Math 212: Content and Practice Outcomes

Tricia Bevans and Dev Sinha

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We view Math 212 as a basis from which pre-service teachers can engage in the mathematical work of teaching, including providing age-appropriate explanations, formatively assessing student understanding, accurately assessing the correctness of non-standard solutions, providing differentiated instruction, asking accessible questions which provoke thought and discussion, and modifying pace and emphasis accounting for student understanding and the demands of mathematics they will take later.

Serving as a basis for such work does not mean that we suggest structuring the course around exactly this work. Indeed, in the transition to the Common Core it can be safely assumed that pre-service teachers will have little if any experience in learning mathematics in the deeper ways we will be demanding of students, so it is essential to provide such experience to them as adult learners of mathematics itself. Moreover, within the model of a liberal education, students engage in core disciplines first as foundational before moving on to how that discipline is addressed within their own area. Finally teachers must draw on their own adult-level understanding – which is more concise, abstract and connected than child-level understanding – as they navigate with child-level language and arguments. Thus, our choice is to focus more on the adult-level understanding in this course.

We will be using these outcomes for a proficiency-based grading system, but as primary goals they inform all of the content, instruction and other design aspects of the course.

1 Content-focused Outcomes

With each outcome we include some examples that could be part of demonstrating proficiency of the outcome, but the list is not exhaustive. For each of these outcomes, students must exhibit proficiency both with numbers and with variables, for example showing why $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ in those terms. The latter proficiency shows a college-level proficiency through a deeper capacity to organize the ideas.

1. **Fractions**

   (a) Understand the meaning of fractions and fraction equivalence:

      i. Recognize, interpret, and draw connections between different meanings of fractions. For example, demonstrate two fractions are equivalent using visual representations, in order to establish algebraic procedures.

      ii. Demonstrate the equality of fractions defined through unit fractions or as the result of division, and show misunderstandings that arise from prematurely using fractions as proportions. For example, show how proportions do not "add" as numbers do, and explain why in terms of the definition of fractions, attending to the whole.

      iii. Give appropriate visual models for each meaning including representations on the number line. Explain strengths and limitations of the various meanings and models.

   (b) Understand addition, subtraction and comparison of fractions:

      i. Use various arguments to compare one fraction to another. For example, provide examples where a given comparison strategy is more or less appropriate than another.

      ii. Give both visual and algebraic arguments for why one fraction is greater than another and explain the connections between the two arguments. For example, explain why "adding as proportions" (or, informally, "adding across") always produces a fraction in between two fractions.

      iii. Extend the meaning of addition and subtraction of whole numbers to fractions with like and unlike denominators including mixed numbers.

      iv. Perform and justify addition and subtraction of fractions with appropriate visual models and algebra.

      v. Recognize common misconceptions about fraction addition and subtraction. Address the misconception in both college-level and age-appropriate ways.

   (c) Understand multiplication and division of fractions:

      i. Use properties of operations, especially associativity and commutativity, to extend the meaning of multiplication from whole numbers to fractions. In particular, autonomously produce arguments starting with phrases such as "If the commutative law for multiplication for whole numbers is going to continue to work,".

      ii. Use appropriate strategies/algorithms to multiply fractions and justify each method. In particular, give more than one visual model to represent multiplication of fractions. For example, explain why $\frac{a}{b} \times c = c \times \frac{a}{b}$ using visual models for each.

      iii. Distinguish between partitive and measurement division.

   iv. Use various methods to divide fractions and show they are equivalent to one another.
   v. Use appropriate visual models and connect them to division methods.

(d) Identify and Create contexts for fraction operations:

   i. Identify when addition/subtraction/multiplication is an appropriate operation for solving a given word problem.
   ii. Construct a realistic context that gives rise to a particular fraction division computation.
   iii. Recognize various possible uses for the "remainder" that may arise in measurement fraction division. In particular, understand both the "standard remainder" given by the standard arithmetic definition (given by one's calculator) and the "absolute remainder", and use visual models to provide arguments about each.

2. **Negative numbers and exponents**

(a) Understand the meaning of negative numbers, and justify computations involving them.

   i. Represent negative values with an appropriate visual model or manipulative, and identify contexts that involve negative values.
   ii. Represent computations with an appropriate visual model. For example, give a visual model for multiplication of a negative number by a positive.
   iii. Justify that a negative multiplied by a negative is a positive. In particular, autonomously produce arguments starting with phrases such as "If the distributive law for positive numbers is going to continue to work, then...". Illustrate such arguments with a visual model.

(b) Calculate with and justify properties of exponents:

   i. Argue for the definition of negative and fractional exponents by extending the law of exponents (which may need to be established if it wasn't in MA 211). In particular, autonomously produce arguments starting with phrases such as "If the law of exponents for whole numbers is going to continue to work, then $2^{\frac{3}{2}}$ must be...".
   ii. Use exponential notation to justify the algebraic steps for the standard algorithm for multiplication of decimals.

3. **Decimals, base-b-imals, and the real number line**

(a) Understand the meaning of decimals and base-b-imals:

   i. Be able to produce and fully label a "base b number line."
   ii. Convert between decimals, "b-imals" and fractions.
   iii. Predict whether a number will have a repeating b-imal representation in a given base.

(b) Perform and justify computations involving decimals:

   i. Use place value understanding and properties of operations to justify arithmetic of decimal or "b-imal" values.

ii. Use an area model to justify "moving the decimal" shortcuts for multiplication and division.

