Minerva Mini-Course Lecture 5 Exercises

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Khovanov homotopy types in general

1. If K is an alternating (or, more generally, quasi-alternating) knot (or link), then the Khovanov homology $Kh_{i,j}(K;R)$, with coefficients in any ring R, is supported in gradings $2i-j=\sigma(K)\pm 1$, where $\sigma(K)$ is the signature of K. Using this, prove that if K is an alternating knot then, for each $j\in\mathbb{Z}$, its Khovanov spectrum $X_j(K)$ is a wedge sum of Moore spectra. (Recall that a Moore space is a simply-connected space $M(G,n)$ with $\widetilde{H}_n(M(G,n))\cong G$ and $\widetilde{H}_i(M(G,n))=0$ for $i\neq n$. A Moore spectrum is the suspension spectrum of a Moore space, or a formal desuspension of one. If the spectrum stuff is new, just view $X_j(K)$ as a CW complex well-defined up to suspension and homotopy equivalence, and prove that some suspension of it is homotopy equivalent to a wedge sum of Moore spaces.)

In particular, problem 1 shows that the Khovanov stable homotopy type has nothing new to say about alternating knots.

2. Prove that, for spectra, $X\times Y\simeq X\vee Y$. Equivalently, prove that, up to homotopy equivalence, the wedge sum of spectra is both a product and a coproduct.

The Burnside category

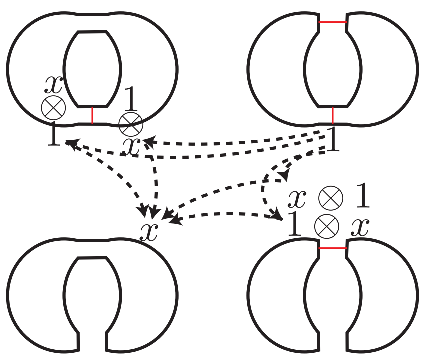
3. Prove that the category of sets is a subcategory of the Burnside category, as follows. The objects are the same. Given a map $f\colon X\to Y$ of sets, there is an $Y\times X$ matrix of sets whose (y,x) entry has one element if $f(x)=y$ and is empty otherwise.
4. In lecture, I asserted that a (strictly unitary, lax) 2-functor $\underline{2}^n\to\mathcal{B}$ is determined by its values on vertices, edges, and faces, and these values on faces satisfy a compatibility condition for each 3-cube. That compatibility condition is that a certain hexagon commutes; without tracing through the definitions, figure out what the compatibility must be (it's the only thing one could plausibly write down).

The ladybug matching

5. The interesting new data used to construct the Khovanov spectrum instead of Khovanov homology came from the *ladybug matching*, the bijections between matrices for certain faces. Specifically, this comes into play for faces that look like the disjoint union of the following picture with some other circles (which are not changing:) **Write**

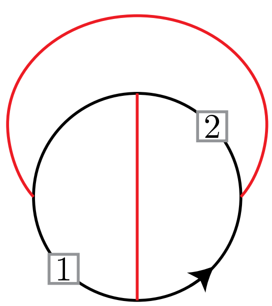
$$\begin{tikzcd} S^1 \amalg S^1 \arrow[d, "\text{merge}"] & S^1 \arrow[l, "\text{split}"'] \arrow[d, "\text{split}"] \\ S^1 & S^1 \amalg S^1 \arrow[l, "\text{merge}"] \end{tikzcd}$$

and the initial generator labels the circle 1 (so it maps to $1 \otimes X + X \otimes 1$ by both the horizontal and vertical maps, and those both map to X in the bottom left). For example:



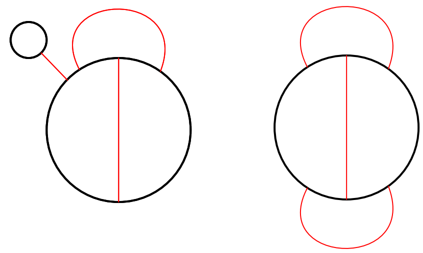
(Here, the red segments indicate where there were crossings previously.)

The bijection corresponding to such a configuration is defined as follows. Consider the two arcs on the initial S^1 that you get to by walking along a red segment and then turning right. Number them (arbitrarily) 1 and 2:



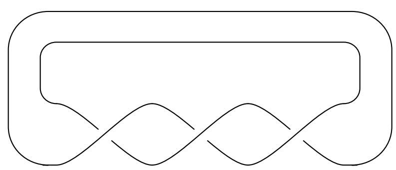
(a) Explain how to use this numbering to define the desired bijection across this face.

(b) Check that this bijection satisfies the compatibility from Problem 4 for the following 3-dimensional cubes:



Diagrams of spaces

6. The description I gave in the lecture of turning a diagram in the Burnside category into a diagram of spaces was deceptive. For example, consider the Khovanov cube associated to the trefoil



(So, the 111-resolution consists of two concentric circles.) We will focus on the part with quantum grading -5. (In the 111-resolution, this corresponds to the generator where both circles are labeled 1.) The induced cube of spaces should have the form

$$\begin{tikzcd}
& S^n \vee S^n \arrow[dd] & & S^n \arrow[ll] \arrow[dd] \\
S^n \arrow[ur] \arrow[dd] & & S^n \vee S^n \arrow[ll] \arrow[ur] \arrow[dd] & \\
& S^n \vee S^n \vee S^n & & S^n \vee S^n \arrow[ll] \\
S^n \vee S^n \arrow[ur] & & S^n \arrow[ll] \arrow[ur] &
\end{tikzcd}$$

from my description in lecture, there's no reason not to take $n = 1$. Prove, however, that there is no way to lift the Khovanov cube to a commuting cube of this form with $n = 1$.

7. In lecture, we considered a commuting square of spaces of the form

$$\begin{tikzcd} X_{01} \arrow[d, "\beta"] & X_{11} \arrow[l, "\gamma"'] \arrow[d, "\delta"] \\ X_{00} & X_{10} \arrow[l, "\alpha"] \end{tikzcd}$$

and we defined its *iterated mapping cone* to be $\mathrm{Cone}(\overline{\beta\delta}\colon \mathrm{Cone}(\gamma) \to \mathrm{Cone}(\alpha))$, where $\overline{\beta\delta}$ is the map induced by β and δ. That is, $\mathrm{Cone}(\gamma) = \big([0,1] \times X_{11}\big) \cup_\gamma X_{01}$, and $\overline{\beta\delta}$ is given by $Id \times \delta$ on $[0,1] \times X_{11}$, and by β on X_{01}.

(a) Prove that $\overline{\beta\delta}$ specified maps respect the equivalence relation, so this iterated mapping cone is well-defined.
(b) Prove that if we took the mapping cone in the opposite order, taking the cones of β and δ first, we would have gotten the same space.
(c) Generalize this construction to n-dimensional cubes, not just squraes.

8. Suppose that the square in the previous problem only homotopy commutes.

(a) Prove that the map $\overline{\beta\delta}$ no longer respects the identifications, so the iterated mapping cone is not well-defined.
(b) Fix a homotopy h between $\beta \circ \gamma$ and $\alpha \circ \delta$. Explain how to incorporate h in the definition of the iterated mapping cone, so one gets a well-defined space in this case.
(c) Give an example showing that different choices of h can give non-homotopy equivalent iterated mapping cones.