arc A in $\{(x,y,z) \mid x \geq 0\} \subset \mathbb{R}^3$ with both endpoints on the z-axis. Then the *spin of* A is the set

$$\{(x\cos(\theta), y, z, x\sin(\theta)) \mid (x,y,z) \in A,\ 0 \leq \theta \leq 2\pi\} \subset \mathbb{R}^4.$$

Exercise. Prove that the spin of A is a 2-sphere.

(Note that the word "spin" is used for lots of things in mathematics, mostly not connected to this.)

23: Another view of ribbon knots

A disk in $\mathbb{R}^3$ with boundary a knot K has only *ribbon singularities* if the following holds. Think of the disk as the image of a map $f : D^2 \to \mathbb{R}^3$ (so that the restriction of f to the boundary winds around K once). First, we require $f(u,y) = (f_1(u,v), f_2(u,v), f_3(u,v))$ to be an *immersion*, meaning that its total derivative matrix

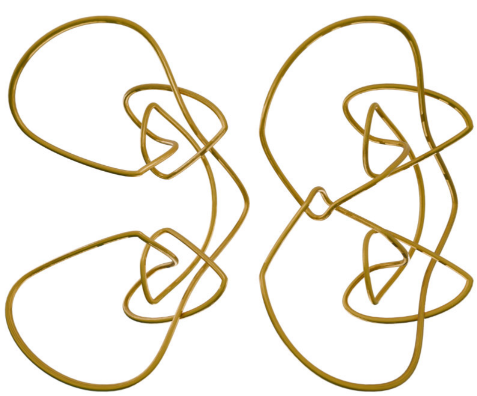
$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{bmatrix}$$

has rank 2 everywhere. (This is a smoothness condition.) Second, for every point $p \in \mathbb{R}^3$, $f^{-1}(p)$ consists of 0, 1, or 2 points. (That is, f has *only double point singularities*.) Finally, let $A \subset D^2$ be the preimages of the double points (so $f|_{D^2 \setminus A}$ is injective). Then we require that A is a union of disjoint arcs in D^2 each of which is either entirely in the interior of D^2 or has both endpoints on the boundary of D^2. (What is not allowed is for one endpoint of a double point arc to be in the interior and the other on the boundary.)

24: Another ribbon example

Symmetric union is an operation on diagrams, not on knots—and even for diagrams, it depends on some choices. I believe the following is a correct description; see the original paper (Kinoshita, Shin'ichi and Terasaka, Hidetaka, "On unions of knots", Osaka Mathematics Journal, 1957, 131–153) for a definitely correct description.

Start with a knot diagram K, and let $m(K)$ be the result of reflecting K across the y-axis, say. (I used the x-axis on the slides.) Put $m(K)$ above K and take their connected sum at, say, the right-most local maximum of K and the corresponding local minimum of $m(K)$. Then pick up some other local maxima of K and corresponding local minima of $m(K)$ and twist K with $m(K)$ near those points. The example from the slides, shown here rotated, does this twisting at only one point, with one full twist.



26: An exotic pair of disks

Suppose D and D' are smoothly embedded disks in B^4 with boundary the same knot $K \subset S^3$.

- We say D and D' are *exotic relative boundary* if
  1. There is a homeomorphism $\phi : B^4 \to B^4$ so that $\phi|_{S^3} = Id$ and $\phi(D) = D'$, but
  2. There is no diffeomorphism $\phi : B^4 \to B^4$ so that $\phi|_{S^3} = Id$ and $\phi(D) = D'$.
- We say that D and D' are *absolutely exotic* if
  1. There is a homeomorphism $\phi : B^4 \to B^4$ so that $\phi(D) = D'$, but
  2. There is no diffeomorphism $\phi : B^4 \to B^4$ so that $\phi(D) = D'$.

I think the more natural-seeming notion to newcomers is the notion of absolutely exotic, but the more natural one to experts is of exotic relative boundary. In the slides, I am not always careful about which of the two I mean when I write "exotic".

