Representations of $G$ and $L^1(G)$

INTRODUCTION: This note was motivated by lectures by Gabriel Nagy at Kansas State. We refer frequently to D&E, which refers to *Principles of Harmonic Analysis* by Anton Deitmar and Siegfried Echterhoff (Springer 2009). We also refer to H&R, this being volumes 1 and 2 of Hewitt & Ross’s *Abstract Harmonic Analysis*. Observe that a version of Theorem 1 below is in H&R, 22.10 and 22.7, but that its proof is much too complicated. It uses the fact (21.13) that every $*$-representation on a Hilbert space is a direct sum of cyclic subrepresentations.

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This is about “duality” between norm-continuous representations $\pi$ of the algebra $L^1(G)$ by bounded linear operators $B(V)$, $V$ a Banach space, and bounded continuous group representations $U_x$ of $G$ on $V$. Here $G$ is an LC-group with left Haar measure $\mu$. One direction is, of course, easy: From $U_x$ one builds $\pi$ by

$$
\pi(f)v := \int_G f(x)U_x(v)d\mu(x) \quad \text{for} \quad f \in L^1(G), v \in V,
$$

(0)
a $V$-valued integral. Let $(e_j)$ be a nice approximate identity which D&E call a Dirac net. As noted in D&E 6.2.2, $\lim_j \pi(e_j)v = v$ for all $v \in V$. In other words, the net of operators $\pi(e_j)$ on $V$ converges to the identity operator on $V$ in the strong operator topology.

Now we begin with an algebra representation $\pi$ of $A = L^1(G)$ satisfying for some finite $C \geq 1$,

$$
\| \pi(f) \|_{op} \leq C \| f \|_1 \quad \text{for} \quad f \in L^1(G).
$$

(1)

Following D&E (and Nagy), we say that $\pi$ is non-degenerate if

$$
\pi(A)V = \{\pi(f)v : f \in L^1(G), v \in V\} \text{ spans a dense subspace of } V,
$$

(2)

where $\pi(A)V = \{\pi(f)v : f \in L^1(G), v \in V\}$. (If $V$ is a Hilbert space and $\pi$ is a $*$-representation, then (2) is plainly the same as saying $\pi(f)(v) = 0$ for all $f \in L^1(G)$ implies $v = 0$.)

Lemma 1. If $\pi$ is non-degenerate as above, then $\pi(A)V = V$.

Proof. We define $(f, v) \to f \cdot v$ from $A = L^1(G) \times V \to V$, where $f \cdot v = \pi(f)v$. Then $V$ is a Banach $L^1(G)$-module (H&R 32.14). In particular, $(f * g) \cdot v = \pi(f * g)v = \pi(f)(\pi(g)v) = f \cdot (\pi(g)v) = f \cdot (g \cdot v)$ and

$$
\| f \cdot v \| = \| \pi(f)v \| \leq \| \pi(f) \|_{op} \cdot \| v \| \leq C \| f \|_1 \cdot \| v \|.
$$

The Dirac net $(e_j)$ is a bounded approximate unit for $L^1(G)$, so the Cohen Factorization Theorem (H&R 32.22) tells us that $\pi(A)V = A \cdot V$ is a closed linear subspace of $V$, and thus $\pi(A)V = V$. □

For Lemma 2 and Theorem 1 below, we will give two proofs, one using Lemma 1 and the other not.
Lemma 2. If $\pi$ is non-degenerate, then $\lim_j \pi(e_j)v = v$ for all $v \in V$.

