Research Statement:
Combinatorial Methods in Algebra and Geometry

Jacob P. Matherne

My research is at the intersection of algebra, geometry, and combinatorics. My dissertation work under Pramod Achar involved studying singularities of certain topological spaces via **perverse sheaves**. Concurrently, I began working with Greg Muller on **cluster algebras**, a subject that has an easier point of entry due to its computable examples, but which has deep connections throughout mathematics and physics.

This early blend of almost diametrically-opposed styles of math shaped my research direction. My work gravitated toward interpreting complicated topological or algebraic questions as combinatorial ones. Currently, I am focused on adapting algebro-geometric ideas to combinatorics via **matroids**. My collaborators and I are developing an **“intersection cohomology”** theory for matroids, allowing for powerful algebro-geometric tools to be used for solving classical combinatorial problems about matroids (Section 1.1).

Some of my research problems lend themselves to **explicit computations** which allow **undergraduates and beginning graduate students** to make early contributions to solving problems in various areas of mathematics. Moreover, experiments can often be done using **computer software** like Macaulay2, GAP, or SageMath. My research statement is divided into four broad sections: matroids (Section 1), cluster algebras (Section 2), representation theory (Section 3), and research with undergraduates (Section 4).

---

**Section 1.1** *(joint with Tom Braden (UMass), June Huh (IAS), Nicholas Proudfoot (Oregon), and Botong Wang (Wisconsin))* We aim to introduce topological techniques into the field of matroid theory. As applications, we hope to prove the non-negativity of Kazhdan–Lusztig polynomials of matroids, and to settle Dowling and Wilson’s 1974 “Top-Heavy Conjecture” on the lattice of flats of a matroid.

**Section 1.2** *(joint with Eric Bucher (Xavier), Christopher Eppolito (graduate student - Binghamton), and Jaiung Jun (Iowa))* We give a cancellation-free antipode formula for the matroid-minor Hopf algebra.

**Section 2.1** *(joint with Greg Muller (Oklahoma))* We develop an algebro-geometric algorithm which gives a presentation for upper cluster algebras in terms of generators and relations. This algorithm is computable in finite-time, and I have implemented it into Sage with my collaborators.

**Section 3.1** *(joint with Alexander Garver (LaCIM), Kiyoshi Igusa (Brandeis), and Jonah Ostroff (Washington))* We introduce a combinatorial model for exceptional sequences of type A quiver representations.

**Section 3.2** *(joint with Pramod Achar (Louisiana State) and Maitreyee Kulkarni (IAS))* We introduce a combinatorial model for computing the Fourier–Sato transform on type A quiver representation varieties.

**Section 3.3** This was my dissertation work which gives a geometric, functorial relationship between representations of an algebraic group and representations of the corresponding Weyl group at the level of mixed, derived categories of sheaves on the affine Grassmannian and nilpotent cone of the Langlands dual group.

**Section 4.1** *(joint with Alexander Garver (LaCIM), Stefan Grosser (undergraduate - UMass), and Alejandro Morales (UMass))* We give a determinant formula for the number of linear extensions of a certain class of tree posets. The investigation of these posets was inspired by the project in Section 3.1.

**Section 4.2** *(joint with Portia Anderson (first-year graduate student - Cornell, prior Smith College) and Julianna Tymoczko (Smith College)) We produce bases for the module of algebraic splines for all graphs with at most two edge-labels, and for all cycles whose edges are labeled with degree-two polynomial ideals.

---

1 Each of the broad sections can be read independently, so the reader is invited to pick his or her favorite subject and go directly to that page.
1 Matroids

A matroid is a gadget which generalizes the notion of linear (in)dependence of vectors in a vector space; more specifically, a matroid is an object $M$ defined by a finite set $I$ together with a collection of subsets satisfying some “dependence axioms” (see [Oxl11]). Examples include certain finite collections of vectors in a vector space, matroids arising from hyperplane arrangements, and matroids arising from graphs.

Every matroid $M$ is equipped with a graded poset called its lattice of flats $L(M)$. For example, the linear subspace $V := (x_1 + x_2 + x_3 = 0) \subseteq \mathbb{C}^I = \{1,2,3\}$ defines a hyperplane arrangement $\mathcal{A}$ (hence a matroid) after intersecting it with the coordinate hyperplanes. All possible intersections of the hyperplanes are the flats, and the partial order is given by reverse inclusion, as illustrated below.

For each flat $F \in L(M)$, we can define two new matroids:

- $M^F$ is a certain matroid on $F$ called the localization of $M$ at $F$, and
- $M_F$ is a certain matroid on $I \setminus F$ called the contraction of $M$ at $F$.

The figure below shows an example (here $M$ is the uniform matroid of rank 3 on $I = \{1,2,3,4\}$) of these two constructions in terms of the lattice of flats $L(M)$.

The localization and contraction of matroids will play a key role in the two problems described below.

1.1 Two problems in matroid theory

1.1.1 The “Top-Heavy Conjecture” for matroids

As mentioned in Section 1, every matroid $M$ is equipped with a graded poset (graded by the rank of the flats) called its lattice of flats $L(M)$. Denote by $L(M)^k$ the collection of flats of rank $k$ in $L(M)$. These are just the flats $k$ levels up from the bottom of $L(M)$.
Conjecture 1.1.1 ("Top-Heavy Conjecture", Dowling–Wilson 1974 [DW74, DW75]). Let $M$ be a rank $d$ matroid. For any $k \leq d/2$, we have

$$\#L(M)^k \leq \#L(M)^{d-k}.$$ 

Example 1.1.2. When the matroid arises from a hyperplane arrangement, the rank of a flat is its codimension as a subspace. For the first matroid in Section 1.1, the poset is vertically-symmetric, so the Top-Heavy Conjecture is trivially satisfied, and for the second matroid, the Top-Heavy Conjecture holds because $4 \leq 6$; in other words, by intersecting our collection of four planes in $\mathbb{C}^3$, we obtained six lines.

Despite being simple to state and understand, the Top-Heavy Conjecture remained unsolved for many years. Forty-three years later, in 2017, June Huh and Botong Wang verified the conjecture in the case of realizable matroids (i.e. matroids which arise from a hyperplane arrangement).

