Minerva Mini-Course Lecture 3 Exercises

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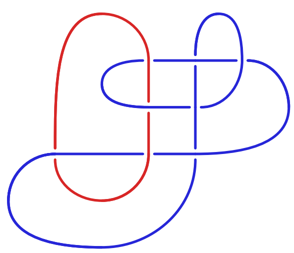
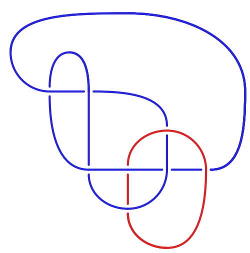
The linking number

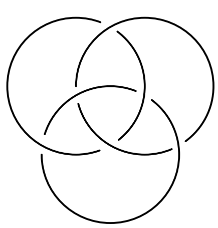
1. In the lecture, I defined the linking number of K_1 and K_2 to be the class that K_2 represents in $H_1(\mathbb{R}^3 \setminus K_1) \cong \mathbb{Z}$. Prove that this agrees, up to a sign, with the sum, over crossings of K_1 over K_2, of the sign of the crossing. Also, prove that both definitions are symmetric: $\mathrm{lk}(K_1, K_2) = \mathrm{lk}(K_2, K_1)$. (With the right orientation conventions, the "up to sign" isn't necessary, but tracking the signs is a little tedious.)

2. Compute the linking numbers of the following links (for some choices of orientations of the components):

   (Images from SnapPy. I chose these links somewhat randomly, and haven't actually done this exercise.)

3. Consider the Borromean rings $K_1 \cup K_2 \cup K_3$:

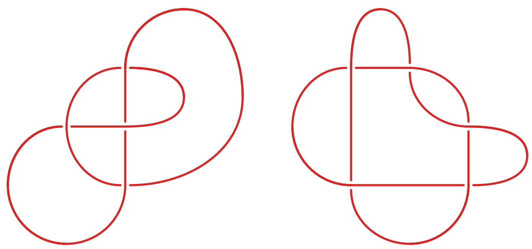
   

   Show that $\mathrm{lk}(K_i, K_j) = 0$ for $i \neq j$ (this is easy). Then prove that the Borromean rings are linked by computing $\pi_1(\mathbb{R}^3 \setminus (K_1 \cup K_2))$ and considering the image of K_3 in this group.

(Aside: this picture is misleading, in that it is not actually possible to build the Borromean rings using round circles. That is, the Borromean rings do not exist.)

Seifert matrices, the Alexander polynomial

4. Compute Seifert matrices and Alexander polynomials of the following knots:



(These are the figure-8 knot 4_1 and the torus knot $5_1 = T(5, 2)$. Pictures from SnapPy.)

5. I defined the *Alexander polynomial* of K to be

$$\Delta_K(t) = \det(t^{1/2}A - t^{-1/2}A^T),$$

where A is a Seifert matrix for K. Prove that $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$, not just $\mathbb{Z}[t^{1/2}, t^{-1/2}]$.

6. Prove that the Seifert matrix A satisfies $\det(A^T - A) = 1$. (Hint: show that $A^T - A$ is given by the intersection pairing on $H_1(\Sigma)$. Then either use Poincar'e duality to show that the intersection pairing has this property for any basis, or observe that it has this property for a simple choice of basis, and the determinant is basis-independent.)

7. The *width* of the Alexander polynomial is the maximum degree appearing in it minus the minimum degree appearing in it; since we have normalized our Alexander polynomials to be symmetric under $t \to t^{-1}$ (why?), the width is twice the maximum degree. Show that the half the width of $\Delta_K(t)$ is a lower bound on the genus of any Seifert surface for K.

8. Prove that $A - tA^T$ is a presentation matrix for $H_1(X_\infty)$ over $\mathbb{Z}[t, t^{-1}]$.

9. Continuing from the prevous problem, deduce that $H_1(X_\infty)$ is a finitely generated, torsion module over $\mathbb{Z}[t, t^{-1}]$. So, we can write $H_1(X_\infty; \mathbb{Q}) = \mathbb{Q}[t, t^{-1}]/p_1(t) \oplus \cdots \oplus \mathbb{Z}[t, t^{-1}]/p_k(t)$ for some polynomials $p_1(t), \ldots, p_k(t)$. Prove that $\Delta_K(t) = \pm a t^n p_1(t) \cdots p_k(t)$.

10. In lecture, I asserted:

Lemma. Any two Seifert matrices for K differ by the following moves:

i) Elementary enlargment / reduction:

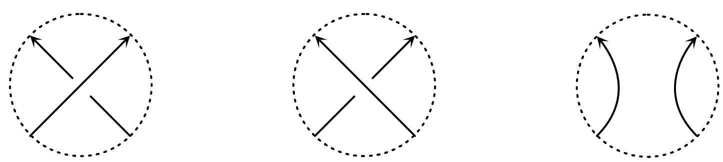
$$A \longleftrightarrow \begin{bmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{or} \qquad A \longleftrightarrow \begin{bmatrix} A & 0 & 0 \\ \xi^T & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Here, ξ is a column.

ii) $A \to P^T A P$ where $\det(P) = 1$.

(For a proof, see for instance Lickorish's book.) Assuming this lemma, prove that $\Delta_K(t)$ is a knot invariant.

11. Prove that the Alexander polynomial satisfies the following *oriented skein relation*: if L_+, L_-, and L_0 agree outside a ball, and inside the ball are given by



L_+ $\qquad$ L_- $\qquad$ L_0

then $\Delta_{L_+} - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\Delta_{L_0}$. (Hint: choose Seifert surfaces adapted to these local moves. For a solution, see Lickorish's book. Also, note that L_0 is a 2-component link; our definition of the Alexander polynomial from a Seifert matrix still works.)

12. Use the oriented skein relation to compute the Alexander polynomials for the knots 4_1 and 5_1 above.

Slice knots

13. The following knot diagram has an obvious genus 1 Seifert surface. Turn that Seifert surface into a slice disk by finding an unknotted curve on it along which you can do surgery in B^4. Why doesn't this strategy work to prove that the trefoil knot is slice (which is false)?