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Duals of Hardy spaces on homogeneous groups

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Hardy spaces on homogeneous groups were introduced and studied by Folland and Stein [3]. The purpose of this note is to show that duals of Hardy spaces H^p, $0 < p \leq 1$, on homogeneous groups can be identified with Morrey–Campanato spaces. This closes a gap in the original proof of this fact in [3].

1 Introduction

We begin by reviewing some definitions. Let G be a homogeneous group, i.e., G is a connected and simply connected nilpotent Lie group which is endowed with a family of dilations $\{\delta_r\}_{r>0}$. We recall that a family of dilations on the Lie algebra $\mathfrak{g}$ of G is a one parameter family of automorphisms of $\mathfrak{g}$ of the form $\{\exp(A \log r) : r > 0\}$, where A is diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues $1 = d_1 \leq d_2 \leq \cdots \leq d_n$, $n = \dim(G)$. Then the exponential map from $\mathfrak{g}$ to G defines the corresponding family of dilations $\{\delta_r\}_{r>0}$ on G. We will often use the abbreviated notation $\delta_r x = rx$ for $x \in G$ and $r > 0$.

We fix a homogeneous norm on G, i.e., a continuous map $|\cdot| : G \to [0, \infty)$ that is C^∞ on $G \setminus \{0\}$ and satisfies

$$\begin{aligned} &\left|x^{-1}\right| = |x| \quad \text{for all} \quad x \in G, \\ &|\delta_r x| = r|x| \quad \text{for all} \quad x \in G,\ r > 0, \\ &|x| = 0 \iff x = 0. \end{aligned}$$

The ball $B(r, x)$ of radius $r > 0$ and center $x \in G$ is defined as

$$B(r, x) = \left\{y \in G : \left|x^{-1}y\right| < r\right\},$$

and we denote by γ be the minimal constant such that

$$|xy| \leq \gamma(|x| + |y|) \quad \text{for all} \quad x, y \in G.$$

If ψ is a function on G and $t > 0$, we define its dilate $D_t\psi$ as

$$D_t\psi(x) = t^{-Q}\psi(\delta_{1/t}x) = t^{-Q}\psi(x/t),$$

where

$$Q = d_1 + \cdots + d_n$$

is the homogeneous dimension of G. The dilate $D_t\psi$ is also denoted by ψ_t. The (left) translate of ψ by $x_0 \in G$ is defined as

$$\tau_{x_0}\psi(x) = \psi\big((x_0)^{-1}x\big).$$

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Given a multiindex $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we set

$$|I| = i_1 + \cdots + i_n, \quad d(I) = d_1 i_1 + \cdots + d_n i_n.$$

Let Δ be the additive semi-group of $\mathbb{R}$ generated by $0, d_1, d_2, \ldots, d_n$. That is, $\Delta = \{d(I) : I \in \mathbb{N}^N\}$. Let $\eta_1, \ldots, \eta_n$ be a basis for the linear polynomials on G such that η_i is homogeneous of degree d_i. Then every polynomial P can be written uniquely as

$$P = \sum_I a_I \eta^I, \quad \eta^I = \eta_1^{i_1} \ldots \eta_n^{i_n}, \quad a_I \in \mathbb{C}.$$

The homogeneous degree of $P = \sum_I a_I \eta^I$ is defined as

$$\deg(P) = \max\{d(I) : a_I \neq 0\}.$$

Given $s \in \Delta$, we denote the space of polynomials of homogeneous degree $\leq s$ by

$$\mathcal{P}_s = \{P \in \mathcal{P} : \deg(P) \leq s\}.$$

We recall that $\mathcal{P}_s$ is invariant under left and right translations; see [3, Proposition 1.25].