Theorem 1.1.3 (Huh–Wang 2017 [HW17]). Let $M$ be a realizable matroid of rank $d$. For any $k \leq d/2$, we have

$$\#L(M)^k \leq \#L(M)^{d-k}.$$ 

The proof of this theorem makes heavy use of the topology of a certain singular projective variety $Y$ (see Section 1.1.3 for the definition of $Y$) defined from a hyperplane arrangement $\mathcal{A}$. When the matroid is not realizable, all geometric techniques are missing; however, one can emulate the topology in a purely combinatorial setting and investigate the following problem. We will detail our progress toward it in Section 1.1.4.


In the next section, we explain another result which makes sense for arbitrary matroids but has only been proven in the realizable case.

1.1.2 Kazhdan–Lusztig polynomials of matroids

Just as Kazhdan–Lusztig (KL) theory for Coxeter groups guided the development of geometric representation theory over thirty-five years, I expect that the geometric study of KL polynomials of matroids will strengthen matroid theory’s connection with other areas and be indicative of the direction in which matroid theory will develop in the future.

In [EPW16], Elias, Proudfoot, and Wakefield associated to each matroid $M$ a polynomial $P_M(t)$ called the KL polynomial of $M$. These polynomials share many analogies with the classical KL polynomials $\text{KL79}$ for Coxeter groups, but exhibit some interesting differences. Both types of polynomials have a purely combinatorial definition—while KL polynomials for Coxeter groups are defined in terms of more elementary polynomials called $R$-polynomials, every matroid has a characteristic polynomial $\chi_M(t)$ which plays this role. In the classical setting, Polo showed that every polynomial with non-negative coefficients and constant term 1 occurs as a KL polynomial $\text{KL79}$ for some symmetric group $\mathfrak{S}_n$. In stark contrast, it is conjectured that the $P_M(t)$ are real-rooted $\text{GPY16}$, which implies that their coefficients form a log-concave sequence.

When the Coxeter group is in fact a finite Weyl group, there is a geometric interpretation of the classical KL polynomials. They are the $\text{dimensions}$ of intersection cohomology stalks of Schubert varieties, which implies non-negativity of their coefficients. In a similar way, when $M$ is realizable (i.e. arises from a hyperplane arrangement), Elias, Proudfoot, and Wakefield identified $P_M(t)$ with the $\text{dimensions}$ of intersection cohomology stalks of a certain singular projective variety $Y$. (We call $Y$ the Schubert variety of a hyperplane arrangement—see Section 1.1.3 for the definition of $Y$.)

Theorem 1.1.5 ([EPW16]). If $M$ is a realizable matroid, then $P_M(t) = \sum_{i \geq 0} t^i \dim \text{IH}_{(\infty, \ldots, \infty)}^i(Y)$.

In this way, the $P_M(t)$ have non-negative coefficients when $M$ is realizable. The question of non-negativity of the KL polynomials for arbitrary Coxeter groups was conjectured in [KL79], but remained unsolved for thirty-five years. In 2014, Elias and Williamson settled this question in the affirmative by using sophisticated

---

2 Note that in an open neighborhood of the most singular point $(\infty, \ldots, \infty)$, the projective variety $Y$ is isomorphic to a well-studied affine conical variety called the reciprocal plane.
diagrammatic combinatorics \cite{EW17, EW16} to give an algebraic proof of the decomposition theorem \cite{BBD82} via the Hodge-theoretic properties of Soergel bimodules \cite{EW14}. Braden, Huh, Proudfoot, Wang, and I are working toward proving the analogous conjecture for matroids. We will detail our progress toward it in Section 1.1.4.

**Problem 1.1.6** (Braden–Huh–M.–Proudfoot–Wang). For an arbitrary matroid \( M \), the coefficients of the KL polynomial \( P_M(t) \) are nonnegative.

Despite the simplicity of this statement, just as in the classical setting, investigating this conjecture leads to deep connections relating topology, combinatorics, and representation theory. A recent example of work in the same vein is that of Adiprasito, Huh, and Katz \cite{AHK18}. They proved, using deep topological arguments, that the coefficients of the characteristic polynomial \( \chi_M(t) \) of an arbitrary matroid form a log-concave sequence; thereby settling a long-standing conjecture of Rota, Heron, and Welsh \cite{Rot71, Her72, Wel76}.

We include here, for convenience, a table summarizing the above discussion, and more.

<table>
<thead>
<tr>
<th>KL theory for Coxeter groups</th>
<th>KL theory for matroids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coxeter group ( W )</td>
<td>matroid ( M )</td>
</tr>
<tr>
<td>Weyl group ( W )</td>
<td>realizable matroid ( M )</td>
</tr>
<tr>
<td>Bruhat poset</td>
<td>lattice of flats ( L(M) )</td>
</tr>
<tr>
<td>( R )-polynomial</td>
<td>characteristic polynomial ( \chi_M(t) )</td>
</tr>
<tr>
<td>Hecke algebra</td>
<td>?</td>
</tr>
<tr>
<td>Polo</td>
<td>real-rooted</td>
</tr>
<tr>
<td>Schubert variety</td>
<td>Schubert variety ( Y ) of a hyperplane arrangement</td>
</tr>
<tr>
<td>Elias–Williamson \cite{EW14}</td>
<td>future work of Braden–Huh–M.–Proudfoot–Wang</td>
</tr>
</tbody>
</table>

### 1.1.3 The realizable case

Because the topology of the realizable case informs our strategy for solving Problems 1.1.4 and 1.1.6 in the general case, I will briefly explain the proof of the “Top-Heavy Conjecture” and the non-negativity of the KL polynomials for realizable matroids (Theorems 1.1.3 and 1.1.5 above).

Consider the maps

\[ V \hookrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \prod_{H \in \mathcal{A}} \mathbb{A}^1 \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1, \]

and define \( Y \) to be the closure of \( V \) inside \( \prod_{H \in \mathcal{A}} \mathbb{P}^1 \). We call \( Y \) the *Schubert variety of the hyperplane arrangement* \( \mathcal{A} \) because it plays an analogous role in the KL theory of matroids that a Schubert variety in the flag variety plays in the KL theory of Coxeter groups. One of the most important analogies is that \( Y \) admits a stratification by affine spaces \cite[Lemmas 7.5 and 7.6]{PXY18}:

\[ Y = \bigsqcup_{F \in L(M)} Y_F, \]

where \( \dim Y_F = \text{rk } F \). Thus we obtain the following result.

