A combinatorial Fourier transform for quiver representation varieties in type A

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Consider the quiver $\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet$.

Notation:

1. $E(w)$ - space of representations for dimension vector $w = (w_1, \ldots, w_n)$
2. $G(w) = \text{GL}(w_1) \times \cdots \times \text{GL}(w_n)$
3. $w^* = (w_n, \ldots, w_1)$ - the reverse dimension vector
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Can we give a combinatorial description of the Fourier–Sato transform:

$$D^b_{G(w)}(E(w)) \xrightarrow{T} D^b_{G(w^*)}(E(w^*))$$

$q_2! q_1^*(\mathcal{F})[\dim E(w)]$

for simple perverse sheaves $\mathcal{F}$?
Outline

1. Quiver representation varieties
2. Some combinatorics
3. Fourier–Sato transform
4. Combinatorial Fourier transform
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Consider the type $A_n$ equioriented quiver

$$Q_n = \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet.$$ 

**A quiver representation** is:

- A finite-dimensional $\mathbb{C}$-vector space $M_i$ for each vertex.
- A linear map $x_i$ for each arrow.
Quiver representations

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\( \text{Rep}(Q_n) \) - abelian category of finite-dimensional complex representations of \( Q_n \)

Above, \( \text{dim}(M) = (\dim M_1, \dim M_2, \ldots, \dim M_n) \in \mathbb{Z}_{\geq 0}^n \).
Fix a dimension vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$.

A quiver representation variety $E(\mathbf{w})$ is the space of all quiver representations for a fixed dimension vector $\mathbf{w}$.

Note that $E(\mathbf{w})$ is an affine variety:

$$E(\mathbf{w}) \cong \mathbb{A}^{w_1w_2 + w_2w_3 + \cdots + w_{n-1}w_n}.$$
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\]

\( G(\mathbf{w}) = \text{GL}(w_1) \times \cdots \times \text{GL}(w_n) \) acts on \( E(\mathbf{w}) \) by

\[
(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_{n-1}) = (g_2 x_1 g_1^{-1}, \ldots, g_n x_{n-1} g_{n-1}^{-1})
\]

giving it a stratification by orbits.

Note that two points \( x, y \in E(\mathbf{w}) \) are in the same \( G(\mathbf{w}) \)-orbit if and only if they are isomorphic objects of \( \text{Rep}(Q_n) \).
Classifying the orbits

Theorem (Gabriel’s Theorem)

There is a bijection

\[
\frac{\text{indec. objects in } \text{Rep}(Q_n)}{\sim} \longleftrightarrow \text{pos. roots for } A_n \text{ root system}.
\]
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\]

To an indecomposable representation

\[
R_{ij} = 0 \to \cdots \to 0 \to \mathbb{C} \overset{\text{id}}{\longrightarrow} \cdots \overset{\text{id}}{\longrightarrow} \mathbb{C} \to 0 \to \cdots \to 0.
\]

we associate its dimension vector, the positive root

\[
\gamma_{ij} = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)_{\text{position } i, \text{ position } j}.
\]
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\gamma_{ij} = (0, \ldots, 0, \underbrace{1}_{\text{position } i}, \ldots, \underbrace{1}_{\text{position } j}, 0, \ldots, 0).
\]

**Corollary**

There is a bijection

\[
\{ G(w)\text{-orbits in } E(w) \} \leftrightarrow B(w) := \{ b_{ij} | \sum b_{ij} \gamma_{ij} = w \}.
\]
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Define the set $P(w)$ of triangular arrays of nonnegative integers such that:

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- Ladders are weakly decreasing.
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We will write $y_{ij}$ for the entry in the $i^{th}$ chute and $j^{th}$ column.

There is a partial order on $\mathbf{P}(w)$ defined by

$$Y \leq_{\text{comb}} Y' \iff \text{ for all } i \text{ and } j, \sum_{k=1}^{j} y_{ij} \leq \sum_{k=1}^{j} y'_{ij}.$$
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If $Y \in P(w)$ and $Z \in P(v)$, then we can form the entry-wise sum $Y + Z \in P(w + v)$. 

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Classifying the orbits combinatorially

**Lemma (Achar–Kulkarni–M.)**

There is a bijection

\[
B(w) := \{ b_{ij} \mid \sum b_{ij} \gamma_{ij} = w \} \leftrightarrow P(w).
\]
Let \( \mathbf{w} = (1, 1, 2) \).
If $Y \in \mathbf{P}(w)$, we write $\mathcal{O}_Y$ for the corresponding $G(w)$-orbit in $E(w)$.

Denote by $M(Y)$ a representation in the orbit $\mathcal{O}_Y$. 
Some observations from the combinatorics

If $Y \in \mathbf{P}(\mathbf{w})$, we write $O_Y$ for the corresponding $G(\mathbf{w})$-orbit in $E(\mathbf{w})$.

Denote by $M(Y)$ a representation in the orbit $O_Y$.

Lemma (Achar–Kulkarni–M.)

