Closed subgroups of compactly generated LCA groups are compactly generated

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I first saw the result stated in the title, where LCA abbreviates "locally compact abelian," in Sidney A. Morris's 1972 paper [7]. The theorem is also Exercise 1 on page 93 of Morris's book [8]. It is proved in (23.13) in Markus Stroppel's book [12]. When I saw this theorem in Morris's paper, I was sure it was new to me, and I still believe it was new to me at that time. When I saw this theorem in Stroppel's 2006 book, I was again sure it was new to me, but of course it wasn't.

Alas, on May 29, 2018, I was very surprised to receive an email, from Professor Hatem Hamrouni at the University of Sfax (Tunisia), indicating that the theorem that many of us thought was due to Morris was anticipated by Martin Moskowitz in 1967 [9]. Both Sid Morris and I are pleased that the history has been clarified. I will outline Moskowitz's proof in Appendix 2.

But what if the group is not abelian? One might hope that closed subgroups of compactly generated locally compact groups[1] are always compactly generated, but this is not true. See Appendix 1. But there are interesting partial results. Except for Appendix 2, this article is a report of joint email work about ten years ago involving Sadahiro Saeki,[2] Karl Hofmann, Bob Burckel, Sid Morris and me.

I first discuss compactly generated groups in more generality. My goal is to present the results as they might appear in Hewitt & Ross [2] if we'd known them, and the unspecified numbered items refer to that book. By (5.39.i), if G is a locally compact group and H is a normal closed subgroup, and if H and G/H are compactly generated, then G is also compactly generated. In addition, if G is compactly generated, then G/H is also, because if the compact set A generates G, then its image $\pi(A)$ under the natural map $\pi : G \to G/H$ is a compact subset of G/H that generates G/H.

[1] All topological groups in this paper are assumed to be Hausdorff spaces.

[2] Saeki was a very productive analyst at Kansas State University for over 25 years. He returned to Tokyo after his retirement. His result that might most interest readers of this article settled the L^p conjecture (from the early 1960s): if G is a locally compact group and $p > 1$, and if $L^p(G)$ is closed under convolution, then G is compact. See [10]. Saeki's proof is elegant and elementary, unlike some earlier partial results.

The following elegant result, due to Sadahiro Saeki, has many nice consequences. The proof is similar to, but more streamlined than, the proof of Corollary B below given by MacBeath and Swierczkowski [6].

Theorem 1. Let H be a closed subgroup of a topological group G such that HK is a compactly generated subgroup of G for some compact set K. Then H is compactly generated. [Note that HK is automatically closed by (4.4).]

Proof. By replacing K by a bigger set, we may assume that K is symmetric and generates HK. Clearly $A : H \cap K^3$ is a compact subset of H. We will show that H is equal to the subgroup L generated by A. Note that $K \subseteq K^3$, since K^2 contains the identity.

First we show that $HK \subseteq LK$. Since $HK = \cup_{n=2}^{\infty} K^n$, it suffices to show that $K^n \subseteq LK$ for $n = 2, 3, 4, \ldots$. Consider x in K^2. Since $x \in HK$, we have $x = hk$ where $h \in H$ and $k \in K$. Then $h = xk^{-1} \in K^2 K = K^3$, so $h \in K^3 \cap H = A$ and $x = hk \in AK \subseteq LK$. Thus $K^2 \subseteq LK$. If $K^n \subseteq LK$, then

$$K^{n+1} = K^n K \subseteq (LK)K = LK^2 \subseteq L(LK) = LK.$$

Therefore all $K^n \subseteq LK$ by induction.

Since $HK \subseteq LK$, clearly $H \subseteq LK$. To show $H \subseteq L$, consider any $h \in H$. Then $h = xk$ where $x \in L$ and $k \in K$. Then $k = x^{-1}h \in LH \subseteq HH = H$, and so $k \in H \cap K \subseteq H \cap K^3 = A$. It follows that $h = xk \in LA \subseteq LL = L$. Thus $H = L$ and H is generated by A.

Corollary A. If H is a closed subgroup of a compactly generated topological group G such that $G = HK$ for some compact set K, then H is compactly generated.

