Minerva Mini-Course Lecture 7 Exercises

Robert Lipshitz

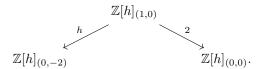
November 10, 2025

Reduced Khovanov homology and the s-invariant

1. I asserted previously that for a knot K and a field \mathbb{F} , the deformed Khovanov homology $Kh^h(K;\mathbb{F})$ has the form $\mathbb{F}[h] \oplus \mathbb{F}[h] \oplus [T]$ where the two $\mathbb{F}[h]$ -summands are generated by elements in bigradings $(0, s_{\mathbb{F}}(K) - 1)$ and $(0, s_{\mathbb{F}}(K) + 1)$ and \mathcal{T} is a torsion $\mathbb{F}[h]$ -module. Deduce that the reduced deformed Khovanov homology $\widetilde{Kh}^h(K;\mathbb{F})$ has the form $\mathbb{F}[h] \oplus \mathcal{T}'$ where the $\mathbb{F}[h]$ is generated in bigrading (0,0) and \mathcal{T}' is a torsion $\mathbb{F}[h]$ -module.

Bockstein-refined s-invariants

2. We discussed an algebraic example where $s_{\beta} \neq s_{\mathbb{F}_2}$, where the reduced deformed Khovanov complex is given by:



Consider the dual complex (over $\mathbb{Z}[h]$), which would be the invariant of the mirror knot if this complex came from a knot. What is s_{β} for the dual complex?

3. There are many other Bockstein homomorphisms. For example, there is a Bockstein homomorphism $\tilde{\beta}$ associated to the coefficient sequence

$$0 \to \mathbb{Z} \stackrel{2}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

Give an algebraic example where $s_{\widetilde{\beta}} \neq s_{\beta}$.

General refined s-invariants

4. Prove the corollary I asserted in lecture: Let K be a knot with $s_{\mathbb{F}_2}(K) = 0$ and $q \colon \widetilde{Kh}^h(K; \mathbb{F}_2) \to \widetilde{Kh}(K; \mathbb{F}_2)$ the map induced by the quotient map of

complexes. Suppose that for any non-torsion $x\in \widetilde{Kh}_{0,0}^h(K),$ $\mathrm{Sq}_2(q(x))\neq 0.$ Then K is not slice.

- 5. Prove that for any natural Khovanov homology operation $\eta, \, s_{\eta}$ gives a slice genus bound.
- 6. Prove that if K is a squeezed knot, then $s(K)=s_{\eta}(K)$ for any K. (Hint: show first that $s_{\eta}(T_{p,q})=s(T_{p,q})=(p-1)(q-1)$. See Feller-Lewark-Lobb's paper for a solution.)
- 7. In the lecture, I only considered Khovanov homology operations η that preserve the quantum grading. Extend the discussion to operations that shift the quantum grading.

Applications

8. Prove that the knot 9_{42} bounds a disk D in \overline{CP}^2 with $[D]=1\in\mathbb{Z}=H_2(\overline{CP}^2\setminus B^4,S^3)$. So, s_{Sq_2} does not obstruct such disks.