Tuesday, March 17, 2009

Calculus I (Math 251, CRN 23180), Final

Name:______________________________

1	2	3	4	5	6	7	8	9	10	TOTAL

INSTRUCTIONS: Look through the exam before you start, and answer the questions that seem easiest to you first. Work on the exam sheets – you may use the backs of the pages if you need more space. Show all your work!!

There are 100 points distributed among 10 questions. You may find the exam too long to finish. If that is so, don't panic. Just make sure you start with the problems that you know how to do.

(1) Let $f(x) = x^2 + 3x$.

(a) Write down the limit definition of $f'(a)$ for this function. Do not evaluate the limit.

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{(x^2 + 3x) - (a^2 + 3a)}{x - a}$$

(b) Using the limit definition, calculate $f'(2)$.

$$\lim_{x\to 2} \frac{x^2 + 3x - 10}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+5)}{x - 2} = \lim_{x\to 2}(x + 5) = 7.$$

(2) Assume a falling object falls $4.9t^2$ meters after t seconds.

(a) If you drop an object from a 490 meter tall building, how many seconds does it take to hit the earth?

(b) Give a formula for the height $h(t)$) above the earth of an object dropped at time 0 from a 490 meter tall building. Be sure to give the range of t for which this formula is valid.

(c) What is the average velocity between the time $t = 0$ and the time the object hits the earth?

(d) What is the instantaneous velocity at time $t = 5$?

(a) *We want to know when it has fallen 490 meters, so when*

$$4.9t^2 = 490.$$

This will be true when $t^2 = 100$*, which is when* $t = 10$.

(b) $h(t) = 490 - 4.9t^2$. *This will make sense for* $t \in [0, 10]$.

(c) *The average velocity will be* $\frac{h(10)-h(0)}{10-0} = -490/10 = -49$ *meters per second.*

(d) *The instantaneous velocity at* $t = 5$ *is* $h'(5) = -9.8 \cdot 5 = -49$ *meters per second.*

(3) (a) Find the linear approximation to $f(x) = \sqrt[5]{x}$ at $a = 32$.
(b) Use this to give an approximation to $\sqrt[5]{35}$.
(c) Use your calculator to get a closer approximation to $\sqrt[5]{35}$.

(a) $f(32) = 2$. $f'(32) = (1/5)32^{(-4/5)} = (1/5)(1/2^4) = 1/80$. *So the linear approximation is*

$$y - 2 = \frac{1}{80}(x - 32) \text{ or } y = \frac{1}{80}x + \frac{8}{5}.$$

(b) *At* 35*, we get* $\frac{1}{80}35 + \frac{8}{5} = 2.0375$.
(c) *The actual value of* $\sqrt[5]{35}$ *is approximately* 2.03617.

(4) Calculate the following limits
(a)

$$\lim_{x \to -\infty} \frac{1 - 2x - 5x^2 - 2x^3}{5x^3 + x - 1}.$$

Dividing top and bottom by x^3 *and taking limits gives* $-2/5$.

(b)

$$\lim_{x \to \infty} \frac{\ln(x)}{\log_2(x)}$$

Using L'Hôpital:

$$\lim_{x \to \infty} \frac{1/x}{1/(x \ln(2))} = \ln(2).$$

(c)

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Using L'Hôpital:

$$\lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = -1/2.$$

(d)

$$\lim_{x \to 1^-} \frac{x^2 + 1}{x^2 - 1}$$

The numerator approaches 2, the denominator approaches 0 *from below, so the limit is* $-\infty$.

(e)

$$\lim_{x \to 0^+} x \ln(x)$$

Using L'Hôpital:

$$\lim_{x \to 0^+} \frac{\ln(x)}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0.$$

(5) Consider the function $x^3 - 3x^2 - 9x + 2$. Find the maximum and minimum values for this function on the interval $[-4, 2]$.

The maximum and minimum will be either at end points or critical points. The derivative is $3x^2 - 6x - 9$. This is 0 when $x = -1, 3$. At -4, the value is -74. At 2 it is -20. At -1 it is 7. So the minimum value is -74 and the maximum value is 7.