Suppose that $0 < p \leq 1$, $1 \leq q \leq \infty$ and $s \in \Delta$. We say that a triplet (p, q, s) is *admissible* if $p < q$ and

$$s \geq \max\{s' \in \Delta : s' \leq Q(1/p - 1)\}.$$

We say that a function a is a (p, q, s)-*atom*, where (p, q, s) is admissible, if

$$\begin{aligned}
&\operatorname{supp} a \subset B(x_0, r) \quad \text{for some} \quad x_0 \in G, \quad r > 0, \\
&||a||_q \leq |B(x_0, r)|^{1/q - 1/p}, \\
&\int_G a(x) P(x)\, dx = 0 \quad \text{for all} \quad P \in \mathcal{P}_s.
\end{aligned}$$

The *atomic Hardy space* $H^p_{q,s}$ is the set of all tempered distributions f such that $f = \sum \lambda_i a_i$ (convergence in $\mathcal{S}'$) such that the a_i are (p, q, s)-atoms, $\lambda_i \geq 0$, and $\sum \lambda_i^p < \infty$. $H^p_{q,s}$ is actually independent of q and s ([3, Theorem 3.30]) and so may be denoted simply by H^p.

Let $\mathcal{B}$ denote the collection of all open balls in G. If $l \geq 0$, $1 \leq q \leq \infty$, and $s \in \Delta$, we define the *Campanato space* $C^l_{q,s}$ to be the space of all locally L^q functions u on G so that

$$\begin{aligned}
||u||_{C^l_{q,s}} &:= \sup_{B \in \mathcal{B}} \inf_{P \in \mathcal{P}_s} |B|^{-l} \left(\frac{1}{|B|} \int_B |u(x) - P(x)|^q dx \right)^{1/q} < \infty \quad (q < \infty), \\
||u||_{C^l_{\infty,s}} &:= \sup_{B \in \mathcal{B}} \inf_{P \in \mathcal{P}_s} |B|^{-l} \operatorname{ess\,sup}_{x \in B} |u(x) - P(x)| < \infty \quad (q = \infty).
\end{aligned}$$

We identify two elements of $C^l_{q,s}$ if they are equal almost everywhere. (Note: The space called $C^l_{q,s}$ here is called $C^{Ql}_{q,s}$ in [3].)

2 Duals of Hardy spaces

The main goal of this note is to prove that the dual of the Hardy space $H^p_{q,s}$ is isomorphic to the Campanato space $C^{1/p-1}_{q',s}/\mathcal{P}_s$, where (p, q, s) is an admissible triplet and $1/q + 1/q' = 1$. This result in the setting of Hardy spaces on homogeneous groups was obtained by Folland and Stein [3, Chapter 5]. However, careful examination of the arguments in [3] reveals a gap in the first part of the proof of [3, Theorem 5.3]. The trouble is that uniform boundedness of a functional on atoms does not guarantee that the functional is bounded on H^p; see [2]. Hence, the operator norm of a functional L on H^p is given by the supremeum of $|La|$ over all atoms a, as asserted in [3, Lemma 5.1], only when the functional is known a priori to be continuous. To remedy this situation we will apply a rather subtle approximation argument inspired by [4, Chapter III.5], see also [1, Section 8].

We will need some simple observations about Campanato spaces. First, note that for any $t > 0$, the substitution $s = r/t$ gives

$$\begin{aligned}
||u_t||_{C^l_{q',s}} &= \sup_{x_0\in G,\, r>0}\inf_{P\in\mathcal{P}_s}|B(x_0,r)|^{-l}\biggl(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}\bigl|t^{-Q}u(x/t)-P(x)\bigr|^{q'}dx\biggr)^{1/q'}\\
&= \sup_{x_0\in G,\, s>0}\inf_{P\in\mathcal{P}_s}\bigl(t^Q|B(x_0,s)|\bigr)^{-l}\biggl(\frac{1}{|B(x_0,s)|}\int_{B(x_0,s)}t^{-Qq'}\bigl|u(x)-t^QP(tx)\bigr|^{q'}dx\biggr)^{1/q'}\\
&= t^{-Q(l+1)}||u||_{C^l_{q',s}}.
\end{aligned} \tag{2.1}$$

