# Minerva Mini-Course Lecture 8 Exercises

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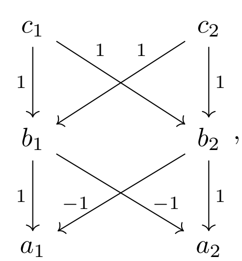
November 24, 2025

## 1-Framed flow categories and spaces

1. Write precisely how to get a CW complex with cells in dimension $N$, $N+1$, and $N+2$ (for $N$ large) from a framed 1-flow category. (I gave all the steps during lecture, so arguably this is really a "clean up your notes" problem; but probably it will take a little more thought than that should.)
2. Show that for a framed 1-flow category $\mathcal{F}$ with $\mathcal{F}_n = \{a\}$, $\mathcal{F}_{n+1} = \emptyset$, and $\mathcal{F}_{n+2} = \{c\}$ and $\mathcal{F}(c,a) = S^1$, the induced stable homotopy type is either $\Sigma^{n-2}CP^2$ or $S^n \vee S^{n+2}$, depending on the framing of the $S^1$.
3. Similarly, show that if $\mathcal{F}$ is a framed 1-flow category with $\mathcal{F}_n = \{a\}$, $\mathcal{F}_{n+1} = \{b\}$, and $\mathcal{F}_{n+2} = \{c\}$; $\mathcal{F}(b,a)$ two points with the same sign; $\mathcal{F}(c,b) = \emptyset$; and $\mathcal{F}(c,a)$ a nontrivially framed circle, then the induced stable homotopy type is $\Sigma^{n-1}\mathbb{R}P^4/\mathbb{R}P^2$.
4. Find a framed 1-flow category that gives $\mathbb{R}P^4/\mathbb{R}P^1$, and one that gives $\mathbb{R}P^2 \wedge \mathbb{R}P^2$. (At this point, you have examples giving all interesting stable homotopy types with cells in three adjacent gradings.)
5. During the lecture, I simplified a 1-framed flow category with the shape:




all 0-dimensional moduli spaces consisting of one point (each), with framing as indicated by the sign above, and all 1-dimensional spaces consisting of intervals. For example, $\mathcal{F}(c_1, a_1)$ consists of an interval whose boundary is $\mathcal{F}(c_1, b_1) \times \mathcal{F}(b_1, a_1) \amalg \mathcal{F}(c_1, b_2) \times \mathcal{F}(b_2, a_1)$. Fill in the boundaries of the rest of the 1-dimensional moduli spaces. (This is a check that you understood the definitions, and should be easy if so.)
6. Continuing the previous problem, give a consistent story for how the 1-dimensional manifolds change when you perform various handleslides

(without worrying about their framings).

7. Now, use Section 4 of Lobb-Orson-Schütz's "Khovanov homotopy calculations using flow category calculus" to track the framings, assuming all the framings of the initial intervals are 0 (in their conventions). Is the resulting space a suspension of $CP^2$ or of $S^2 \vee S^4$?