The next corollary was proved by MacBeath and Swierczkowski [6].

Corollary B. If H is a closed subgroup of a compactly generated locally compact group G, and if the space G/H is compact, then H is compactly generated.

Proof. By (5.24.b), there is a closed compact subset F of G where $G/H = \{xH : x \in F\}$. Thus $G = FH$ and Corollary A applies since clearly the supposition $G = HK$ can be replaced by the supposition $G = KH$.

Corollary C. Let H be a closed subgroup of a topological group G and N be a compact subgroup of G. If H or N is normal in G and HN is compactly generated, then H is compactly generated. [Note that HN is automatically a closed subgroup of G.]

Theorem 2. Let H be a closed subgroup of a topological group G, and let N be a compact normal subgroup of G. The following are equivalent:

(a) H is compactly generated;

(b) HN/N is compactly generated;

(c) HN is compactly generated.

Proof. Let $\pi : G \to G/N$ be the natural projection.

(a) $\Rightarrow$ (b). If A is a compact subset of H that generates H, then $\pi(A)$ is a compact subset of G/N that generates $\pi(H) = HN/N$.

(b) $\Rightarrow$ (c). This follows from (5.39.h). Here's the details with our notation. Suppose $\{xN : x \in X\}$ is a compact subset of HN/N that generates HN/N, where $X \subseteq HN$. By (5.24.a), XN is compact. Then $XN \cup N$ is also compact and it generates HN.

(c) $\Rightarrow$ (a). This follows from Corollary C.

We will call a (necessarily compactly generated) topological group a **Moskowitz-Morris group** or an **M-M group** if each closed subgroup of it is compactly generated.

Corollary D. If G is a topological group and N is a compact normal subgroup such that G/N is an M-M group, then G is an M-M group.

Proof. Let H be a closed subgroup of G. Note that HN/N is a closed subgroup of G/N by (5.18) because $HN/N = \pi(H)$ where π is again the natural projection of G onto G/N. Thus HN/N is compactly generated. By Theorem 2, H is also compactly generated.

Corollary E. If M is an M-M group and K is a compact group, then $M \times K$ is an M-M group.

Proof. Let $N = \{e\} \times K$. Then N is a compact normal subgroup of $M \times K$, and $(M \times K)/N$ is topologically isomorphic with M. Hence $(M \times K)/N$ is an M-M group, and $M \times K$ is an M-M group by Corollary D.

Here is our version of the Moskowitz-Morris theorem.

Theorem 3. Every compactly generated LCA group G is an M-M group.

Proof. By the structure theorem (9.8), we can take $G = \mathbb{R}^c \times \mathbb{Z}^d \times E$ where c, d are nonnegative integers and E is a compact abelian group. Every closed subgroup of $\mathbb{R}^c \times \mathbb{Z}^d$ is a closed subgroup of $\mathbb{R}^m$, where $m = c + d$, so it is compactly generated by (9.11). Thus $\mathbb{R}^c \times \mathbb{Z}^d$ is an M-M group, and Corollary E shows that G is also an M-M group. This completes the proof.

Saeki asks: Is the product of two M-M groups an M-M group? If not, what about when one factor is the real line $\mathbb{R}$?

Next we give a proof of (9.12) that is simpler than the proof in [2].

Theorem (9.12) Let τ be a topological isomorphism of $\mathbb{R}^a \times \mathbb{Z}^b \times F$ into $\mathbb{R}^c \times \mathbb{Z}^d \times E$, where

a, b, c, d are nonnegative integers and F and E are compact groups [not necessarily abelian]. Then $a \le c$ and $a + b \le c + d$.

First a lemma.

Lemma. Let τ be a topological isomorphism of H into $G \times K$, where H and G are locally compact groups and K is a compact group. If H has no compact elements other than the identity, then H is topologically isomorphic to a closed subgroup of G.

Proof. Let π be the projection of $G \times K$ onto G. Since $\tau(H)$ is closed in $G \times K$ by (5.11), the image $\pi(\tau(H))$ is closed in G by (5.18). Also, π is one-to-one on $\tau(H)$. [If (x, k_1) and (x, k_2) in $\tau(H)$ have the same image x in G, then $(e, k_1 k_2^{-1})$ is in $\tau(H)$. Since only the identity of $\tau(H)$ is a compact element, $k_1 = k_2$.]