Next, for any $B\in\mathcal{B}$, let $\pi_B : L^1(B)\to\mathcal{P}_s$ be the natural projection defined by

$$\int_B(\pi_Bf(x))Q(x)\,dx = \int_B f(x)Q(x)\,dx \quad \text{for all} \quad f\in L^1(B), \quad Q\in\mathcal{P}_s.$$

We claim that there is a constant $C = C_s$, independent of f and B, such that

$$\sup_{x\in B}|\pi_Bf(x)| \le C\frac{1}{|B|}\int_B|f(x)|\,dx. \tag{2.2}$$

Indeed, for the fixed ball $B_0 = B(0,1)$, let $\{Q_I : d(I)\le s\}$ be an orthonormal basis of $\mathcal{P}_s$ with respect to the $L^2(B_0)$ norm. Then

$$\pi_{B_0}f = \sum_{d(I)\le s}\biggl(\int_{B_0}f(x)\overline{Q_I(x)}\,dx\biggr)Q_I,$$

so the estimate (2.2) holds for $B = B_0$ with $C = |B_0|\sum_I\bigl(\sup_{x\in B_0}|Q_I(x)|\bigr)^2$. Since $\pi_{B(x_0,r)}f = (\tau_{x_0}\circ D_r\circ\pi_{B_0}\circ D_{1/r}\circ\tau_{(x_0)^{-1}})f$, (2.2) then follows for arbitrary $B = B(x_0,r)\in\mathcal{B}$.

Next, we claim that we can define an equivalent norm on $C^l_{q',s}$ by setting

$$|||u|||_{C^l_{q',s}} = \sup_{B\in\mathcal{B}}|B|^{-l}\biggl(\frac{1}{|B|}\int_B|u(x)-\pi_Bu(x)|^{q'}dx\biggr)^{1/q'} \quad (1\le q'<\infty), \tag{2.3}$$

$$|||u|||_{C^l_{\infty,s}} = \sup_{B\in\mathcal{B}}|B|^{-l}\operatorname{ess\,sup}_{x\in B}|u(x)-\pi_Bu(x)| \quad (q'=\infty). \tag{2.4}$$

Indeed, for any $B\in\mathcal{B}$ and $P\in\mathcal{P}_s$, by the fact that $P = \pi_BP$ and (2.2) we have

$$\begin{aligned}
&\biggl(\frac{1}{|B|}\int_B|u(x)-\pi_Bu(x)|^{q'}dx\biggr)^{1/q'}\\
&\quad\le\biggl(\frac{1}{|B|}\int_B|u(x)-P(x)|^{q'}dx\biggr)^{1/q'}+\biggl(\frac{1}{|B|}\int_B|\pi_B(P-u)(x)|^{q'}dx\biggr)^{1/q'}\\
&\quad\le\biggl(\frac{1}{|B|}\int_B|u(x)-P(x)|^{q'}dx\biggr)^{1/q'}+C\,\frac{1}{|B|}\int_B|u(x)-P(x)|\,dx\\
&\quad\le(C+1)\biggl(\frac{1}{|B|}\int_B|u(x)-P(x)|^{q'}dx\biggr)^{1/q'}.
\end{aligned}$$

Therefore,

$$||u||_{C^l_{q',s}} \le |||u|||_{C^l_{q',s}} \le (C+1)||u||_{C^l_{q',s}} \quad \text{for all} \quad u\in C^l_{q',s}. \tag{2.5}$$

The key ingredients in the proof of the duality theorem are some approximation results for Campanato spaces. To begin with, define the space

$$\Theta^q_s = \bigl\{f\in L^q(G) : \operatorname{supp}f \text{ is compact and } \textstyle\int_G f(x)P(x)\,dx = 0 \text{ for } P\in\mathcal{P}_s\bigr\}. \tag{2.6}$$