**Proposition 1.1.7.** The odd-degree cohomology of \( Y \) vanishes, and \( \dim H^{2k}(Y) = \#L(M)^k \).

Thus, the proof of the “Top-Heavy Conjecture” in the realizable case reduces to producing an injective map \( H^{2k}(Y) \rightarrow H^{2d-2k}(Y) \). If \( Y \) were smooth, the Hard Lefschetz Theorem would provide such an injection. But the non-smoothness of \( Y \) requires moving to intersection cohomology.

**Proof of Theorem 1.1.3** \cite{HW17}. There is a natural map \( H^*(Y) \rightarrow H^*(Y) \) making \( HH^*(Y) \) a module over \( H^*(Y) \), and since \( Y \) has a stratification by affine spaces the map is an injection \cite{BE09}. Let \( \mathcal{L} \in H^2(Y) \) be the class of an ample line bundle. Since \( Y \) is a singular projective variety, the Hard Lefschetz Theorem asserts that we have an isomorphism \( \mathcal{L}^{d-2k} : HH^{2k}(Y) \rightarrow HH^{2d-2k}(Y) \). We have the following commutative diagram

\[
\begin{array}{ccc}
H^{2d-2k}(Y) & \longrightarrow & HH^{2d-2k}(Y) \\
\mathcal{L}^{d-2k}|_{H^{2k}(Y)} & \longrightarrow & \mathcal{L}^{d-2k} \\
H^{2k}(Y) & \longrightarrow & HH^{2k}(Y)
\end{array}
\]
The right vertical arrow is an isomorphism and the two horizontal arrows are injections, so the left vertical arrow is required to be an injection. The proof follows from Proposition 1.1.7.

1.1.4 The general case

The proofs of Theorems 1.1.3 and 1.1.5 inform our approach for arbitrary matroids (even though no geometry exists here).

Let $M$ be a rank $d$ matroid. There is a graded ring $B^\bullet(M)$ called the graded Möbius algebra with basis $\{y_F\}_{F \in L(M)}$ having deg $y_F = \text{rk } F$ and multiplication given by

$$y_F y_G = \begin{cases} y_{F \lor G} & \text{if } \text{rk } F \lor G = \text{rk } F + \text{rk } G, \\ 0 & \text{else.} \end{cases}$$

Theorem 1.1.8 (Huh–Wang 2017 [HW17]). When $M$ is a realizable matroid, then there is an isomorphism of graded rings $B^\bullet(M) \cong H^2\bullet(\bar{Y})$ between the graded Möbius algebra and the cohomology of the Schubert variety of the corresponding hyperplane arrangement.

It remains to produce a certain graded $B^\bullet(M)$-module $I^\bullet(M)$ which plays the role of the intersection cohomology $\text{IH}^\bullet(Y)$, so that one can imitate the proofs of Theorems 1.1.3 and 1.1.5 when no geometry exists.

Semi-wonderful geometry

One approach for producing $I^\bullet(M)$ is to try to write down generators and relations (purely in terms of the lattice of flats $L(M)$) for the $H^\bullet(\bar{Y})$-module $\text{IH}^\bullet(Y)$; unfortunately, intersection cohomology is generally quite complicated, and this does not seem practical.

Instead, we again find inspiration from geometry—there exists a resolution of singularities $\bar{Y} \to Y$ such that the intersection cohomology $\text{IH}^\bullet(Y)$ is a submodule of $H^\bullet(\bar{Y})$, and this latter ring does indeed have a nice combinatorial presentation. Here $\bar{Y}$ is a smooth projective variety (we call the semi-wonderful resolution defined by a canonical iterative sequence of blowups of strata of $Y$.

We describe the combinatorial presentation of $H^\bullet(\bar{Y})$ below.

Let $A^\bullet(M)$ be the graded ring with generators $\{x_F \mid F \in L(M) \text{ is a proper flat}\}$ and $\{y_F \mid F \in L(M) \text{ has } \text{rk } F = 1\}$, modulo the relations

- $x_F x_G = 0$ if $F$ and $G$ are incomparable,
- $y_H = \sum_{H \not\leq F} x_F$, and
- $y_H x_F = 0$ if $H \not\leq F$.

We call $A^\bullet(M)$ the semi-wonderful Chow ring of $M$. Note that we have a natural inclusion $B^\bullet(M) \hookrightarrow A^\bullet(M)$ given by $y_F \mapsto y_F$, so $A^\bullet(M)$ is an algebra over $B^\bullet(M)$.

Theorem 1.1.9 (Braden–Huh–M.–Proudfoot–Wang). When $M$ is a realizable matroid, then there is an isomorphism of graded rings $A^\bullet(M) \cong H^2\bullet(\bar{Y})$.

Remark 1.1.10. In [FY04] the Chow ring of a matroid is introduced, and it is deeply analyzed in [AHK18]. This ring has a similar presentation to $A^\bullet(M)$, and geometrically it is the cohomology $H^\bullet(\bar{Y})$ of de Concini and Procesi’s full wonderful model $\bar{Y}$ (which is the fiber of our semi-wonderful resolution $\bar{Y} \to Y$ over the most singular point of $Y$). For our purposes, working with $A^\bullet(M)$ is better because $\bar{Y} \to Y$ is a stratified map.

\[3\] The reader familiar with the Coxeter group setting should interpret this as an analog of a Bott–Samelson (BS) resolution of a Schubert variety; however, our resolution is canonical—it does not depend on any choices, whereas a BS resolution depends on a choice of reduced word.
**Combinatorial intersection cohomology of matroids** We are now ready to give a construction of the intersection cohomology $I^\bullet(M)$ of a matroid $M$. There is a canonical isomorphism $A^d(M) \cong \mathbb{C}$, and a perfect pairing, we call the Poincaré pairing,

$$A^k(M) \otimes A^{d-k}(M) \to A^d(M) \cong \mathbb{C}$$

for any $0 \leq k \leq d$ given by multiplication in $A^\bullet(M)$.

**Definition 1.1.11.** The intersection cohomology $I^\bullet(M)$ of a matroid $M$ is the unique indecomposable summand of $A^\bullet(M)$ as a $B^\bullet(M)$-module which contains $A^d(M) \cong \mathbb{C}$.

**Conjecture 1.1.12** (Braden–Huh–M.–Proudfoot–Wang). The intersection cohomology $I^\bullet(M)$ of a matroid $M$ satisfies Poincaré duality with respect to the Poincaré pairing on $A^\bullet(M)$.

