## Appendix

The following proof of Tychonoff's Theorem is due to Paul Chernoff (1992), with an improvement by Charles Pugh (2003). Pugh provided Lemma 2 which allows us to avoid the subtle concept of subnet.

**Lemma 1.** A topological space $X$ is compact if and only if every net has a cluster point.

*Proof.* See Theorem 2, Chapter 5, of Kelley's *General Topology*, Van Nostrand, 1955. ♣

**Lemma 2.** Consider a product space $X \times Y$, where $Y$ is compact. If $\langle x_\alpha, y_\alpha \rangle_{\alpha \in A}$ is a net in $X \times Y$ and $\langle x_\alpha \rangle_{\alpha \in A}$ clusters at $x$ in $X$, then for some $y$ in $Y$, $\langle x_\alpha, y_\alpha \rangle_{\alpha \in A}$ clusters at $(x, y)$.

*Proof.* We form a new net in $Y$, directed by a new set

$$B = \{\beta = (\alpha, U_x) : \alpha \in A, U_x \text{ is a neighborhood of } x, \text{ and } x_\alpha \in U_x\},$$

equipped with the product direction $\beta \leq \beta'$ if and only if $\alpha \leq \alpha'$ and $U_x \supseteq U'_x$. This gives a new net $\langle y_\beta \rangle_{\beta \in B}$ in $Y$ defined by $y_\beta = y_\alpha$ whenever $\beta = (\alpha, U_x) \in B$. Since $Y$ is compact, this net clusters at some $y \in Y$.

We claim that $\langle x_\alpha, y_\alpha \rangle_{\alpha \in A}$ clusters at $(x, y)$. Let $U \times V$ be a neighborhood of $(x, y)$, and let $a \in A$ be given. For Some $\alpha \geq a$, $x_\alpha$ is in $U$. Since $\langle y_\beta \rangle_{\beta \in B}$ clusters at $y$ in $Y$, there is $\beta' = (\alpha', U'_x) \geq (\alpha, U)$ such that $y_{\beta'} \in V$. Hence $(x_{\alpha'}, y_{\alpha'})$ is in $U \times V$ with $\alpha' \geq a$, and the net clusters at $(x, y)$. ♣

**Corollary.** If $X$ and $Y$ are compact spaces, then so is $X \times Y$.

*Proof.* Consider a net $\langle x_\alpha, y_\alpha \rangle_{\alpha \in A}$ in $X \times Y$. Since $X$ is compact, $\langle x_\alpha \rangle_{\alpha \in A}$ clusters at some $x$ in $X$ by Lemma 1. Now apply Lemma 2 and Lemma 1 again. ♣

**Tychonoff's Theorem.** The product $X = \prod_{i \in I} X_i$ of compact spaces $X_i$ is compact.

*Proof.* By Lemma 1, it suffices to show that every net $\langle f_\alpha \rangle_{\alpha \in A}$ in $X$ has a cluster point in $X$.

For each nonempty set $J \subseteq I$ and $g$ in $\prod_{i \in J} X_i$, we say $g$ is a *partial cluster point* of $\langle f_\alpha \rangle_{\alpha \in A}$ if $g$ is a cluster point of the net $\langle f_\alpha|_J \rangle_{\alpha \in A}$ of restrictions to $J$.

Let $\mathcal{F}$ be the set of all partial cluster points of $\langle f_\alpha \rangle_{\alpha \in A}$. $\mathcal{F}$ is nonempty because if $J = \{i_0\}$ then $\langle f_\alpha|_J \rangle_{\alpha \in A}$ is a net in $X_{i_0}$, so it has a cluster point since $X_{i_0}$ is compact.

We partially order $\mathcal{F}$ by extension:

$$g_1 \leq g_2 \quad \text{if} \quad \text{dom}(g_1) \subseteq \text{dom}(g_2) \quad \text{and} \quad g_1(i) = g_2(i) \quad \text{for} \quad i \in \text{dom}(g_1).$$

Consider a chain $\mathcal{C}$ in $\mathcal{F}$. By the proposition on page 3, $g_0 = \cup_{g \in \mathcal{C}} g$ is a function with domain $J = \cup_{g \in \mathcal{C}} \text{dom}(g)$.

*Claim.* $g_0$ is in $\mathcal{F}$, i.e., $g_0$ is a partial cluster point of $\langle f_\alpha \rangle_{\alpha \in A}$.

*Prf.* We need to show that $g_0$ is a cluster point of the net $\langle f_\alpha|_J \rangle_{\alpha \in A}$. Consider a basic neighborhood of $g_0$ in $\prod_{i \in J} X_i$ having the form

$$W = \{h \in \prod_{i \in J} X_i : h(i) \in U_i \quad \text{for} \quad i \in F\},$$

where $F$ is a finite subset of $J$ and $U_i$ is open in $X_i$ for each $i \in F$. Since $\mathcal{C}$ is a chain, there exists $g$ in $\mathcal{C}$ so that $F \subseteq \text{dom}(g)$. Consider $\alpha \in A$. Since $g$ is a cluster point of $\langle f_\alpha|_{\text{dom}(g)} \rangle_{\alpha \in A}$ and $F$ is finite, there exists $\beta$ in $A$ so that $\beta \succeq \alpha$ and $f_\beta(i) \in U_i$ for $i \in F$. Therefore $f_\beta|_J$ belongs to $W$. Since $\alpha$ in $A$ is arbitrary and the basic neighborhood $W$ of $g_0$ is arbitrary, this shows that $g_0$ is a cluster point of $\langle f_\alpha|_J \rangle_{\alpha \in A}$. Hence $g_0$ is in $\mathcal{F}$, which verifies the claim.

Since $g_0$ is clearly an upper bound for $\mathcal{C}$, we see that every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. So, by Zorn's Lemma, $\mathcal{F}$ has a maximal element $g^*$. If $\text{dom}(g^*) = I$, then $\langle f_\alpha \rangle_{\alpha \in A}$ has a cluster point in $X$ and we're done.

So assume that $\text{dom}(g^*) = J^*$ is not equal to $I$. Select $k$ in $I \setminus J^*$. Since $g^*$ is in $\mathcal{F}$, it is a partial cluster point of the net $\langle f_\alpha|_{J^*} \rangle_{\alpha \in A}$. The net $\langle f_\alpha \rangle_{\alpha \in A}$, restricted to $J^* \cup \{k\}$, is a net in the product $\displaystyle\prod_{i \in J^*} X_i \times X_k$. So Lemma 2 applies to give a partial cluster point that extends $g^*$, contradicting the maximality of $g^*$. ♣