Since π is a closed mapping by (5.18), it is also a closed mapping of the closed subgroup $\tau(H)$ onto $\pi(\tau(H))$. Since π is one-to-one on this closed subgroup, it is also an open mapping, so that π is a topological isomorphism of $\tau(H)$ onto $\pi(\tau(H))$. Thus the composition $\pi \circ \tau$ is a topological isomorphism of H into G.

Proof of (9.12) First, we show $a + b \le c + d$. Restricting τ to $\mathbb{R}^a \times \mathbb{Z}^b \times \{e\}$ gives a topological isomorphism of $\mathbb{R}^a \times \mathbb{Z}^b$ into $\mathbb{R}^c \times \mathbb{Z}^d \times E \subseteq \mathbb{R}^{c+d} \times E$. Applying the Lemma with $H = \mathbb{R}^a \times \mathbb{Z}^b$, $G = \mathbb{R}^{c+d}$ and $K = E$, we see that H is topologically isomorphic to a closed subgroup of $\mathbb{R}^{c+d}$. Hence $a + b \le c + d$ by (9.11).

To prove $a \le c$, note that restricting τ to $\mathbb{R}^a \times \{\mathbf{0}\} \times \{e\}$ gives a topological isomorphism of $\mathbb{R}^a$ into $\mathbb{R}^c \times \mathbb{Z}^d \times E$. Since $\mathbb{R}^a$ is connected, τ maps $\mathbb{R}^a$ into $\mathbb{R}^c \times \{\mathbf{0}\} \times E$. Applying the Lemma with $H = \mathbb{R}^a$, $G = \mathbb{R}^c$ and $K = E$, we see that $\mathbb{R}^a$ is topologically isomorphic to a closed subgroup of $\mathbb{R}^c$. Hence $a \le c$ by (9.11).

APPENDIX 1

This appendix is concerned with compactly generated locally compact groups that are not M-M groups, i.e., have non-compactly-generated closed subgroups.

As a simple example, we show that the discrete free group G generated by x and y has a subgroup H that is not finitely generated. Let H be the subgroup generated by all $x^n y^n$, $n \ge 1$. For each k, let H_k be generated by all $x^n y^n$ for $n \le k$. Then H is the increasing union of the H_k's. Moreover, $x^{k+1} y^{k+1}$ is not in H_k, since no reduced word in H_k contains a string of x's or x^{-1}'s longer than k. Now consider a finite subset F of H. Then F is a subset of some H_k. The group generated by F does not contain $x^{k+1} y^{k+1}$, so F cannot generate H. Thus H is not finitely generated.

The above example generalizes to all discrete free groups. First we mention the Nielsen-Schreier subgroup theorem: Every subgroup of a free non-abelian group is a free group. See Theorem 7.2.1 in Hall [1], page 28 in Kurosh [4], and page 155 in Specht [11]. Stroppel

[12] states, on page 198, that every free non-abelian group contains subgroups that are not finitely generated, for instance, the commutator subgroup. Kurosh [4], page 36, proves the following: The commutator group of a free group of finite rank is a free group of countably infinite rank.

Finally, we present an interesting example of a non-discrete compactly generated locally compact group, $SL(2,\mathbb{R})$, that has non-compactly-generated closed subgroups. This presentation was supplied by Karl H. Hofmann. As noted in Lang [5], page 209, $SL(2,\mathbb{R})$ is generated by

$$\left\{ u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

where b ranges over $\mathbb{R}$. The compact set $\{u(b) : |b| \le 1\} \cup \{w\}$ generates $SL(2,\mathbb{R})$.