Conjecture 1.1.12 gives a (Björner–Ekedahl-type) inclusion $B^\bullet(M) \hookrightarrow I^\bullet(M)$; indeed, $A^d(M) \subset I^\bullet(M)$ and by Poincaré duality of $I^\bullet(M)$, we know $A^0(M) \subset I^\bullet(M)$. Thus $1_{A^\bullet(M)} \in I^\bullet(M)$, and since $I^\bullet(M)$ is a $B^\bullet(M)$-module, the claim follows. What remains to emulate the proof of Theorem 1.1.3 is Hard Lefschetz for $I^\bullet(M)$.

**Conjecture 1.1.13** (Braden–Huh–M.–Proudfoot–Wang). The intersection cohomology $I^\bullet(M)$ of a matroid $M$ satisfies the Hard Lefschetz Theorem; that is, there exists a class $L \in B^1(M)$ such that for every $k \leq d/2$, the map

$$L^{d-2k} : I^k(M) \to I^{d-k}(M)$$

is an isomorphism.

Now, assuming the validity of Conjectures 1.1.12 and 1.1.13, we obtain a proof of the “Top-Heavy Conjecture” for arbitrary matroids using the same argument as in the realizable case; thereby settling Problem 1.1.4.

Lastly, we expect to have a similar interpretation of the coefficients of the KL polynomials of matroids as graded dimensions of $I^\bullet(M)$.

**Conjecture 1.1.14** (Braden–Huh–M.–Proudfoot–Wang). For an arbitrary matroid $M$, we have $P_M(t) = \sum_{i \geq 0} t^i \dim(I^i(M)/B^i(M) \cdot I^{i-1}(M))$.

**Remark 1.1.15.** The proofs of Conjectures 1.1.12, 1.1.13 and 1.1.14 will require a complicated induction in the style of [EW14] and [AHK18]. The induction hypothesis will include statements about Poincaré duality, Hard Lefschetz, and the Hodge–Riemann bilinear relations for $I^\bullet(M)$ as well as for various intermediate objects we define along the way.

### 1.2 Antipodes of matroid Hopf algebras

Many combinatorial structures can be used as building blocks to construct graded connected Hopf algebras. Examples come from graphs, simplices, polynomials, Young tableaux, and many more. In this project, we focus on finding a cancellation-free antipode formula for the matroid-minor Hopf algebra (first introduced by Schmitt [Sch94]).

When a Hopf algebra arises from combinatorial objects, it is often referred to as a *combinatorial Hopf algebra* (see [GR16] for an excellent treatment of Hopf algebras, especially as they appear in combinatorics). Motivations for studying these algebras appear throughout many diverse areas of mathematics, including: combinatorics, representation theory, mathematical physics, and K-theory. Given a combinatorial Hopf algebra, the computation of its antipode gives combinatorial identities for the objects which built up the algebra. For this reason, finding the simplest expression for the antipode can prove extremely valuable.

In general the antipode is given in terms of satisfying certain commutative diagrams, but in the setting where the Hopf algebra is both graded and connected Takeuchi gave a formula which describes the map explicitly.
Theorem 1.2.1 ([Tak71]). A graded, connected $k$-bialgebra $H$ is a Hopf algebra, and it has a unique antipode $S$ whose formula is given by
\[ S = \sum_{i \in \mathbb{Z}_{\geq 0}} (-1)^{i} \mu^{-1} \circ \pi \circ \Delta^{-1} \]  
where $\mu^{-1} = \eta$, $\Delta^{-1} = \epsilon$, and $\pi : H \to H$ is the projection map defined by extending linearly the map $\pi|_{H_\ell} = \begin{cases} 0 & \text{if } \ell = 0, \\ 1 & \text{if } \ell \geq 1, \end{cases}$ where $H_\ell$ is the $\ell$th graded piece of $H$.

The only drawback to the formula above is that it can contain a large amount of cancellation. The goal in general is then to find refinements of this formula so that we can more easily compute the antipode of $H$. Ideally this leads to a cancellation-free formula. Much work has been done in this area for specific Hopf algebras. Cancellation-free formulas have been found for antipodes of combinatorial Hopf algebras associated to graphs, symmetric functions, shuffles, polynomials, simplicial complexes, and many more [HM12, LP07, Pat16, BS16, BHM16].

Our main result is a cancellation-free formula for computing the antipode of uniform matroids in the matroid-minor Hopf algebra. This is a certain combinatorial Hopf algebra whose elements are matroids, and whose comultiplication map is defined in terms of localization and contraction (as in Section 1).

Theorem 1.2.2 (Bucher–M. 2016 [BM16]). The image of the uniform matroid $U_m^n$ under the antipode map $S$ is given by the following cancellation-free formula. Choose a total ordering $<$ on the ground set of the matroid. Then
\[ S(U_m^n) = \sum_{I,L} (-1)^{n-|L|+1} U_I^{|I|} \oplus U_L^{m-|I|}, \]
where $I,L$ ranges over all pairs of subsets of the ground set such that
- $I$ and $L$ are disjoint, and
- $|I| < m \leq |I| + |L|$, and
- if $|I| + |L| = m$ then $|L| = 1$ and the element in $L$ is the maximal element of $I \cup L$ with respect to $<.$

Our proof of this theorem involves reinterpreting Theorem 1.2.1 for the matroid-minor Hopf algebra; in this case, this reduces to a sum over certain ordered set partitions. To systematically remove cancellations from this sum, we introduce a sign-reversing involution on the collection of ordered set partitions. The idea of introducing a sign-reversing involution is ubiquitous throughout combinatorics, and it was used by Benedetti and Sagan to provide antipode formulas for various other combinatorial Hopf algebras [BS16].

Since our paper appeared on the arXiv, Aguiar and Ardila developed a completely general way to obtain a cancellation-free formula for all matroids in terms of the associated matroid polytope $\AA_{A17}$ via an embedding into the Hopf algebra of generalized permutohedra, thereby completing the problem of cancellation-free antipode formulas for the matroid-minor Hopf algebra. Their methods also provide a unifying framework that applies to graphs, posets, set partitions, linear graphs, hypergraphs, simplicial complexes, building sets, and simple graphs.

However, recently together with Bucher, Eppolito, and Jun, I realized that our methods in [BM16] can be generalized to produce cancellation-free antipode formulas for all matroids.

Problem 1.2.3 (Bucher–Eppolito–Jun–M.). Obtain a cancellation-free antipode formula for all matroids in the matroid-minor Hopf algebra by using sign-reversing involutions.