Proof using Lemma 1. Given $v \in V$, we have $v = \pi(f)w = f \cdot w$ for some $f \in L^1(G)$ and $w \in V$. Therefore

$$\lim_j \pi(e_j)v = \lim_j e_j \cdot w = \lim_j e_j \cdot (f \cdot w) = \lim(e_j * f) \cdot w = f \cdot w = v.$$ 

Proof avoiding Lemma 1. Given $v \in V$ and $\epsilon > 0$, use (2) to select $\sum_{i=1}^n \pi(f_i)v_i$ satisfying

$$\| \sum_{i=1}^n \pi(f_i)v_i - v \| < \frac{\epsilon}{3C}.$$ 

Then $\| \pi(e_j)v - v \|$ is bounded by

$$\| \pi(e_j)v - \sum_{i=1}^n \pi(e_j)\pi(f_i)v_i \| + \| \sum_{i=1}^n \pi(e_j)\pi(f_i)v_i - \sum_{i=1}^n \pi(f_i)v_i \| + \| \sum_{i=1}^n \pi(f_i)v_i - v \|.$$ 

The third and first terms in the sum are bounded by $\epsilon/3$, and the middle term is bounded by

$$\sum_{i=1}^n \| \pi(e_j \ast f_i - f_i)\| \leq C \sum_{i=1}^n \| e_j \ast f_i - f_i \| \cdot \| v_i \|,$$

which is also less than $\epsilon/3$ for sufficiently “large” $j$ in the Dirac net. \qed

NOTE. Thus, for non-degenerate $\pi$: if $\pi(f)v = 0$ for all $f \in L^1(G)$, then $v = 0$. This is the non-degeneracy-like hypothesis in H&R 22.10 and 22.7.

Theorem 1. Given a norm-continuous representation $\pi$ of $A = L^1(G)$, as described above, there exists a representation $U_x$ of $G$, where each $U_x$ is a bounded linear operator on $V$ with norm not greater than $C$, and such that equation (0) holds.

Proof using Lemma 1. For $x \in G$, we propose to define $U_x : V \to V$ by

$$U_x(\pi(f)v) := \pi(L_x(f))v \quad \text{for all} \quad x \in G \quad \text{and} \quad f \in L^1(G),$$

where $L_x$ is left translation by $x^{-1}$ in $L^1(G)$. Since $V = \pi(A)V$, $U_x$ is defined for all $v \in V$, provided that it is well-defined. But note that

$$\pi(L_x(e_j))(\pi(f)v) = \pi(L_x(e_j \ast f)v = \pi(L_x(e_j \ast f))v \to \pi(L_xf)v,$$

since $\pi$ is continuous and $e_j \ast f \to f$ in $L^1(G)$. So the right side of (3) is unambiguously determined by the vector $\pi(f)v$. This calculation also shows

$$\| U_x(\pi(f)v) \| \leq \liminf_j \| \pi(L_x(e_j)) \|_{op} \cdot \| \pi(f)v \| \leq C \liminf_j \| L_x(e_j) \|_1 \cdot \| \pi(f)v \| \leq C \| \pi(f)v \|.$$ 

It remains to verify equation (0): $\pi(f)v = \int_G f(x)U_x(v)d\mu(x)$. For fixed $f$, both sides are continuous linear functions of $v$, so it suffices to verify (0) for elements $\pi(g)v$ in $\pi(A)V = V$. So we want

$$\pi(f)(\pi(g)v) = \int_G f(x)U_x(\pi(g)v)d\mu(x).$$
For fixed $v$, both sides are continuous as functions of $f$, so it suffices to consider $f$ in $C_c(G)$. Now we’re done:

$$\int_G f(x)U_x(\pi(g)v)\,d\mu(x) = \int_G f(x)\pi(L_x(g))v\,d\mu(x) = \int_G \pi(f(x)L_x(g))v\,d\mu(x),$$

and, as elaborated on in the next paragraph, by D&E B.6.1(c) this is equal to

$$\pi \left( \int_G f(x)L_x(g)d\mu(x) \right) v = \pi(f \ast g)v = \pi(f)(\pi(g)v),$$

where the first equality follows from D&E B.6.5. Note that $f \ast g(y) = \int_G f(x)g(x^{-1}y)\,d\mu(x) = \int_G f(x)L_x(g)(y)\,d\mu(x)$.