1. In the algebra $M_2(\mathbb{R})$ of all real 2×2 matrices, the set

$$U = \left\{ \begin{pmatrix} 1 + a_{11} & a_{12} \\ a_{21} & 1 + a_{22} \end{pmatrix} : |a_{jk}| < 1 \quad \text{for all} \quad j, k \right\}$$

is an open neighborhood of the identity matrix $\{\mathbf{I}\}$ which meets the subring $M_2(\mathbb{R})$ in $\{\mathbf{I}\}$. Thus $SL(2,\mathbb{Z}) \subseteq M_2(\mathbb{Z})$ meets $U \cap SL(2,\mathbb{R})$ in $\{\mathbf{I}\}$, and is therefore a discrete subgroup. Therefore any subgroup that we find in $SL(2,\mathbb{Z})$ will be discrete in $SL(2,\mathbb{R})$.

2. We consider $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\mathbf{I}\}$ and its discrete subgroup $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm\mathbf{I}\}$. A subgroup G of $PSL(2,\mathbb{Z})$ yields a discrete subgroup Γ of $SL(2,\mathbb{R})$ as a full inverse image of G. Then Γ is finitely generated if and only if G is finitely generated. If F is a free subgroup of G, then it splits in Γ, so Γ includes an isomorphic copy of F.

3. Let G be the subgroup of $PSL(2,\mathbb{Z})$ generated by

$$\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \{\pm\mathbf{I}\}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \cdot \{\pm\mathbf{I}\} \right\}.$$

The first element generates a subgroup isomorphic with $\mathbb{Z}(2)$, and the second a subgroup isomorphic with $\mathbb{Z}(3)$. The next theorem is from [11]. It is stated at the bottom of page 187, with different notation, and proved on page 188.

Theorem.$PSL(2,\mathbb{Z})$ is the free product $\mathbb{Z}(2) * \mathbb{Z}(3)$ of these two subgroups.

4. Since the commutator subgroup of a free product of abelian groups is free (see [11], page 201), $PSL(2,\mathbb{Z})$ and $SL(2,\mathbb{Z})$ contain free subgroups. As noted above, these free subgroups contain non-finitely-generated free subgroups, for example, their commutator subgroups. Thus $PSL(2,\mathbb{Z})$ and $SL(2,\mathbb{Z})$ contain closed, indeed discrete, subgroups which are not finitely generated.

5. More recent results have been published by Hofmann and Neeb [3].

APPENDIX 2

Moskowitz's proof that every closed subgroup of a compactly generated LCA is compactly generated is contained in Theorem 2.6(2) [9]. The key tool is the following nice theorem:

Theorem A. An LCA group G is compactly generated if and only if its character group $\widehat{G}$ has no small subgroups, i.e., there is a neighborhood of 1 containing no nontrivial subgroups.

With this, suppose that H is a closed subgroup of the compactly generated LCA group G. Then $\widehat{G}$ has no small subgroups by Theorem A. Then for any closed subgroup A of $\widehat{G}$, the quotient $\widehat{G}/A$ has no small subgroups, as shown in Theorem 2.6(1) [9]. If A is the annihilator in $\widehat{G}$ of the subgroup H of G, then $\widehat{G}/A$ is the character group of H; see for example (24.11) [2]. Therefore H is compactly generated by Theorem A.

Theorem A is proved using the structure theorem for compactly generated LCA groups (9.8) [2]: such groups have the form $\mathbb{R}^m \times \mathbb{Z}^n \times E$ where m, n are nonnegative integers and E is a compact abelian group. Then $\widehat{G} = \mathbb{R}^m \times \mathbb{T}^n \times D$ where D is the discrete group $\widehat{E}$. It is easy to see that $\mathbb{R}^m \times \mathbb{T}^n \times D$ has no small subgroups, and Moskowitz shows in Theorem 2.4 [9] that every G with no small subgroups has this form. To do this, he observes that $G = \mathbb{R}^m \times H$ where H contains a compact open subgroup C; see (24.30) [2]. Since C also has no small subgroups, it is topologically isomorphic with $\mathbb{T}^n \times F$ where n is a nonnegative integer and F is a finite group; this follows from Lie theory or (9.5) [2]. Since $\mathbb{T}^n$ is open and divisible in H, by 6.22.b [2] it is a direct factor: $H = \mathbb{T}^n \times D$ where D is a discrete group. Thus $G = \mathbb{R}^m \times \mathbb{T}^n \times D$.

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