Some advantages of our technique are that they add more evidence to the power of sign-reversing involutions in the study of combinatorial Hopf algebras (as initiated in [BS16]), and they conjecturally give a cancellation-free formula for matroids over hyperfields (defined in [BB16]) in the hyperfield matroid-minor Hopf algebra (defined in [EJS17]).

2 Cluster Algebras

2.1 Computing upper cluster algebras

Cluster algebras are commutative unital domains generated by distinguished elements called cluster variables which are defined by a combinatorial process called mutation. Many notable spaces are equipped with cluster structures where certain regular functions play the role of cluster variables. For example, the coordinate ring of the space of $m \times n$ matrices is naturally a cluster algebra, and each matrix minor is a cluster variable. In this way, identities among matrix minors are a special case of the theory.

Cluster algebras are generally defined in terms of an infinite generating set; however, the most important cluster algebras may be finitely-generated. For this reason, computing generating sets of cluster algebras and the relations between those generators is an interesting problem. The cluster algebra $A$ is the combinatorially defined object, but from a geometric perspective, there is a more natural algebra to consider: the upper cluster algebra $U$, which was introduced in [BFZ05]. It is defined as an infinite intersection of Laurent polynomial rings, which makes $U$ difficult to work with in general, as it is hard to write down any (even infinite) generating set. Despite this hurdle, all known examples of upper cluster algebras enjoy many nice properties, such as normality and being log Calabi–Yau. Studying $U$ geometrically often gives information about the more intrinsic algebra $A$, since $A \subseteq U$ by the Laurent phenomenon [FZ02].

One obstacle in the theory of cluster algebras has been an almost complete lack of examples in situations where $A \neq U$. In a recent paper [MM15], G. Muller (University of Michigan) and I exploited the algebraic geometry of $U$ to obtain an algorithm for producing a presentation of $U$ (when $A$ is “totally coprime”—a mild technical condition) in terms of generators and relations.

Theorem 2.1.1 (M.-Muller 2015 [MM15]). If $A$ is a totally coprime cluster algebra, then a finite-time algorithm for presenting $U$ exists whenever $U$ is finitely generated.

We used this technique to give presentations of several interesting upper cluster algebras where $A \neq U$. Also, together with Mills, Muller, and Williams, I implemented the algorithm in the computer algebra system Sage (see https://trac.sagemath.org/ticket/18800).

![Fig. 1: Important cluster algebras for which our algorithm gave a presentation of $U$.](https://example.com/fig1)

Theorem 2.1.1 has been used by several other mathematicians in their research—I. Canakci, K. Lee, and R. Schiffler used our work to prove that $A = U$ for the dreaded torus (the last quiver in Figure 1) [CLS14], and more recently, M. Gross, P. Hacking, S. Keel, and M. Kontsevich used our computations to make conjectures about the cluster variety for the once-punctured torus (the first quiver in Figure 1 with $a = 2$) [GHKK14].

3 Representation Theory

3.1 Exceptional sequences and linear extensions

An exceptional sequence $(V_1, \ldots, V_n)$ of quiver representations is a sequence of representations obeying certain strong homological constraints. Exceptional sequences were introduced in [GR87] to study exceptional vector bundles on $\mathbb{P}^2$. Since then, they have been shown to have many useful applications to several areas of mathematics. For example, they have applications in combinatorics because (complete) exceptional sequences are in bijection with maximal chains in the lattice of noncrossing partitions [IT09, HK13]. Furthermore, they
have been shown to be intimately connected to acyclic cluster algebras as their dimension vectors appear as rows of e-matrices [ST13].

Even though they are pervasive throughout mathematics, very little work has been done to give a concrete description of exceptional sequences. Previously, A. Garver (University of Minnesota) and I extended work of T. Araya [Ara13] by classifying exceptional sequences of representations of the linearly-ordered quiver of type A using noncrossing edge-labeled trees in a disk with boundary vertices [GM15].

To extend this classification to type A quivers with any orientation, Igusa, Garver, Ostroff, and I used a more general combinatorial model called a strand diagram. We defined a bijective map Φ, which takes an indecomposable representation to its corresponding strand. It turns out that all of the homological information in the definition of an exceptional sequence is stored in a noncrossing diagram of strands.

Theorem 3.1.1 (Garver–Igusa–M.–Ostroff 2015 [GIMO15]). Let Q be a type A quiver and let U and V be two distinct indecomposable representations of Q.

a) The strands Φ(U) and Φ(V) intersect nontrivially if and only if neither (U,V) nor (V,U) are exceptional pairs.

b) The strand Φ(U) is clockwise from Φ(V) if and only if (U,V) is an exceptional pair and (V,U) is not an exceptional pair.

c) The strands Φ(U) and Φ(V) do not intersect at any of their endpoints and they do not intersect nontrivially if and only if (U,V) and (V,U) are both exceptional pairs.

Theorem 3.1.1 has many immediate consequences. B. Keller proved that any two green-to-red sequences (sequences that are like MGS’s except that mutation at red vertices is allowed) applied to Q with principal coefficients produce isomorphic ice quivers [Kel13]. His proof hinges on deep representation-theoretic and geometric methods, but the statement itself is purely combinatorial. During the FPSAC’13 conference, B. Keller asked for a combinatorial proof—our theorem solves this problem. Another application of Theorem 3.1.1 is that it allowed us to classify all e-matrices for type A cluster algebras in a completely combinatorial way using strand diagrams.

A complete exceptional collection is an unordered list of quiver representations which can be ordered to make a complete exceptional sequence. We have shown that the number of complete exceptional sequences arising from a given complete exceptional collection is equal to the number of linear extensions of a certain poset arising from the corresponding strand diagram. Counting the number of linear extensions of posets is a notoriously difficult problem in combinatorics; however, the class of posets arising from strand diagrams are very nice.

Theorem 3.1.2 (Garver–Igusa–M.–Ostroff 2015 [GIMO15]). The class of posets arising from strand diagrams are exactly the posets P that satisfy the following properties.

- each x ∈ P has at most two covers and covers at most two elements,
- the underlying graph of the Hasse diagram of P has no cycles,
- the Hasse diagram of P is connected.

In particular, zig-zag posets are contained in this class, and their linear extensions are counted by a determinant formula. Together with Alejandro Morales, I have been working towards a formula for the number of linear extensions of all posets in this class (see Section 4.1 for progress in counting linear extensions of a related class of posets).

