Minerva Mini-Course Lecture 7 Exercises

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Reduced Khovanov homology and the s-invariant

1. I asserted previously that for a knot K and a field $\mathbb{F}$, the deformed Khovanov homology $Kh^h(K;\mathbb{F})$ has the form $\mathbb{F}[h] \oplus \mathbb{F}[h] \oplus [T]$ where the two $\mathbb{F}[h]$-summands are generated by elements in bigradings $(0, s_{\mathbb{F}}(K) - 1)$ and $(0, s_{\mathbb{F}}(K) + 1)$ and $\mathcal{T}$ is a torsion $\mathbb{F}[h]$-module. Deduce that the reduced deformed Khovanov homology $\widetilde{Kh}^h(K;\mathbb{F})$ has the form $\mathbb{F}[h] \oplus \mathcal{T}'$ where the $\mathbb{F}[h]$ is generated in bigrading $(0,0)$ and $\mathcal{T}'$ is a torison $\mathbb{F}[h]$-module.

Bockstein-refined s-invariants

2. We discussed an algebraic example where $s_\beta \neq s_{\mathbb{F}_2}$, where the reduced deformed Khovanov complex is given by:

$$\begin{array}{ccc} & \mathbb{Z}[h]_{(1,0)} & \\ {}^{h}\swarrow & & \searrow^{2} \\ \mathbb{Z}[h]_{(0,-2)} & & \mathbb{Z}[h]_{(0,0)}. \end{array}$$

   Consider the dual complex (over $\mathbb{Z}[h]$), which would be the invariant of the mirror knot if this complex came from a knot. What is s_β for the dual complex?

3. There are many other Bockstein homomorphisms. For example, there is a Bockstein homomorphism $\widetilde{\beta}$ associated to the coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

   Give an algebraic example where $s_{\widetilde{\beta}} \neq s_\beta$.

General refined s-invariants

4. Prove the corollary I asserted in lecture: Let K be a knot with $s_{\mathbb{F}_2}(K) = 0$ and $q\colon \widetilde{Kh}^h(K;\mathbb{F}_2) \to \widetilde{Kh}(K;\mathbb{F}_2)$ the map induced by the quotient map of

complexes. Suppose that for any non-torsion $x \in \widetilde{Kh}^{h}_{0,0}(K)$, $\mathrm{Sq}_2(q(x)) \neq 0$. Then K is not slice.

5. Prove that for any natural Khovanov homology operation η, s_η gives a slice genus bound.

6. Prove that if K is a squeezed knot, then $s(K) = s_\eta(K)$ for any K. (Hint: show first that $s_\eta(T_{p,q}) = s(T_{p,q}) = (p-1)(q-1)$. See Feller-Lewark-Lobb's paper for a solution.)

7. In the lecture, I only considered Khovanov homology operations η that preserve the quantum grading. Extend the discussion to operations that shift the quantum grading.

Applications

8. Prove that the knot 9_{42} bounds a disk D in $\overline{CP}^2$ with $[D] = 1 \in \mathbb{Z} = H_2(\overline{CP}^2 \setminus B^4, S^3)$. So, s_{Sq_2} does not obstruct such disks.