To apply D&E B.6.1(c) above, let $\varphi(x) := f(x)L_x(g)$, so that $\varphi : G \to L^1(G)$ and we want

$$\int_G \pi(\varphi(x))v\,d\mu(x) = \pi \left( \int_G \varphi(x)\,d\mu(x) \right)(v). \tag{4}$$

As noted in the first paragraph of the proof of D&E B.6.5, $\varphi$ is continuous with compact support. So is the function $x \to \pi(\varphi(x))v$ from $G$ into $V$, so both integrals in (5) exist. Now, fix $v$. Then $T(f) = \pi(f)v$ clearly defines a continuous linear operator $T : L^1(G) \to V$, and we can apply D&E to $\varphi$ to obtain $\pi \left( \int_G \varphi(x)\,d\mu(x) \right) v =

$$T \left( \int_G \varphi(x)\,d\mu(x) \right) = \int_G (T(\varphi))(x)\,d\mu(x) = \int_G T(\varphi(x))\,d\mu(x) = \int_G \pi(\varphi(x))v\,d\mu(x),$$

which is equation (4). □

Proof avoiding Lemma 1. For $x \in G$, we want $U_x$ to satisfy $U_x(\pi(f)v) = \pi(L_x(f))v$ for $f \in A = L^1(G)$ and $v \in V$, where $L_x$ is left translation by $x^{-1}$ in $L^1(G)$. So we first define $U_x$ on the span of $\pi(A)V$:

$$U_x \left( \sum_{i=1}^n \pi(f_i)v_i \right) := \sum_{i=1}^n \pi(L_x(f_i))v_i. \tag{5}$$

Since left translation satisfies $L_x(e) * f = L_x(e \ast f)$ (H&R 20.11), elements of our Dirac net $(e_j)$ satisfy

$$\pi(L_x(e_j)) \left( \sum_{i=1}^n \pi(f_i)v_i \right) = \sum_{i=1}^n \pi(L_x(e_j) * f_i)v_i = \sum_{i=1}^n \pi(L_x(e_j \ast e_i))v_i.$$

Since $(e_j)$ is an approximate unit in $L^1(G)$, and $L_x$ and $\pi$ are continuous, we see that

$$\lim_{j} \pi(L_x(e_j)) \left( \sum_{i=1}^n \pi(f_i)v_i \right) = \lim_{j} \sum_{i=1}^n \pi(L_x(e_j \ast e_i))v_i = \sum_{i=1}^n \pi(L_x(f_i))v_i. \tag{6}$$

Since the left-hand side of (6) depends only on $\sum_{i=1}^n \pi(f_i)v_i$, this shows that definition (5) is well defined.
Equation (6) implies that
\[ \| \sum_{i=1}^{n} \pi(L_x(f_i))v_i \| \leq \liminf_j \| \pi(L_x(e_j)) \|_{\text{op}} \cdot \| \sum_{i=1}^{n} \pi(f_i)v_i \|. \]

Since \( \| L_x \|_{\text{op}} = 1 \) and \( \| \pi \| \leq C \), we conclude that
\[ \| U_x \left( \sum_{i=1}^{n} \pi(f_i)v_i \right) \| \leq C \| \sum_{i=1}^{n} \pi(f_i)v_i \|. \]

Thus \( U_x \) is a bounded linear operator on the span of \( \pi(A)V \), with norm no greater than \( C \). So each \( U_x \) can be extended to an operator on \( V \), with norm no greater than \( C \).

The rest of the proof, verifying equation (0), is the same as the Proof using Lemma 1. \( \square \)

Here is D&E’s 6.2.3.

**Theorem 2.** If \( \pi \) in Theorem 1 is a \( \ast \)-representation on \( L^1(G) \), acting on a Hilbert space \( V \), then the representation \( U \) on \( G \) is a unitary representation.

**Proof.** This follows from the computation at the bottom of D&E’s page 133 together with Lemma 1. \( \square \)

**References:**