Fig. 2: The indecomposable representations of the type A_2 quiver 1 ← 2 and their corresponding strands.
Problem 3.1.3 (M.–Morales). Count the number of complete exceptional sequences arising from a given complete exceptional collection by developing a method for counting the number of linear extensions of posets arising from strand diagrams.

3.2 Computing the Fourier–Sato transform combinatorially

In this section, I will describe a problem involving perverse sheaves on certain quiver varieties and discuss possible applications to representation theory of quantum loop algebras $\mathcal{U}_q(Lg)$ and cluster algebras.

The Fourier–Sato transform is a geometric version of the Fourier transform for functions from analysis. If $V$ is a complex vector space, then the Fourier–Sato transform is a certain functor $\mathbb{T}$ which gives an equivalence of categories between derived categories of conical sheaves

$$\mathbb{T} : \mathcal{D}^b_{\text{con}}(V) \to \mathcal{D}^b_{\text{con}}(V^*),$$

where $V^*$ is the dual vector space. The functor $\mathbb{T}$ has been used in various forms to much success for more than thirty years—it was used by Laumon in the mid-1980s to significantly shorten Deligne’s proof of the Weil conjectures (see [KW01] for a nice treatment), it gives an alternative construction of the Springer correspondence [HK84, Bry86], and it plays a central role in the theory of character sheaves [Lus87, Mir04]. Despite its usefulness, in practice $\mathbb{T}$ is difficult to compute explicitly. In fact, it has only been explicitly computed in few settings—some examples include the nilpotent cone $\mathcal{N}$ of a semisimple Lie algebra $g$ [AM15] and various stratified spaces of $n \times n$ matrices [BG99].

P. Achar, M. Kulkarni, and I have developed a method for computing $\mathbb{T}$ combinatorially. Let $Q$ be the type $A_n$ quiver $\bullet \to \cdot \to \cdots \to \bullet$. Fix a dimension vector $w$, and consider the space of representations of $Q$ with this dimension vector. This is an affine space $E_w$ which carries an action of a group $G_w$, a product of general linear groups. This action gives a stratification of $E_w$ into $G_w$-orbits, and we have the Fourier–Sato transform

$$\mathbb{T} : \mathcal{D}^b_{G_w}(E_w) \to \mathcal{D}^b_{G_w^*}(E_{w^*})$$

where $w^*$ is the reverse of $w$. In this setting, $\mathbb{T}$ is $t$-exact for the perverse $t$-structure, and it sends (simple) perverse sheaves to (simple) perverse sheaves.

To each $G_w$-orbit, we associate a triangular array of nonnegative integers satisfying certain conditions arising from dominant weights for $GL_n$. These triangular arrays completely determine the $G_w$-orbits.

**Theorem 3.2.1** (Achar–Kulkarni–M. 2016). There is a bijection between $G_w$-orbits in $E_w$ and a certain set $P_w$ of triangular arrays of nonnegative integers.

The combinatorial set $P_w$ carries a wealth of information about the geometry and representation theory of $E_w$—the dimension of $G_w$-orbits and the partial order on them, as well as whether a representation is injective or projective, can be read off the diagram. Most notably, it allows for a combinatorial computation of $\mathbb{T}$.

**Theorem 3.2.2** (Achar–Kulkarni–M. 2018 [AKM18]). There exists a combinatorial algorithm $\mathbb{T} : P_w \to P_{w^*}$ that computes $\mathbb{T}$ for every simple perverse sheaf; i.e., such that $\mathbb{T}(\text{IC}(O_\lambda)) = \text{IC}(O_{\mathbb{T}(\lambda)})$, where $\text{IC}(O_\lambda)$ is the simple perverse sheaf supported on the closure $\overline{O_{\lambda}}$ of the $G_w$-orbit $O_\lambda$ given by $\lambda \in P_w$.

**Corollary 3.2.3** (Achar–Kulkarni–M. 2018 [AKM18]). The map $\mathbb{T}$ and its inverse $\mathbb{T}'$ both give new algorithms for computing the Knight–Zelevinsky multisegment duality (defined in [KZ90]).

We give a few examples of $\mathbb{T}$ below for the $A_3$ quiver with dimension vector $w = (3, 3, 3)$. Note that the first example is the combinatorial version of the geometric fact that $\mathbb{T}$ takes the unique zero-dimensional orbit to the unique dense orbit.

$$\mathbb{T}
\begin{pmatrix}
3 & 3 & 3 \\
3 & 3 & 3
\end{pmatrix}
= 
\begin{pmatrix}
0 & 2 & 3 \\
2 & 3 & 3
\end{pmatrix}
\quad
\mathbb{T}
\begin{pmatrix}
3 & 3 & 3 \\
3 & 0 & 3
\end{pmatrix}
= 
\begin{pmatrix}
0 & 3 & 3 \\
3 & 3 & 3
\end{pmatrix}
\quad
\mathbb{T}
\begin{pmatrix}
3 & 3 & 3 \\
0 & 3 & 3
\end{pmatrix}
= 
\begin{pmatrix}
0 & 3 & 3 \\
3 & 3 & 3
\end{pmatrix}.$$

Recently, Nakajima used the Fourier–Sato transform on his graded quiver varieties to prove a character formula for representations of quantum loop algebras $\mathcal{U}_q(Lg)$, as well as to give a monoidal categorification of certain cluster algebras [Nak11]. In his proof, he computes the Fourier–Sato transform of only a single object. Theorem 3.2.2 allows for the computation of $\mathbb{T}$ for every object, at least for the quiver $Q$. 

October 2018 Research Statement – Jacob P. Matherne 10
Problem 3.2.4. Get information about representations of \( \mathcal{U}_q(L\mathfrak{g}) \) by using \( T \), our combinatorial description of \( T \). What does our explicit computation of \( T \) say about the cluster algebras involved?

This problem is particularly interesting to me as cluster algebras have been another important research interest of mine—I have results about their structure theory (see Section 2.1) and their relationship with quiver representations (see Section 5.1). Solving this problem would blend geometric representation theory, quivers, and cluster algebras (several of my research interests).

3.3 Derived geometric Satake equivalence, Springer correspondence, and small representations

Let \( G \) be a semisimple complex algebraic group and \( T \) be a maximal torus. The Weyl group \( W \) acts on the zero weight space \( V^T \) of any representation \( V \) of \( G \), giving a functor \( \Phi_G : \text{Rep}(G) \to \text{Rep}(W) \).