(6) Calculate the following derivatives

(a)

$$\frac{d}{du}\frac{1}{\sqrt{16-9u^2}}$$

$$(-1/2)(16-9u^2)^{-3/2}(-18u)$$

(b)

$$\frac{d}{dt}\sqrt{\frac{t-1}{t+1}}$$

$$\frac{1}{2\sqrt{\frac{t-1}{t+1}}}$$

(c)

$$\frac{d}{dx}3^{5x+1}$$

$$3^{5x+1}\ln(3)\cdot 5$$

by the chain rule.

(d)

$$\frac{d}{dx}[\sin(x)+(1+x^4)^3]^5$$

$$5[\sin x+(1+x^4)^3]^4(\cos x+3(1+x^4)^2 4x^3)$$

(e)

$$\frac{d}{dx}\ln(\sin(2x))$$

$$\frac{\cos(2x)2}{\sin(2x)} = 2\cot(2x)$$

(7) Find the rectangle of perimeter 100 with the maximum area. *Let the sides of the rectangle be x and y. $2x + 2y = 100$, so $y = 50 - x$. The area of the rectange is*

$$A(x) = xy = x(50-x).$$

The domain is $0 < x < 50$. $A'(x) = 50 - 2x$. So the critical point is when $x = 25$. $A(0) = A(50) = 0$. $A(25) = 625$. So the rectangle is a 25 foot by 25 foot rectangle.

(8) You wish to find the radius of a cylindrical can holding 1 liter (1000 cubic centimeters) using the smallest amount of metal (minimizing the total area of the top, bottom and sides).

(You may use that the volume of a can of height h and radius r is $\pi r^2 h$.)

We have $\pi r^2 h = 1000$, so $h = \frac{1000}{\pi r^2}$. The area is given by

$$A(r) = 2\pi r h + 2\pi r^2 = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2.$$

This formula makes sense for $0 < r < \infty$. $A'(r) = 4\pi r - \frac{2000}{r^2}$. The critical point is when $4\pi r^3 = 2000$, so when $r^3 = 500/\pi$, which is when $r = \sqrt[3]{500/\pi}$. $A'(r)$ is negative for smaller r and positive for larger r, so this is a minimum.

(9) Suppose you have a leak which is causing a circular puddle whose area is growing at 3 cubic inches per second. At what rate is the radius changing when the area is 49π inches? *Let $A(t) = \pi r(t)^2$. Then*

$$3 = A'(t) = 2\pi r(t) r'(t).$$

When $A(t) = 49\pi$, $r = 7$. So we get

$$3 = 2\pi 7 r' \text{ so } r' = \frac{3}{14\pi}$$

inches per second.

(10) Consider the function $f(x) = \frac{x^2+1}{x^2-1}$.

Find out where this function is increasing and decreasing, where any critical points are, where the function is convex up and convex down, any points of inflection, horizontal and vertical asymptotes, and graph the function.

$$f'(x) = -\frac{4x}{(x^2-1)^2}$$

The domain of our function is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. The derivative is negative when $x > 0$ (and not 1) and positive when $x < 0$ (and not -1). so the function is increasing on $(-\infty,-1)$ and on $(-1,0)$ and decreasing on $(0,1)$ and $(1,\infty)$, and 0 is a critical point. $f(0) = -1$ and is a local maximum.

$$\lim_{x\to -1^-} f(x) = \infty, \lim_{x\to -1^+} f(x) = -\infty, \lim_{x\to 1^-} f(x) = -\infty, \lim_{x\to 1^+} f(x) = \infty.$$

Horizontal asymptotes are 0.

$$f''(x) = \frac{4(1+3x^2)}{(x^2-1)^3}.$$

This is positive on $(-\infty, -1)$ *and on* $(1, \infty)$ *and negative on* $(-1, 1)$*. So the function is concave up on* $(-\infty, -1)$ *and* $(1, \infty)$ *and concave down on* $(-1, 1)$*.*