(c) Understand the real number line:

i. Distinguish between rational and irrational numbers with appropriate justification.

ii. Prove that a number has an eventually repeating b-imal expansion if and only if it is a rational number. In particular, know how long one might need to wait before the expansion repeats.

iii. Explain informally the denseness of the rationals in the real numbers. In particular, explain two ways to find a number between any two rational numbers and then arbitrary real numbers.

iv. Be able to produce a list which eventually would contain any rational number (with repeats, which can be "tossed out"). Given any list of decimal numbers, show there is a decimal number which is not on the list.

2 Practice-focused outcomes

As you engage in this content, we expect you to engage in mathematical practices. A good analogy is to be made with music, where there is a distinction to be made between listening to music and producing it. Accounting for these practices will happen more in class rather than in exams or writing assignments. Nonetheless, they are essential in your achieving proficiency in the content above.

1. **Understand and engage in a range of mathematical reasoning.**

   Math teachers need to be well-versed in a range of reasoning which can support understanding. In order roughly from least to most rigorous, we see some such reasoning as follows.

   (a) Background examples and discussion. It takes some mathematical work to recognize an example as germane or not to a general kind of statement (e.g. is $532 + 309 = 509+332$ an example of commutativity?). While providing examples does not serve as an argument (except when providing counterexamples) this kind of work, sometimes called contextualizing and decontextualizing, is an important step in reasoning.

   (b) Finding models, making sketches, and solution planning. These are all skills important to launch full immersion into reasoning.

   (c) Analyzing examples. This is an intermediate step between providing relevant examples and providing a fully generalized example. A key step is understanding whether an example works is fully generalizable.

   (d) Identify relevant definitions. Relevant definitions (including model-based definitions) are an essential part of mathematical argument.

   (e) Arguments with pictures. These can range in level of rigor and completeness depending on how they are structured.

   (f) Generalized examples. This type of argument is a bit of a lost art (historically, it was used by mathematicians such as Pascal). If one starts with an example and then explains fully how relevant features of that example hold in general, one can provide a fully rigorous argument.

   (g) Arguments with variables and more formalism. While these are the lingua franca for modern mathematics, and in particular essential for undergraduate math majors to master, they should be employed strategically in this context.

   We should delineate between the first four and the last three. The first four are key ingredients of reasoning, which we would expect any students with some effort to be able to summon when relevant. But it is only the last three which can support fully rigorous argument, and thus for example be considered complete as homework.

   For example, a student proficient in the practice of argument would be able to go from understanding the proof that an even number added to an even number is even to a proof that a multiple of three added to a multiple of three gives another multiple of three. They would be able to engage multiple such proofs, for example proof by picture (and explanation), generalized example, and using variables. Similarly, having seen that a number is a multiple of four if its last two base-ten digits are, they would be able to show that the same holds for multiples of eight in base twelve.

2. **Gain experience in the Mathematical Practices.**

   Reasoning is addressed above as particularly important, but the full range of mathematical practices such as problem-solving, modeling, being precise, and seeing structure are essential for students to experience. We strongly recommend looking at the Mathematical Practices in the Common Core document, but we include them here for quick reference.

   (a) Make sense of problems and persevere in solving them.
   (b) Reason abstractly and quantitatively.
   (c) Construct viable arguments and critique the reasoning of others.
   (d) Model with mathematics.
   (e) Use appropriate tools strategically.
   (f) Attend to precision.
   (g) Look for and make use of structure.
   (h) Look for and express regularity in repeated reasoning.

   These practices serve both understanding what is ultimately challenging material (contrary to what some may be led to believe by the word "elementary") and serve future teachers in understanding the outcomes desired in mathematics education reform.

   For example, students engaging in the practice of seeing structure will be able to explain why one can multiply numbers by adding together numbers on a list obtained by repeated doubling. They will persevere when engaging in long-division in other bases, and for example see there must be an arithmetic error by checking their answer through multiplication and then finding their error. They will attend to precision in how they describe numbers in other bases, and checking that answers given are valid in that base.

3. **Link concrete to abstract**

   Proficient Math 212 students will be able to take a general statement and provide an example. Conversely, they will look at some collection of examples or other phenomena and be able to formulate a statement which captures essential features of the collection. Such students understand the interplay between conjectures and theorems.

   These students will see how meanings are developed, first in more concrete ways connecting to previous experience and then more abstract ways. They will understand how connecting different approaches reinforces all, and by doing so creates a robust meaning. They will begin to identify what meanings can serve student understanding in some situations, as a foundation for a lifelong refinement of this teaching practice.

   For example, a student proficient in linking concrete to abstract would be able to give at least four different models for numbers and be able to connect any two of them.

4. **Begin to see the coherence and global structure of mathematics**

   A full understanding of this coherence and global structure is an ambitious aspiration. There are two concrete ways in which students in Math 211 can make significant steps towards such vision. One is a view which is also mentioned in our content standards. It is the Meaning-Method-Mastery framework, first developed by Cody Patterson. This development can be seen both within isolated strands of mathematics and as the strands interact.

   Another way in which students will see coherence is through awareness of Algebraic Thinking in elementary mathematics. The Common Core prescribes using of arithmetic as a venue to reflect on some of the main ideas of algebra. For example, by understand the role of "canceling" (or "compensation") in arithmetic strategies, students are better able to see a purpose for similar moves in algebra. Math 211 students will start to see these connections, for example through giving adult-level descriptions including variables for phenomena an arithmetic.

   A student engaging in the practice of seeing coherence might for example explain why the number line is important in early grades, when it doesn't necessarily serve understand of the mathematics at hand. They may write a sequence of problems which call for the partitive missing factor model of division, which use the same (or very similar) modeling context but are appropriate to grades 3, 4, 5 and 6. They could explain the connection between "FOIL" and multiplication of two-digit numbers.