Achar, Henderson, and Riche recently constructed a geometric lift of this functor \cite{AH13, AHR15}. Its construction begins with a result of Lusztig \cite{Lus81} that for \( \text{GL}_n(\mathbb{C}) \) the nilpotent cone \( \mathcal{N} \) (the variety of nilpotent \( n \times n \) matrices) can be embedded in the affine Grassmannian \( \text{Gr}_{\mathbb{C}} \), an infinite-dimensional analog of the Grassmannian of \( k \) planes in \( n \) space. To generalize this result to other groups, one needs to restrict to a certain finite-dimensional closed subvariety \( \text{Gr}^{sm} \) of \( \text{Gr} \). There is a certain open subvariety \( \mathcal{M} \subset \text{Gr}^{sm} \) and a finite map \( \pi : \mathcal{M} \to \mathcal{N} \), which can be viewed as a generalization of Lusztig’s embedding for other groups. The map \( \pi \) gives rise to a functor \( \Psi_G : \text{Perv}(\text{Gr}_{\mathbb{C}}) \to \text{Perv}(\mathcal{N}) \).

To explain in what sense \( \Psi_G \) is a geometric lift of \( \Phi_G \), we invoke two major theorems in geometric representation theory: the geometric Satake equivalence and the Springer correspondence.

(a) There is an equivalence of categories \( S : \text{Perv}(\text{Gr}_{\mathbb{C}}) \to \text{Rep}(G) \), where \( \text{Perv}(\text{Gr}_{\mathbb{C}}) \) is the category of \( \text{Gr}_{\mathbb{C}} \)-equivariant perverse sheaves on the affine Grassmannian \( \text{Gr}_{\mathbb{C}} := \tilde{G}(\mathbb{C})/\tilde{G}(\mathcal{D}) \), \( \tilde{G} \) is the Langlands dual group, \( \mathcal{D} = \mathbb{C}(t) \), and \( \mathcal{O} = \mathbb{C}[t] \) \cite{Lus83, Gin95, MV07}.

(b) There is a functor \( S : \text{Perv}(\mathcal{N}) \to \text{Rep}(W) \), where \( \text{Perv}(\mathcal{N}) \) is the category of \( \mathcal{G} \)-equivariant perverse sheaves on the nilpotent cone \( \mathcal{N} \subset \mathfrak{g} \) \cite{Spr76, Lus81, BM81}.

Theorem 3.3.1 \cite{AH13, AHR15}. The functor \( \Psi_G \) is a geometric lift of \( \Phi_G \); i.e., there exists a commutative diagram:

\[
\begin{array}{ccc}
\text{Perv}(\text{Gr}_{\mathbb{C}}) & \xrightarrow{\sim} & \text{Rep}(G) \\
\downarrow{\Psi_G} & & \downarrow{\Phi_G} \\
\text{Perv}(\mathcal{N}) & \xrightarrow{\sim} & \text{Rep}(W)
\end{array}
\]

Parallel to this development, derived versions of (a) and (b) were developed—Bezrukavnikov and Finkelberg gave an equivalence \( \text{der}S : D^b_{\text{Gr}_{\mathbb{C}}} \to D^b_{\text{Gr}} \) \cite{BF08}, and Rider established the equivalence \( \text{der}S : D^b_{\text{Gr},\text{Spr}}(\mathcal{N}) \to D^b_{\text{Gr}}(\mathfrak{g}^*) \) \cite{Rid13}. Here, \( \mathfrak{g}^* \) and \( \mathfrak{h}^* \) are both affine varieties, so the categories on the right-hand side are graded modules over their coordinate rings. The word “mix” refers to a geometric analog of grading. My dissertation work involved producing a derived analog of \( \Phi_G \) and showing that its derived lift is \( \Psi_G \).

Theorem 3.3.2 \cite{Mat16}. There exists a commutative diagram of derived categories:

\[
\begin{array}{ccc}
D^b_{\text{Gr}_{\mathbb{C}}} & \xrightarrow{\text{der}S} & D^b_{\text{Gr}}(\mathfrak{g}^*) \\
\downarrow{\Psi_G} & & \downarrow{\text{der}\Phi_G} \\
D^b_{\text{Gr},\text{Spr}}(\mathcal{N}) & \xrightarrow{\sim} & D^b_{\text{Gr}}(\mathfrak{h}^*)
\end{array}
\]

The solution to this problem involves two major steps.

1. Establish the result for all groups \( G \) of semisimple rank 1.
2. Show that each of the functors involved commutes with restriction to such a group.

Over fields \( \mathbb{F} \) of positive characteristic, representation theory is much more difficult—for example, the categories involved are not semisimple in general. In his thesis, Mautner proved that for \( G = \text{GL}_n(\mathbb{F}) \) the category \( \text{Perv}_G(\mathcal{N}) \) is equivalent to the category of finitely-generated modules for a certain Schur algebra \([\text{Man}10]\). Very recently, tremendous progress has been made toward producing analogs of Springer theory in positive characteristic \([\text{AHJR}16, \text{AHJR}14, \text{AHJR}15]\). It is motivating to consider a positive characteristic version of Theorem 3.3.2 for its possible applications to positive characteristic Springer theory and representations of Schur algebras.

**Problem 3.3.3.** Develop an analog of Theorem 3.3.2 for sheaves in positive characteristic.

### 4 Research with undergraduates

I would like to share two research projects that involve undergraduate students. The project in Section 4.1 began as a joint-independent-study course that Alejandro Morales and I offered to undergraduate math major Stefan Grosser at UMass. The project in Section 4.2 grew out of a year of weekly meetings in Smith College’s postbaccalaureate (and undergraduate) research group on algebraic splines.

#### 4.1 Counting linear extensions of posets

The question of counting the number of linear extensions of a partially ordered set (poset) has applications to fields as diverse as computer science and social science. In computer science, linear extensions appear in sorting algorithms: if one could efficiently compute the number of linear extensions of a poset, then one would know at every step of a sorting algorithm which two elements to compare next. In social sciences, they appear when ranking any partially-ordered data, for example, job candidates or products for selling.

