Math 213: Content and Practice Outcomes

Tricia Bevans, Tammy Nezol, and Dev Sinha

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We view Math 213 as a basis from which pre-service teachers can engage in the mathematical work of teaching, including providing age-appropriate explanations, formatively assessing student understanding, accurately assessing the correctness of non-standard solutions, providing differentiated instruction, asking accessible questions which provoke thought and discussion, and modifying pace and emphasis accounting for student understanding and the demands of mathematics they will take later.

Serving as a basis for such work does not mean that we suggest structuring the course around exactly this work. Indeed, in the transition to the Common Core it can be safely assumed that pre-service teachers will have little if any experience in learning mathematics in the deeper ways we will be demanding of students, so it is essential to provide such experience to them as adult learners of mathematics itself. Moreover, within the model of a liberal education, students engage in core disciplines first as foundational before moving on to how that discipline is addressed within their own area. Finally teachers must draw on their own adult-level understanding – which is more concise, abstract and connected than child-level understanding – as they navigate with child-level language and arguments. Thus, our choice is to focus more on the adult-level understanding in this course.

We will be using these outcomes for a proficiency-based grading system, but as primary goals they inform all of the content, instruction and other design aspects of the course.

1 Content-focused Outcomes

With each outcome we include some examples that could be part of demonstrating proficiency of the outcome, but the list is not exhaustive. For each of these outcomes, students must exhibit proficiency both with examples and in general, where appropriate. For example, a student might give several examples of rectangles and use measurements to *hypothesize* that the diagonals are equal, but must also show using a logical proof that this is true for *any* rectangle. The latter proficiency shows a college-level proficiency through a deeper capacity to organize the ideas.

1. **Ratios and Proportional Reasoning**

   (a) Understand the meaning of ratios and proportional relationships:

      i. Recognize, describe, and represent ratios, rates, and proportional relationships using appropriate vocabulary and notation.

      ii. Use visual and algebraic models to represent proportional relationships (e.g. tables, graphs, tape diagrams, and double number lines, equations) and use them to solve problems.

      iii. Identify relevant features and draw connections between different visual and algebraic models. For example show the multiplicative and additive structure in both a table and graph of the same proportional relationship or explain how the unit rate in one representations connects to how it is represented in another representation.

   (b) Apply proportional reasoning to real-world contexts:

      i. Use multiple methods to compare proportional relationships in context. For example, use two tables of ratios of quantities of yellow and red paint in more than one way to determine if the colors represented in the tables are the same. (see RP progression page 6)

      ii. Identify contexts that can/cannot be represented by proportional relationships. (e.g. statistics and probability, mixtures)

      iii. Create and solve real-world problems that use proportional reasoning and interpret results in context.

   (c) Apply proportional reasoning to mathematical concepts:

      i. Use dimensional analysis to convert units in both one and multiple dimensions

      ii. Connect proportions to geometry concepts (e.g. arc length, similarity)

2. **Polygons, Angles, Lines, and Symmetry**

   (a) Identify characteristics of and classify geometric figures:

      i. Determine if a figure is a subset of a larger category (e.g. a square is also a rectangle)

      ii. Determine if a figure has line or rotational symmetry and identify the line of symmetry or the measure for the angle of rotational symmetry.

   (b) Use properties of geometric figures, angles, and symmetry in proofs and problem solving:

      i. Prove the sum of the interior angles of a triangle is $180°$

ii. Prove additional characteristics about polygons using basic characteristics of lines, angles, and symmetry. For example, find the sum of the interior/exterior angles of a polygon.
iii. Use symmetry to structure proofs involving repeated arguments

(c) Use, derive and/or prove formulas and theorems about polygons

i. Accurately use appropriate formulas in problem solving (e.g. Pythagorean theorem and area formulas)
ii. Derive area formulas for two-dimensional polygons and circles and extend them to get surface area formulas.
iii. Derive formulas for volume of three-dimensional geometric figures.
iv. Apply area and Pythagorean formulas to real-world contexts
v. Provide multiple proofs of the Pythagorean Theorem.

(d) Understand constructions of angles, lines, and geometric figures

i. Construct and justify construction of lines with various characteristics including parallel and perpendicular through a given point using compass and straightedge, paper folding, or geometry software such as Geogebra.
ii. Construct and justify construction of angles with various characteristics including those with $90°$ and other angle measures, bisected angles using compass and straightedge, paper folding, or geometry software such as Geogebra.

3. **Congruence and Similarity**

(a) Understand congruence

i. Give the meaning of congruence using both construction and transformations and draw connections between the two approaches.
ii. Use triangle congruence theorems in proofs
iii. Prove two figures are congruent using translations, rotations, and/or reflections with or without a coordinate plane.

(b) Understand similarity

i. Give the meaning of similarity using both construction and transformations and draw connections between the two approaches.
ii. Use triangle similarity theorems in proofs
iii. Prove two figures are congruent using translations, rotations, reflections and/or dilations with or without a coordinate plane.

(c) Apply triangle congruence and similarity to real-world and mathematical problems

(d) Understand constructions using congruent and similar triangles

i. Construct a triangle congruent to another using triangle congruence properties using compass and straightedge and demonstrate when there is an ambiguous case.
ii. Construct congruent or similar figures using compass straightedge, paper folding, or geometry software and justify why the constructions work.
iii. Identify and perform other constructions that use triangle congruence as justification.