Lemma 2.1 *Suppose* $u \in C^l_{q',s}$*, where* $l \geq 0$*,* $1 \leq q' \leq \infty$*, and* $s = 0, 1, \ldots$ *Fix a nonnegative function* $\varphi \in C^\infty$ *with compact support and* $\int \varphi = 1$*, and let* $\varphi_r(x) = r^{-Q}\varphi(x/r)$*. Then*

$$\int_G f(x)(u * \varphi_r)(x)\,dx \longrightarrow \int_G f(x)u(x)\,dx \quad \text{as} \quad r \longrightarrow 0 \quad \text{for all} \quad f \in \Theta^q_s, \tag{2.7}$$

and

$$||u * \varphi_r||_{C^l_{q',s}} \leq |||u|||_{C^l_{q',s}} \quad \text{for all} \quad r > 0. \tag{2.8}$$

P r o o f. If $q' < \infty$, (2.7) holds since $u * \varphi_r \to u$ in $L^{q'}_{\text{loc}}(G)$ as $r \to 0$. If $q' = \infty$, $u * \varphi_r$ is uniformly bounded on compact sets and converges a.e. to $u(x)$ by [3, Theorem 2.6], so (2.7) holds by the dominated convergence theorem. Next, given $B \in \mathcal{B}$ and $r > 0$, define a function P_r by

$$P_r(x) = \int_G \pi_{y^{-1}B} u\big(y^{-1}x\big)\varphi_r(y)\,dy.$$

Since we can write $\pi_{y^{-1}B} u\big(y^{-1}x\big) = \sum_{d(I)\leq s} c_\alpha(y)\eta^I\big(y^{-1}x\big)$ and the coefficients $c_\alpha(y)$ are continuous functions of y, P_r is a polynomial of homogeneous degree $\leq s$. By the Minkowski inequality,

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |(u*\varphi_r)(x) - P_r(x)|^{q'}\,dx\right)^{1/q'} \\
&\quad = \left(\frac{1}{|B|}\int_B \left|\int_G \big(u\big(y^{-1}x\big) - \pi_{y^{-1}B}u\big(y^{-1}x\big)\big)\varphi_r(y)\,dy\right|^{q'} dx\right)^{1/q'} \\
&\quad \leq \int_G \left(\frac{1}{|B|}\int_B \big|u\big(y^{-1}x\big) - \pi_{y^{-1}B}u\big(y^{-1}x\big)\big|^{q'} dx\right)^{1/q'} |\varphi_r(y)|\,dy \\
&\quad = \int_G \left(\frac{1}{|y^{-1}B|}\int_{y^{-1}B} |u(z) - \pi_{y^{-1}B}u(z)|^{q'} dz\right)^{1/q'} \varphi_r(y)\,dy \\
&\quad \leq |||u|||_{C^l_{q',s}} |B|^l.
\end{aligned}$$

This proves (2.8). □

Lemma 2.2 *Let* $\psi \in C^\infty$ *be such that* $\operatorname{supp}\psi \subset B(0,1)$*,* $0 \leq \psi(x) \leq 1$*, and* $\psi(x) = 1$ *for* $x \in B(0, 1/2)$*. There exist* $C > 0$ *and* $\tilde{s} \in \Delta$ *with* $\tilde{s} \geq s$ *such that*

$$||(u - \pi_{B_0}u)\psi||_{C^l_{q',\tilde{s}}} \leq C\,||u||_{C^l_{q',s}} \quad \text{for all} \quad u \in C^l_{q',s}, \tag{2.9}$$

where $B_0 = B(0, \gamma(2\gamma + 1))$*.*

P r o o f. Suppose $u \in C^l_{q',s}$ with $|||u|||_{C^l_{q',s}} \leq 1$. For brevity, we only consider the case $q' < \infty$; the case $q' = \infty$ uses a similar argument. Let $U = u - \pi_{B_0}u$. Since $\operatorname{supp}\psi \subset B_0$,

$$\int_G |U(x)\psi(x)|^{q'}\,dx \leq \int_{B_0} |U(x)|^{q'}\,dx \leq |B_0|^{lq'+1} < \infty. \tag{2.10}$$