It is well-known throughout combinatorics (and perhaps all of math) that counting linear extensions of arbitrary posets is hard—this counting problem is known to be \#P-complete \([\text{BW}91]\), which means that we really should not expect a closed formula for the number of linear extensions of an arbitrary poset. In fact, even for the Boolean lattice (all subsets of an \( n \)-element set ordered by inclusion), the number of linear extensions is only known up to \( n = 7 \). However, certain classes of posets have nice formulas for their number of linear extensions: for rooted trees the number of linear extensions is given by a simple product formula \([\text{Knu}73]\), for zig-zag posets (certain posets associated to permutations) the number of linear extensions is given by the determinant of a matrix \([\text{Sta}12]\). Both rooted trees and zig-zags are examples of posets called trees (their Hasse diagrams have no cycles), but the collection of all trees is much larger than the union of these two classes.

**Problem 4.1.1** (Garver–Grosser–M.–Morales). With respect to counting linear extensions, find the line that divides badly-behaved posets and well-behaved posets.

Although no explicit formula for counting linear extensions of tree posets exists, there is a hint that this problem may be in fact be manageable—in 1990, Atkinson gave a non-explicit recursive algorithm (but not a closed formula) for counting the number of linear extensions of tree posets in polynomial time \([\text{Atk}90]\). This hint of decency motivates our work on tree posets.

**Problem 4.1.2** (Garver–Grosser–M.–Morales). Define the subclass \( \mathcal{Q} \) of all tree posets which admit an explicit determinant formula.

Here are some examples of families of posets for which we have found a well-defined strategy for producing a determinant formula.

(a) rooted trees (more generally \( d \)-complete posets \([\text{KY}17]\) and zig-zag posets

(b) posets obtained from zig-zag posets by attaching the minimal element of rooted trees (or more generally a \( d \)-complete posets) to an element of the zig-zag poset
If $T$ is a tree poset (with say $n$ elements), we write $e(T)$ for the number of linear extensions of $T$. The general idea for our strategy is to cleverly pick edges of our original poset $T$ to fold so that we obtain a rooted tree, then to use this information to systematically build a matrix whose determinant is $e(T)$. More precisely, given a covering relation $x \nearrow y$ in $T$, we have by inclusion-exclusion that

$$e(T) = e(T - \{x \nearrow y\}) - e(T - \{x \nearrow y\} + \{x \searrow y\}).$$

Since $T$ is a tree, we can iterate this process until only a disjoint union of rooted trees remains, and each rooted tree has a product formula [Knu73]. Unfortunately, the resulting alternating formula for $e(T)$ has exponentially-many terms

$$e(T) = \sum_{S \subset \{1,2,\ldots,n\}} (-1)^{|S|} e(T_S).$$

Garver, Grosser, Morales, and I are working to determine when this alternating sum can be turned into a determinant.

**Example 4.1.3.** We give an example of such a determinant formula for a poset which does not belong to the families of (a) and (b) above. Consider the poset $T$ with Hasse diagram

By folding the red edges and taking certain subgraphs of the folded graph, we build a matrix

$$M_T = \begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}.
$$

Note that each poset in the matrix $M_T$ is a rooted tree and has a product formula. We abuse notation and write $M_T$ for the matrix whose entries are $e(T')$ for each rooted tree poset $T'$ appearing in $M_T$. Then we have that the number of linear extensions of $T$ is given by

$$e(T) = n! \det M_T = 6! \det \begin{pmatrix}
1 & 1/3 & 1/3 \\
1 & 1/3 & 1/3 \\
0 & 1 & 1
\end{pmatrix} = 12.$$

When Problem 4.1.2 has been solved, it will be interesting to think about the intersection of the class $Q$ with the class $P$ from Theorem 3.1.2 in Section 3.1.

**Problem 4.1.4** (Garver–Grosser–M.–Morales). Find the intersection of class $P$ from Section 3.1.2 and class $Q$. Use this to solve Problem 3.1.3.

### 4.2 Finding bases for modules of splines

Splines were first developed for applications in analysis and applied mathematics. In this setting, they are defined as piecewise polynomial functions on a triangulation of some specified region of the plane, usually being required to agree up to some degree of differentiability on the overlaps. If we dualize this definition, we obtain a graph with edge-labels carrying the overlap-agreement conditions. An (algebraic) spline is a generalization of this notion: given a ring $R$ and graph $G$ whose edges are labeled by ideals of $R$, a spline on $G$ is an assignment of an element of $R$ to each vertex of $G$ with the condition that labels on adjacent vertices $u,v$ differ by an element of the ideal labeling the edge $uv$.

Algebraic splines were first introduced by Billera [Bil88] and later reformulated by Gilbert, Tymoczko, and Viel [GT16].

---

4 This condition is called the “GKM-condition” because it appears in Goresky, Kottwitz, and MacPherson’s combinatorial computation of the torus-equivariant cohomology of certain varieties [GKM98].
Example 4.2.1. Let $R$ be a commutative ring with identity. Below is an edge-labeled (by principal ideals in $R$) graph, together with a spline (in cyan) on this edge-labeled graph.

\[ ij \xrightarrow{(i-j)} j^2 \]

\[ (j) \xrightarrow{(j-i^2)} j \]

A key question in both the classical analytic side and the algebraic side is to determine the rank of the $R$-module $R_G$ of all splines for an edge-labeled graph $G$.

**Problem 4.2.2** (Anderson–M.–Tymoczko). Given a commutative ring $R$ with identity and a graph $G$, compute the rank of $R_G$. Moreover, if the edge-labeling ideals are finitely generated, compute an explicit minimal generating set for $R_G$ as an $R$-module.

We have constructed explicit bases for $R_G$ for various graphs and rings whenever the edge-labeling ideals are finitely-generated.

**Theorem 4.2.3** (Anderson–M.–Tymoczko). We assume $R$ is a commutative unique factorization domain with identity. We have explicitly constructed bases for the $R$-module of splines $R_G$ in the following situations:

- All graphs $G$ and all rings $R$ with only one distinct edge-label,
- All graphs $G$ and all rings $R$ with two distinct edge-labels (these are “singular splines” in the classical analytic theory), and
- $G$ an $n$-cycle, $R$ the ring of polynomials in two variables, and edge-labels homogeneous degree-two ideals (these are “pinwheel triangulations” in the classical analytic theory).

There are a large number of directions ripe for investigation—one can pick his or her favorite family of graphs, and try to compute the rank of the module $R_G$ of all splines. These problems are especially accessible for undergraduate research because the main tool is linear algebra, and easy-to-use Macaulay2 software is available online for running experiments to produce minimal generating sets for $R_G$ in examples.

**References**


