Minerva Mini-Course Lecture 2 Exercises

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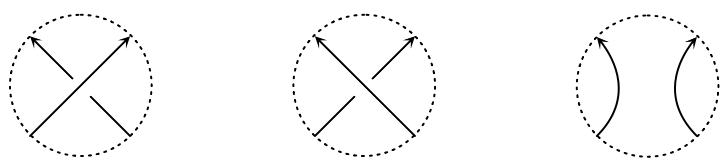
The Jones polynomial

1. Consider three oriented link diagrams that agree outside a disk, and in that disk they differ as shown:

   L_+ $\quad$ L_- $\quad$ L_0

   Show that $q^{-2}V_{L_+}(q) - q^2 V_{L_-}(q) = (q^{-1} - q)V_{L_0}(q)$.

2. Prove that the Jones polynomial is determined by three properties: it is a knot invariant, it takes a particular value $V_U(q)$ on the unknot U (for our normalization, $V_U(q) = q + q^{-1}$), and the *oriented skein relation* from the previous problem. (Hint: first show that these determine its value on n-component unlinks. Then use the exercise from last time that you can turn any link diagram into a diagram for the unlink by changing enough crossings, together with induction on the number of crossings.) Prove moreover that the value we chose for $V_U(q)$ is the only one so that if $L = L_1 \amalg L_2$ is a disjoint union of two links, then $V_L(q) = V_{L_1}(q)V_{L_2}(q)$.

Arguably, problem 2 demystifies the Jones polynomial, except that the oriented skein relation remains a mystery. The Alexander polynomial, which we will discuss next week, satisfies a similar oriented skein relation, but can also be defined using basic algebraic topology. So, one could imagine a world where the Jones polynomial was discovered by varying that skein relation to see what other polynomials arose.

3. Prove that the Jones polynomial is invariant under all three Reidemeister moves.

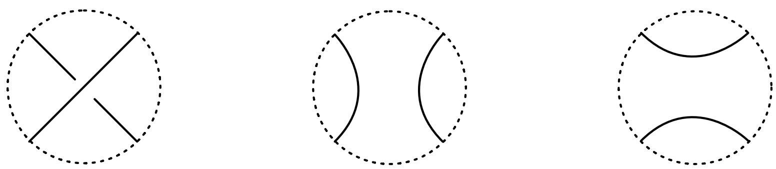
4. The *Kauffman bracket* of a link diagram L is defined by

$$\langle L \rangle = \sum_{v \in \{0,1\}^N} (-q)^{|v|}(q+q^{-1})^{|L_v|},$$

where L has N crossings and L_v is the complete resolution corresponding to the vector v. Prove that the Kauffman bracket is characterized by three properties: $\langle L_1 \amalg L_2 \rangle = \langle L_1 \rangle \langle L_2 \rangle$, $\langle U \rangle = q + q^{-1}$ (where U is an unknot diagram with no crossings), and the *unoriented skein relation*, that

$$\langle L \rangle = \langle L_0 \rangle - q \langle L_1 \rangle,$$

where L, L_0, and L_1 agree outside a disk, and in the disk are as follows:



L $\qquad$ L_0 $\qquad$ L_1

5. Prove that the Kauffman bracket is invariant under the second and third kinds of Reidemeister moves, but changes by a power of q under the first kind. (What power?) In fact, show that the Kauffman bracket is an invariant of *framed links*, i.e., links with a choice of nonvanishing normal vector field. (Also, if you are new to this, define precisely what isotopy of framed links should mean.)

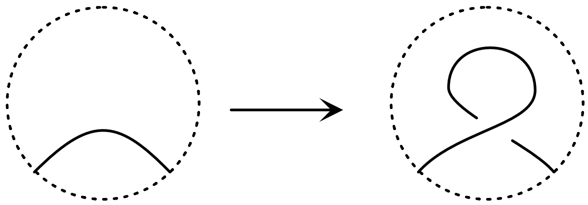
Khovanov homology

1. Let M be a closed, orientable n-manifold and $\mathbb{F}$ a field. The cup product makes $H^*(M;\mathbb{F})$ into a ring. The diagonal map $\Delta : M \to M \times M$ and the Künneth Theorem induce a map $\Delta_* : H_*(M;\mathbb{F}) \to H_*(M;\mathbb{F}) \otimes H_*(M;\mathbb{F})$, giving $H_*(M;\mathbb{F})$ a comultiplication. Via Poincar'e Duality, this also gives $H^*(M;\mathbb{F})$ a comultiplication. Prove that this multiplication and comultiplication satisfy the Frobenius relation from lecture. (Non-hint: this is probably tricky, without building up some more algebra first.)

2. Prove that the differential on the Khovanov complex does, in fact, preserve the quantum grading.

3. Check that the Khovanov cube commutes, using the Frobenius algebra relations and a case check. (There are more elegant ways to do this, but this is very convincing.)

4. Prove that Khovanov homology is invariant under the first kind of Reidemeister move, i.e.,



(Hint: there are two ways to resolve the crossing. The 0-resolution gives the original diagram, and the 1-resolution has an extra circle Z. Show that all the generators over the 0-resolution and the generators over the 1-resolution that label Z by 1 form an acyclic subcomplex, and consider the quotient by that subcomplex.)

5. Compute the Khovanov homology of the trefoil and the deformed Khovanov complex of the trefoil. (Actually, if you do this for both trefoils, this is twice as much fun.)

The s-invariant

11. Continuing the previous problem, compute the s-invariant of the trefoil.
12. Verify for the trefoil and the figure 8 knot that, over $\mathbb{Z}[X, h, h^{-1}]/(X^2 = Xh)$, the deformed Khovanov complex has homology $\mathbb{Z}[h] \oplus \mathbb{Z}[h]$. Do this by writing the complex in the basis $\{A, B\}$ from lecture and seeing how ti decomposes into simple pieces.
13. An orientation of a link diagram L induces a particular preferred complete resolution, the *oriented resolution*. Show that for the deformed Khovanov homology over $\mathbb{Z}[X, h, h^{-1}]/(X^2 = Xh)$, there is one generator for each orientation resolution, as follows. Fix an orientation, and let L_v be the corresponding oriented resolution. A checkerboard coloring of the regions in $\mathbb{R}^2 \setminus L$ induces a checkerboard coloring of L_v. Label a circle in L_v by A if the circle is oriented as the boundary of the black region, and by B otherwise. Prove the result is a cycle, and in fact that these cycles freely generate the homology.