400/500 ALGEBRA SEQUENCE

This is a guide to our 400/500 algebra sequence intended primarily for instructors.

The most important point is that this is supposed to be a 400 level course suitable for our undergraduate math majors. The pace needs to be reasonable for an undergraduate seeing abstract algebra for the first time. Several tempting topics are really not appropriate at all: avoid modules completely, avoid Zorn's lemma. Go slowly especially at the beginning, do lots of simple examples, less is more!

Here is a list of topics which we suggest should be covered. It is probably appropriate to mix these topics up as suits the text chosen and the instructor's taste. We recommend the instructor extracts a course- and text-specific syllabus based loosely on the list below.

Number theory. Euclidean algorithm, infinite number of primes, modular arithmetic, inverses, as needed for cyclic groups.

Groups. Cyclic groups, groups of symmetries of polygons and polyhedra, dihedral groups, symmetric groups, matrix groups. Direct products. Conjugacy classes, centers, conjugation in S_n. Subgroups, cosets, Lagrange, normal subgroups and homomorphisms, Three homomorphism theorems and examples; equivalence relations and equivalence classes are in here of course with considerable time spent emphasizing the need to check well-definedness. Alternating groups, simplicity. Group actions with some simple applications, e.g. center of a p-group, Cayley's theorem, orbit counting lemma, finite subgroups of SO_3, Sylow theorems (optional). Fundamental theorem of finite abelian groups (proof optional but definitely NOT done via theory of modules over PIDs). Generating groups, esp. symmetric and alternating, using transpositions or 3-cycles. Derived groups, solvability. Simplicity of A_n.

Rings. Always with 1. Subrings and ideals. Three homomorphism theorems and examples. Polynomial rings, Euclidean algorithm, Euclidean domains. Fields of fractions of integral domains. Maximal ideals, for getting new fields. Prime elements. PIDs, UFDs. Gauss' lemma. Eisenstein.

Linear algebra. Definition of a field. Vector space, basis, dimension for finite dimensional vector spaces via the exchange lemma. Subspaces. This is meant to be a minimal amount of linear algebra needed for field theory rather than a serious course in linear algebra, but perhaps some brief mention of determinants or dual spaces may be appropriate. Remember there is now a 400 level abstract algebra sequence which serious undergraduates should aim to take. For graduate students linear algebra is reviewed extensively in 600 algebra in case they have not seen enough of it before.

Field theory. Finite extensions, ruler and compass. Adjoining roots, splitting fields. Classification of finite fields, multiplicative groups of finite fields are cyclic. Algebraic extensions, algebraic closure in countable case, fundamental theorem of algebra. Galois extensions and correspondence. Fundamental theorem of Galois theory. Easy examples of computing Galois groups of polynomials. Proof that polynomials solvable by radicals have solvable Galois group with explicit example of unsolvable quintic; proof of the converse is optional. Further topics like cyclomotomic polynomials, symmetric polynomials are optional.

Suggested text books. We do not want to insist on particular text books, but it is important to choose one so that the students have a reference to fall back on. Here are three suggestions which may be appropriate; we have not tried any of them ourselves but they look promising and roughly at the intended level.

- F. Goodman, Algebra, abstract and concrete. This book is available free of charge at

  `http://homepage.math.uiowa.edu/~ goodman/algebrabook.dir/algebrabook.html.`

  Beware that the book covers too much material, in particular the section on modules should be skipped.
- J. J. Rotman, A first course in abstract algebra. Chapters 1 through 5 plus the odd optional topic from chapter 6 seem appropriate.
- J. Gallian, Contemporary abstract algebra. Skip the bar codes but otherwise seems reasonable.

—Jon Brundan, Victor Ostrik, Arkady Vaintrob (June 2013).