29: Exotic closed surfaces

The space RP^2 on the slide, the *real projective plane*, is the result of gluing a disk to a Möbius band by a homeomorphism of their boundaries. (The boundary of a Möbius band is a single circle.) It is the closed (connected) surface with the largest Euler characteristic, i.e., which can be built with the smallest number of handles (bands). Equivalently, it is the result of identifying opposite pairs of points on the 2-sphere. (Proving this gives a homeomorphic surface is a good exercise, if you've seen a little topology but not done it before.)

34: unknotting number.

We defined $g_4(K)$ to be the minimum genus of any smooth, orientable surface in B^4 with boundary K. This is called the *slice genus* or 4-*ball genus*. By definition, a knot is slice if and only if $g_4(K) = 1$.

The *unknotting number* $u(K)$ is the minimum number of times a knot has to pass through itself (transversely) in order to get the unknot.

Exercise. Prove that $g_4(K) \leq u(K)$.

Exercise. Prove that, in any knot diagram, you can change some subset of the crossings (exchanging under and over strands) to turn the diagram into a diagram for the unknot.

Exercise. Prove that $u(K)$ is the minimum, over all diagrams, of the number of crossing changes in that diagram needed to get a diagram for the unknot.

More exercises

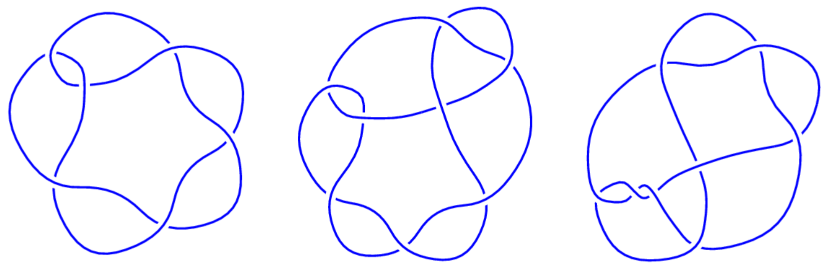
1. The paper arXiv:2205.15283 of Hayden-Kim-Miller-Park-Sundberg has a number of beautiful pictures of surfaces in 3-space, one of which was on the classifi-

cation of surfaces slide during the lecture. Compute the Euler characteristics of several of them—say, the surfaces in Figure 5, 9, and 20—and check if they are orientable, and then use the Classification of Surfaces theorem to identify them with model surfaces.

2. Prove that the connected sum of a knot with its mirror image always bounds a smooth disk in 4-space and, in fact, is a ribbon knot.

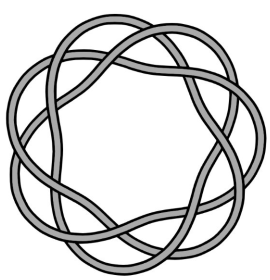
3. Prove the following knots are slice (bound smooth disks in the 4-ball), by

   1. Finding a movie for such a disk.
   2. Finding a ribbon disk in 3-space.
   3. Presenting the knot as a symmetric union.



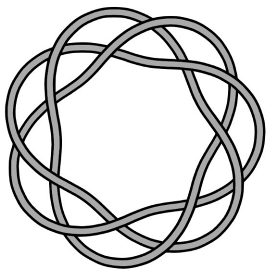
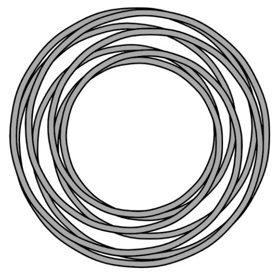
(All three parts are tricky. The diagrams are from KnotInfo; these are the knots 6_1, 8_8, and 8_9.)

4. Find the best upper bound you can on the unknotting number of $T(3,7)$, the $(3,7)$-torus knot:



(Picture produced by Blender.)

5. How are $T(3,7)$ and $T(7,3)$ related?

6. Prove the surface from slide 28 is non-orientable.
7. Prove equivalence of the two definitions of ribbon disks: as embedded disks in the 4-ball given by movies with no local maxima, and as disks in 3-space that are embedded except for ribbon singularities.