Therefore, if $B = B(x_0, r) \in \mathcal{B}$ with $r \geq 1$, then

$$|B|^{-l}\left(\frac{1}{|B|}\int_B |U(x)\psi(x)|^{q'}\,dx\right)^{1/q'} \leq \left(\frac{|B_0|}{|B|}\right)^{l+1/q'} \leq C < \infty.$$

Hence, to show (2.9) it is enough to estimate the integral of $U\psi$ over balls $B = B(x_0, r)$ with $0 < r < 1$. Moreover, we can assume that $B \cap B(0,1) \neq \varnothing$, since otherwise $U\psi = 0$ on B. Consequently, we are limited to balls $B \subset B_0$. Let $P_1 = \pi_B U$. By (2.2) and (2.10),

$$\left(\frac{1}{|B|}\int_B |P_1(x)|^{q'}\,dx\right)^{1/q'} \leq C\left(\frac{1}{|B|}\int_B |U(x)|^{q'}\,dx\right)^{1/q'} \leq C_1|B|^{-1/q'}. \tag{2.11}$$

Let $P_2(x)$ be the left Taylor polynomial of ψ at x_0 of homogenous degree s_0 (i.e., the polynomial whose left-invariant derivatives at the origin of homogeneous degree $\le s_0$ agree with the corresponding derivatives of f at x_0), where $s_0 \in \Delta$ is chosen to satisfy $s_0 \ge Q(l + 1/q')$. By the Taylor inequality ([3, Theorem 1.37] and the remark following it), the remainder satisfies

$$\left|\psi(x) - P_2\left(x_0^{-1}x\right)\right| \le C_2\left|x_0^{-1}x\right|^{s_0} \quad \text{for} \quad x \in B \subset B(x_0, 1),$$

with C_2 independent of x_0. Finally, let $P(x) = P_1(x)P_2\left(x_0^{-1}x\right)$, which is a polynomial of homogeneous degree at most $\tilde{s} = s + s_0$. By (2.11),

$$\begin{aligned}
&\left(\int_B |U(x)\psi(x) - P(x)|^{q'}\,dx\right)^{1/q'}\\
&\le \left(\int_B |[U(x) - P_1(x)]\psi(x)|^{q'}\,dx\right)^{1/q'} + \left(\int_B \left|P_1(x)\left[\psi(x) - P_2\left(x_0^{-1}x\right)\right]\right|^{q'}dx\right)^{1/q'}\\
&\le ||\psi||_\infty\left(\int_B |U(x) - P_1(x)|^{q'}\,dx\right)^{1/q'} + \sup_{x\in B}\left|\psi(x) - P_2\left(x_0^{-1}x\right)\right|\left(\int_B |P_1(x)|^{q'}\,dx\right)^{1/q'}\\
&\le |B|^{l+1/q'} + C_1C_2r^{s_0}.
\end{aligned}$$

But $r^{s_0} = C_3|B|^{s_0/Q} \le C_3|B|^{l+1/q'}$, so (2.9) is proved with $\tilde{s} = s + s_0$. □

Lemma 2.3 *Suppose* $u \in C^l_{q',s}$, *where* $l \ge 0$, $1 \le q' \le \infty$, *and* $s = 0, 1, \dots$ *There exist* $\tilde{s} \ge s$, *a constant* $C > 0$ *independent of* u, *and a sequence of test functions* $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{S}$ *so that*

$$||u_k||_{C^l_{q',\tilde{s}}} \le C\,||u||_{C^l_{q',s}} \quad \textit{for all} \quad k \in \mathbb{N}, \tag{2.12}$$

$$\lim_{k\to\infty}\int_G f(x)u_k(x)\,dx = \int_G f(x)u(x)\,dx \quad \textit{for all} \quad f \in \Theta^q_s, \quad 1/q + 1/q' = 1. \tag{2.13}$$

