Minerva Mini-Course Lecture 4 Exercises

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The s-invariant

1. Problem 11 from Lecture 2's exercises actually corresponds to a topic covered in Lecture 4. Problems 12 and 13 from Lecture 2 correspond to material I have skipped so far, but might cover in a future lecture.
2. Given link diagrams D and D', the *disjoint union* $D \amalg D'$ just means drawing D next to D', with no crossings between them. Prove that the Khovanov complex $CKh(D \amalg D')$ is isomorphic to the tensor product $CKh(D) \otimes_{\mathbb{Z}} CKh(D')$. What does this imply about Khovanov homology? What are the analogous statements for the Bar-Natan deformation $CKh_h(D \amalg D')$?
3. Given a link diagram D, the *mirror* $m(D)$ of D is the result of reversing all the crossings (i.e., reflecting across the projection plane). Show that the Khovanov complex of $m(D)$ is the dual of the Khovanov complex of D. How do the gradings work?
4. Prove that if K is a knot, then $s(m(K)) = -s(K)$.
5. Extend our proof that the s-invariant is an obstruction to being topologically slice to prove: If K bounds a genus-g, orientable surface Σ in B^4, then $|s(K)| \leq 2g$. (Hint: this should be easy.)

Satellites

6. Prove that, up to isotopy in $\mathbb{R}^3 \setminus K$, the Seifert longitude of K is independent of choice of Seifert surface.

7. The Whitehead double of the trefoil I drew in lecture is *not* the diagram on the left below:

   Rather, it has several extra twists in the box of the diagram on the right. Do an isotopy from the diagram I drew to this diagram with extra twists, and see how many twists there are.

8. Explain how to turn (a bunch of copies of) a Seifert surface for K and a surface with boundary on $P \cup \partial(S^1 \times D^2)$ into a Seifert surface for K_P. (This leads to a proof of the Alexander polynomial formula for satellites.)

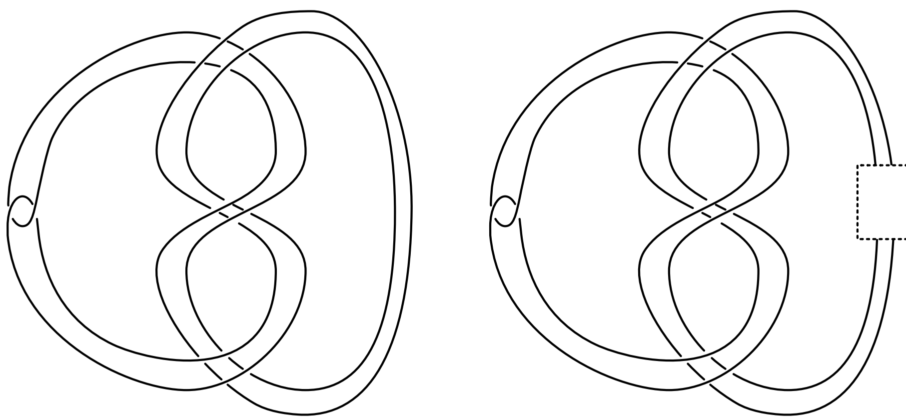

Figure 1: Twisted Whitehead doubles of the trefoil

A lemma from the construction of an exotic $\mathbb{R}^4$

9. Suppose M and N are n-dimensional, oriented manifolds with boundary, and $\phi_0, \phi_1 : \partial M \to \partial N$ are isotopic homeomorphisms (meaning there is a 1-parameter family of homeomorphisms $\phi_t : \partial M \to \partial N$ connecting them). Prove that $M \cup_{\phi_0} N$ and $M \cup_{\phi_1} N$ are homeomorphic. (Here, $M \cup_{\phi_0} N = M \amalg N / \sim$ where $\partial M \ni m \sim \phi_0(m) \in \partial N$, and similarly for ϕ_1.)

   Also, we didn't really need the conditions that M and N are manifolds, or that the subspaces being glued were their boundaries. In what more general setting does your proof work?