4. **Lines and Systems of Linear Equations**

   (a) Understand the geometric and algebraic structure of lines

      i. Connect lines to proportional relationships. For example, use triangle similarity to understand slope.
      ii. Use more than one method for finding the equation of a line. In particular, use similar triangles and slope.
      iii. apply transformations to lines. In particular, explain the slopes of parallel and perpendicuar lines and show lines are congruent using transformations

   (b) Solve basic systems of linear equations

      i. Solve a system of linear equations graphically and algebraically
      ii. Recognize mathematical and real-world contexts where a system of equations is an appropriate solution path, including imposing the structure of a coordinate plane to find alternate solutions to a geometric problem.
      iii. Interpret the solution of a system of equations in the context of a real-world application.

2 Practice-focused outcomes

As you engage in this content, we expect you to engage in mathematical practices. A good analogy is to be made with music, where there is a distinction to be made between listening to music and producing it. Accounting for these practices will happen more in class rather than in exams or writing assignments. Nonetheless, they are essential in your achieving proficiency in the content above.

1. **Understand and engage in a range of mathematical reasoning.**

   Math teachers need to be well-versed in a range of reasoning which can support understanding. In order roughly from least to most rigorous, we see some such reasoning as follows.

   (a) Background examples and discussion. It takes some mathematical work to recognize an example as germane or not to a general kind of statement (e.g. is $532 + 309 = 509+332$ an example of commutativity?). While providing examples does not serve as an argument (except when providing counterexamples) this kind of work, sometimes called contextualizing and decontextualizing, is an important step in reasoning.

   (b) Finding models, making sketches, and solution planning. These are all skills important to launch full immersion into reasoning.

   (c) Analyzing examples. This is an intermediate step between providing relevant examples and providing a fully generalized example. A key step is understanding whether an example works is fully generalizable.

   (d) Identify relevant definitions. Relevant definitions (including model-based definitions) are an essential part of mathematical argument.

   (e) Arguments with pictures. These can range in level of rigor and completeness depending on how they are structured.

   (f) Generalized examples. This type of argument is a bit of a lost art (historically, it was used by mathematicians such as Pascal). If one starts with an example and then explains fully how relevant features of that example hold in general, one can provide a fully rigorous argument.

   (g) Arguments with variables and more formalism. While these are the lingua franca for modern mathematics, and in particular essential for undergraduate math majors to master, they should be employed strategically in this context.

   We should delineate between the first four and the last three. The first four are key ingredients of reasoning, which we would expect any students with some effort to be able to summon when relevant. But it is only the last three which can support fully rigorous argument, and thus for example be considered complete as homework.

   For example, a student proficient in the practice of argument would be able to go from understanding the proof that an even number added to an even number is even to a proof that a multiple of three added to a multiple of three gives another multiple of three. They would be able to engage multiple such proofs, for example proof by picture (and explanation), generalized example, and using variables. Similarly, having seen that a number is a multiple of four if its last two base-ten digits are, they would be able to show that the same holds for multiples of eight in base twelve.

2. **Gain experience in the Mathematical Practices.**

   Reasoning is addressed above as particularly important, but the full range of mathematical practices such as problem-solving, modeling, being precise, and seeing structure are essential for students to experience. We strongly recommend looking at the Mathematical Practices in the Common Core document, but we include them here for quick reference.

   (a) Make sense of problems and persevere in solving them.
   (b) Reason abstractly and quantitatively.
   (c) Construct viable arguments and critique the reasoning of others.
   (d) Model with mathematics.
   (e) Use appropriate tools strategically.
   (f) Attend to precision.
   (g) Look for and make use of structure.
   (h) Look for and express regularity in repeated reasoning.

   These practices serve both understanding what is ultimately challenging material (contrary to what some may be led to believe by the word "elementary") and serve future teachers in understanding the outcomes desired in mathematics education reform.

   For example, students engaging in the practice of seeing structure will be able to explain why one can multiply numbers by adding together numbers on a list obtained by repeated doubling. They will persevere when engaging in long-division in other bases, and for example see there must be an arithmetic error by checking their answer through multiplication and then finding their error. They will attend to precision in how they describe numbers in other bases, and checking that answers given are valid in that base.

3. **Link concrete to abstract**

   Proficient Math 213 students will be able to take a general statement and provide an example. Conversely, they will look at some collection of examples or other phenomena and be able to formulate a statement which captures essential features of the collection. Such students understand the interplay between conjectures and theorems.

   These students will see how meanings are developed, first in more concrete ways connecting to previous experience and then more abstract ways. They will understand how connecting different approaches reinforces all, and by doing so creates a robust meaning. They will begin to identify what meanings can serve student understanding in some situations, as a foundation for a lifelong refinement of this teaching practice.

   For example, a student proficient in linking concrete to abstract would be able to give at least four different models for numbers and be able to connect any two of them.

4. **Begin to see the coherence and global structure of mathematics**

   A full understanding of this coherence and global structure is an ambitious aspiration. There are two concrete ways in which students in Math 211 can make significant steps towards such vision. One is a view which is also mentioned in our content standards. It is the Meaning-Method-Mastery framework, first developed by Cody Patterson. This development can be seen both within isolated strands of mathematics and as the strands interact.

   Another way in which students will see coherence is through awareness of Algebraic Thinking in elementary mathematics. The Common Core prescribes using of arithmetic as a venue to reflect on some of the main ideas of algebra. For example, by understand the role of "canceling" (or "compensation") in arithmetic strategies, students are better able to see a purpose for similar moves in algebra. Math 211 students will start to see these connections, for example through giving adult-level descriptions including variables for phenomena an arithmetic.

   A student engaging in the practice of seeing coherence might for example explain why the number line is important in early grades, when it doesn't necessarily serve understand of the mathematics at hand. They may write a sequence of problems which call for the partitive missing factor model of division, which use the same (or very similar) modeling context but are appropriate to grades 3, 4, 5 and 6. They could explain the connection between "FOIL" and multiplication of two-digit numbers.