Proof. First suppose $u \in C^l_{q',s} \cap C^\infty$. Let $\tilde{u}_k = D_{2^{-k}}u$ and $u_k = D_{2^k}((\tilde{u}_k - \pi_{B_0}\tilde{u}_k)\psi)$, where B_0 is as in Lemma 2.2. By (2.1) and (2.9),

$$||(\tilde{u}_k - \pi_{B_0}\tilde{u}_k)\psi||_{C^l_{q',\tilde{s}}} \le C\,||\tilde{u}_k||_{C^l_{q',s}} = C\,2^{kQ(l+1)}||u||_{C^l_{q',s}}.$$

Therefore (2.12) holds, since

$$||u_k||_{C^l_{q',\tilde{s}}} = 2^{-kQ(l+1)}||(\tilde{u}_k - \pi_{B_0}\tilde{u}_k)\psi||_{C^l_{q',\tilde{s}}} \le C\,||u||_{C^l_{q',s}}. \tag{2.14}$$

Moreover,

$$u_k(x) = u(x) - (D_{2^k} \circ \pi_{B_0} \circ D_{2^{-k}})u(x) = u(x) - \pi_{B(0,2^k\gamma(2\gamma+1))}u(x) \quad \text{for} \quad x \in B\left(0, 2^{k-1}\right). \tag{2.15}$$

Thus (2.13) also holds.

To end the proof we remove the assumption that $u \in C^\infty$. Given $u \in C^l_{q',s}$, define the sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{S}$ by $u_k = D_{2^k}((\tilde{u}_k - \pi_{B_0}\tilde{u}_k)\psi)$, where $\tilde{u}_k = D_{2^{-k}}(u * \varphi_k)$ with φ_k as in Lemma 2.1. Combining (2.8) and (2.14) yields (2.12), whereas (2.7) and (2.15) yield (2.13), completing the proof of of Lemma 2.3. □

Lemma 2.4 *Suppose that* (p, q, s) *is admissible and* $f \in \Theta^q_s$, *where* Θ^q_s *is given by* (2.6). *Suppose* $u \in C^l_{q',s}$, $1/q + 1/q' = 1$, $l = 1/p - 1$. *There exists* $\tilde{s} \ge s$ *such that if* f *is decomposed into* $f = \sum_{i=1}^\infty \lambda_i a_i$, *where* $\sum_{i=1}^\infty |\lambda_i|^p < \infty$ *and the* a_i*'s are* $(p, q, \tilde{s})$*-atoms, then*

$$\int fu = \sum_{i=1}^\infty \lambda_i \int a_i u. \tag{2.16}$$

P r o o f. Let a be a (p,q,s)-atom supported on a ball $B \in \mathcal{B}$ and $u \in C^l_{q',s}$. Since $\int ua = \int (u-P)a$ for all $P \in \mathcal{P}_s$ then by the standard calculation we have

$$
\begin{aligned}
\left| \int ua \right| &= \inf_{P \in \mathcal{P}_s} \left| \int (u-P)a \right| \\
&\le \left(\int_B |a|^q \right)^{1/q} \left(\inf_{P \in \mathcal{P}_s} \int |u-P|^{q'} \right)^{1/q'} \\
&\le |B|^{1/q-1/p} |B|^{l+1/q'} |B|^{-l} \left(\frac{1}{|B|} \inf_{P \in \mathcal{P}_s} \int |u-P|^{q'} \right)^{1/q'} \\
&\le ||u||_{C^l_{q',s}}.
\end{aligned}
\tag{2.17}
$$

