Minerva Mini-Course Lecture 6 Exercises

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The Burnside category, spectra, and so on

1. Let $\mathscr{C}$ be a category in which the only isomorphisms are the identity maps. (This is not needed, but makes the definition a little simpler. The cube category is an example.) A *homotopy coherent functor* from a $\mathscr{C}$ to the category of based topological spaces consists of:

   - A space $F(x)$ for each object $x \in \mathscr{C}$ and
   - For each sequence $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k$ of composable morphisms, a continuous map

     $$F(f_k, \dots, f_1)\colon [0,1]^{k-1} \times F(x_0) \to F(x_k)$$

     (sending $[0,1]^{k-1}$ times basepoint to basepoint).

   These must satisfy that

   $$F(f_k, \dots, f_1)(t_{k-1}, \dots, t_{i+1}, 0, t_{i-1}, \dots, t_1) \\ = [F(f_k, \dots, f_{i+1})(t_{k-1}, \dots, t_{i+1})] \circ [F(f_i, \dots, f_1)(t_{i-1}, \dots, t_1)]$$

   $$F(f_k, \dots, f_1)(t_{k-1}, \dots, t_{i+1}, 1, t_{i-1}, \dots, t_1) \\ = F(f_k, \dots, f_{i+1} \circ f_i, \dots, f_1)(t_{k-1}, \dots, t_{i+1}, t_{i-1}, \dots, t_1).$$

   Using the construction from lecture, prove that any functor from the cube category $\underline{2}^n$ to the Burnside category can be lifted to a homotopy coherent functor from $\underline{2}^n$ to spaces.

2. Assuming you know what the words mean, prove that the map $\mathcal{B} \to \mathsf{Spectra}$ extends to a functor of $(\infty, 1)$-categories from the Burnside category to CW spectra.

Mapping cones

3. In the lecture, we refined the Khovanov cube to a functor $\underline{2}^n \to \mathcal{B}$, then used this to construct a homotopy coherent cube of spaces. We then took the iterated mapping cone to get the Khovanov stable homotopy type.

For convenience, assume that instead of a homotopy coherent cube of spaces we got a strict cube from this construction. Prove that the cellular chain complex of the iterated mapping cone agrees with the usual Khovanov complex. (Maybe work over $\mathbb{Z}/2$ to avoid having to trace signs throughout.)

Some algebraic topology exercises

(These are taken from a forthcoming book; please let me know if you notice errors. Also, some of these are challenging.)

4. Construct an explicit isomorphism $\widetilde{H}^n(X;G) \cong [X, K(G,n)]$ by showing that any cellular n-cocycle α can be represented by a cellular map f, and two such maps f, f' are homotopic, and if α is a coboundary, then f is nullhomotopic. (Hint: Construct all the maps and homotopies inductively cell-by-cell.)
5. Prove that $K(G,n)$ is unique up to homotopy equivalence.
6. Fix $n \geq 3$ and let K_n^n be the n-skeleton of $K(\mathbb{Z}/2,n)$ (with respect to a minimal cell structure). Prove that the map $\pi_{n+1}(K_n^{(n)}) \to \pi_{n+1}(K_n^{(n+1)})$ is an isomorphism. (Hint: Since $n \geq 3$, $\pi_m(K_n^{(n+1)}, K_n^{(n)}) \cong \pi_m(K_n^{(n+1)}/K_n^{(n)})$ for $m \leq n+2$ by excision for homotopy groups (see Hatcher). Then use the homotopy long exact sequence for the pair $(K_n^{(n+1)}, K_n^{(n)})$.)
7. Continuing from the previous problem, Prove $\pi_{n+2}(K_n^{(n+2)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$, with the generator of $\mathbb{Z}/2$ represented by a map $S^{n+2} \to K_n^{(n+1)}$ whose composition with the quotient map $K_n^{(n+1)} \to K_n^{(n+1)}/K_n^{(n)}$ is the Hopf map η, and the generator of $\mathbb{Z}$ represented by a map $S^{n+2} \to K_n^{(n+2)}$ whose composition with the quotient map $K_n^{(n+2)} \to K_n^{(n+2)}/K_n^{(n+1)}$ is a degree 2 map. (Hint: Use the long exact sequence for the pair $(K_n^{(n+2)}, K_n^{(n+1)})$ and observe that the image of d'_{n+3} is same as the image of f_{n+2} from the previous exercise.)

The first two Steenrod squares as obstruction classes

8. Prove that the class $\gamma(a)$ we constructed in lecture is a cocycle, and is well-defined up to a coboundary.
9. Prove that γ is natural and stable.
10. Compute γ for $\mathbb{R}P^2$, and conclude that $\gamma = \mathrm{Sq}^1$. Alternatively, give a more direct proof of this using the fact that Sq^1 is a Bockstein homomorphism.
11. Prove that the class η constructed in lecture defines a stable cohomology operation. (This is probably a bit tricky.)
12. Compute η for a generator of $H^2(CP^2;\mathbb{Z}/2)$. Also for a generator of $H^2(RP^4/RP^1;\mathbb{Z}/2)$ and $H^3(RP^5/RP^2;\mathbb{Z}/2)$.
13. Prove that η is not identically zero (e.g., using the previous exercise), and deduce that $\eta = \mathrm{Sq}^2$.