1. $O_Y$ is the unique closed orbit in $E(\mathbf{w})$ if and only if $Y$ is the unique minimal element of $\mathbf{P}(\mathbf{w})$.
2. If $Y \in \mathbf{P}(\mathbf{w})$ and $Z \in \mathbf{P}(\mathbf{v})$, then $M(Y + Z) \simeq M(Y) \oplus M(Z)$.
3. $M(Y)$ is an injective object in $\text{Rep}(Q_n)$ if and only if $Y$ is constant along ladders.
4. $M(Y)$ is a projective object in $\text{Rep}(Q_n)$ if and only if $Y$ has nonzero entries only in the last ladder.
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See Kashiwara–Schapira (Section 3.7) for more details.

Can we give a combinatorial description of the Fourier–Sato transform:

$$D^b_{G(w)}(E(w)) \xrightarrow{\mathbb{T}} D^b_{G(w^*)}(E(w^*))$$

$$\mathcal{F} \xrightarrow{q_2!q_1^*} q_2!q_1^*(\mathcal{F})[\dim E(w)]$$

for simple perverse sheaves $\mathcal{F}$?
Fourier–Sato transform

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for simple perverse sheaves \(\mathcal{F}\)?

\[
E(w) \times E(w^*)
\]

\[
\{(x, y) \in E(w) \times E(w^*) \mid \text{Re}(\langle x, y \rangle) \leq 0\}
\]
Some properties and applications of the Fourier transform

**Properties:**
- $t$-exact for the perverse $t$-structure and sends simples to simples.
- Equivalence of categories
- "almost" an involution
- Compatible with convolution; i.e. $\mathbb{T}(\mathcal{F} \star \mathcal{G}) = \mathbb{T}(\mathcal{F}) \star \mathbb{T}(\mathcal{G})$. 

**Applications:**
- Used in the 1980s to shorten Deligne's proof of the Weil conjectures (Laumon).
- The Springer correspondence (Hotta–Kashiwara, Evens–Mirković)
- Character sheaves (Lusztig, Mirković)
- Character formula for quantum loop algebras uses Fourier transform on graded quiver varieties (Nakajima).
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Theorem (Achar–Kulkarni–M.)

There is a bijection

\[ P(w) \xrightarrow{T} P(w^*) \]

defined inductively by

\[
T \begin{pmatrix}
Y' \\
y_1, n \\
y_{n, 1} \\
\vdots
\end{pmatrix} = \tau_n^{y_{1, n}} \tau_{n-1}^{y_{2, n-1} - y_{1, n}} \cdots \tau_1^{y_{n, 1} - y_{n-1, 2}}
\]

where \( T(a) = a \).
Define $\tau_j : P(w) \rightarrow P(w + e_1 + \ldots + e_j)$ by:

- Add 1 as far down the $j^{th}$ chute as possible, drawing an impassable vertical line there.
- Repeat for chutes $j - 1, \ldots, 1$ not crossing lines.
Example of $T$

\[
T\left( \begin{array}{c}
1 \\
0 \\
2 \\
\end{array} \right) = \begin{array}{c}
1 \\
\end{array}
\]

\[
T\left( \begin{array}{c}
1 \\
0 \\
0 \\
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0 \\
1 \\
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\[
T\left( \begin{array}{c}
1 \\
0 \\
1 \\
2 \\
\end{array} \right) = \tau_2 \tau_1 \left( \begin{array}{c}
0 \\
0 \\
0 \\
1 \\
\end{array} \right) = \begin{array}{c}
2 \\
0 \\
0 \\
1 \\
\end{array}
\]

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T\left( \begin{array}{c}
1 \\
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\]
Running example
Main conjecture

Conjecture (Achar–Kulkarni–M.)

The bijection \( \mathcal{T} : \mathbf{P}(w) \rightarrow \mathbf{P}(w^*) \) determines \( \mathcal{T} : D^b_G(w)(E(w)) \rightarrow D^b_G(w^*)(E(w^*)) \) for simple perverse sheaves; that is,

\[
\mathcal{T}(\text{IC}(\mathcal{O}_Y)) = \text{IC}(\mathcal{O}_{\mathcal{T}(Y)}).
\]
Main conjecture

Conjecture (Achar–Kulkarni–M.)

The bijection $\mathcal{T} : \mathcal{P}(w) \rightarrow \mathcal{P}(w^*)$ determines $\mathcal{T} : D^b_{G(w)}(E(w)) \rightarrow D^b_{G(w^*)}(E(w^*))$ for simple perverse sheaves; that is,

$$\mathcal{T}(\text{IC}(\mathcal{O}_Y)) = \text{IC}(\mathcal{O}_{T(Y)}).$$

Proof idea:

Since $\mathcal{T}(\mathcal{F} \star \mathcal{G}) = \mathcal{T}(\mathcal{F}) \star \mathcal{T}(\mathcal{G})$, we can use induction on the dimension vector. The proof should follow from a careful study of the combinatorics of $\star$ as well as the interplay between $\leq_{\text{geom}}$ and $\leq_{\text{comb}}$. 
The End

Thanks!