Next, suppose that $f \in \Theta^q_s$ is decomposed into $f = \sum_{i=1}^\infty \lambda_i a_i$, where $\sum_{i=1}^\infty |\lambda_i|^p < \infty$ and the a_i's are $(p,q,\tilde{s})$-atoms, where $\tilde{s} \ge s$ is the same as in Lemma 2.3. Suppose also that $u \in C^l_{q',s}$, $1/q + 1/q' = 1$, $l = 1/p - 1$. Let $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{S}$ be the sequence guaranteed by Lemma 2.3. For every $k \in \mathbb{N}$ we have

$$
\int f u_k = \sum_{i=1}^\infty \lambda_i \int a_i u_k, \tag{2.18}
$$

since convergence in H^p implies convergence in $\mathcal{S}'$ by [3, Proposition 2.15]. By (2.13)

$$
\lim_{k \to \infty} \int_G a_i(x) u_k(x)\,dx = \int_G a_i(x) u(x)\,dx \quad \text{for all} \quad i \in \mathbb{N}.
$$

By (2.12) and (2.17) we have $|\int u_k a_i| \le ||u_k||_{C^l_{q',\tilde{s}}} \le C\,||u||_{C^l_{q',s}}$. Since $\sum_{i=1}^\infty |\lambda_i| \le \left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{1/p} < \infty$ we can take the limit as $k \to \infty$ in (2.18) by the dominated convergence theorem applied to counting measure on $\mathbb{N}$. This shows (2.16). $\square$

At last we are in a position to prove the duality theorem.

Theorem 2.5 *Suppose* (p,q,s) *is admissible. Then*

$$
\left(H^p_{q,s}\right)^* \cong C^l_{q',s}/\mathcal{P}_s, \quad \textit{where} \quad 1/q + 1/q' = 1, \quad l = 1/p - 1.
$$

More precisely, if $u \in C^l_{q',s}$ *and* f *is a finite linear combination of* (p,q,s)*-atoms, let* $L_u f = \int uf$. *Then* L_u *extends continuously to* $H^p_{q,s}$, *and every* $L \in (H^p_{q,s})^*$ *is of this form. Moreover,*

$$
||u||_{C^l_{q',s}} = ||L_u||_{(H^p_{q,s})^*} \quad \textit{for all} \quad u \in C^l_{q',s}. \tag{2.19}
$$

P r o o f. The fact that any bounded functional L on $H^p_{q,s}$ must be of the form $L = L_u$ for some $u \in C^l_{q',s}$ was already shown in [3].

Conversely, suppose $u \in C^l_{q',s}$. Our goal is to demonstrate that the functional $L_u f = \int uf$ defined initially for $f \in \Theta^q_s$, where Θ^q_s is given by (2.6), extends to a bounded functional on $H^p_{q,s}$ and $||L_u||_{(H^p_{q,s})^*} \le ||u||_{C^l_{q',s}}$. We emphasize again that boundedness of L_u on atoms (2.17) alone does not guarantee boundedness on the entire space.

Suppose that $f \in \Theta^q_s$. By [3, Theorem 3.30] we can find an atomic decomposition of $f = \sum_{i=1}^\infty \lambda_i a_i$, where

$$
\left(\sum_{i=1}^\infty |\lambda_i|^p \right)^{1/p} \le 2\,||f||_{H^p_{q,\tilde{s}}} \le C\,||f||_{H^p_{q,s}},
$$

and the a_i's are $(p,q,\tilde{s})$-atoms. By (2.17) and Lemma 2.4

$$
|L_u f| \le \sum_{i=1}^\infty |\lambda_i| |L_u a_i| \le ||u||_{C^l_{q',s}} \left(\sum_{i=1}^\infty |\lambda_i|^p \right)^{1/p} \le C\,||u||_{C^l_{q',s}} ||f||_{H^p_{q,s}}.
$$

Therefore, L_u extends uniquely to a bounded functional on $H^p_{q,s}$. Next, we recall that the norm of a bounded functional on $H^p_{q,s}$ is always achieved on atoms; see [3, Lemma 5.1], which holds under the assumption of continuity. Therefore, (2.17) implies $||L_u||_{(H^p_{q,s})^*} \leq ||u||_{C^l_{q',s}}$, which finishes the proof of Theorem 2.5. □

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