Duals of Hardy spaces on homogeneous groups

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Hardy spaces on homogeneous groups were introduced and studied by Folland and Stein [3]. The purpose of this note is to show that duals of Hardy spaces \( H^p \), \( 0 < p \leq 1 \), on homogeneous groups can be identified with Morrey–Campanato spaces. This closes a gap in the original proof of this fact in [3].

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1 Introduction

We begin by reviewing some definitions. Let \( G \) be a homogeneous group, i.e., \( G \) is a connected and simply connected nilpotent Lie group which is endowed with a family of dilations \( \{ \delta_r \}_{r>0} \). We recall that a family of dilations on the Lie algebra \( g \) of \( G \) is a one parameter family of automorphisms of \( g \) of the form \( \{ \exp(A \log r) : r > 0 \} \), where \( A \) is diagonalizable linear operator on \( g \) with positive eigenvalues \( 1 = d_1 \leq d_2 \leq \cdots \leq d_n \), \( n = \dim(G) \). Then the exponential map from \( g \) to \( G \) defines the corresponding family of dilations \( \{ \delta_r \}_{r>0} \) on \( G \). We will often use the abbreviated notation \( \delta_r x = rx \) for \( x \in G \) and \( r > 0 \).

We fix a homogeneous norm on \( G \), i.e., a continuous map \( |\cdot| : G \to [0, \infty) \) that is \( C^\infty \) on \( G \setminus \{0\} \) and satisfies

\[
|x^{-1}| = |x| \quad \text{for all} \quad x \in G,
\]

\[
|\delta_r x| = r|x| \quad \text{for all} \quad x \in G, \ r > 0,
\]

\[
|x| = 0 \iff x = 0.
\]

The ball \( B(r, x) \) of radius \( r > 0 \) and center \( x \in G \) is defined as

\[
B(r, x) = \{ y \in G : |x^{-1}y| < r \},
\]

and we denote by \( \gamma \) be the minimal constant such that

\[
|xy| \leq \gamma(|x| + |y|) \quad \text{for all} \quad x, y \in G.
\]

If \( \psi \) is a function on \( G \) and \( t > 0 \), we define its dilate \( D_t \psi \) as

\[
D_t \psi(x) = t^{-Q} \psi(\delta_{1/t} x) = t^{-Q} \psi(x/t),
\]

where

\[
Q = d_1 + \cdots + d_n
\]

is the homogeneous dimension of \( G \). The dilate \( D_t \psi \) is also denoted by \( \psi_t \). The (left) translate of \( \psi \) by \( x_0 \in G \) is defined as

\[
\tau_{x_0} \psi(x) = \psi((x_0)^{-1}x).
\]

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Given a multiindex \( I = (i_1, \ldots, i_n) \in \mathbb{N}^n \), we set

\[
|I| = i_1 + \cdots + i_n, \quad d(I) = d_1 i_1 + \cdots + d_n i_n.
\]

Let \( \Delta \) be the additive semi-group of \( \mathbb{R} \) generated by \( 0, d_1, d_2, \ldots, d_n \). That is, \( \Delta = \{ d(I) : I \in \mathbb{N}^N \} \). Let \( \eta_1, \ldots, \eta_n \) be a basis for the linear polynomials on \( G \) such that \( \eta_i \) is homogeneous of degree \( d_i \). Then every polynomial \( P \) can be written uniquely as

\[
P = \sum_I a_I \eta^I, \quad \eta^I = \eta_1^{i_1} \cdots \eta_n^{i_n}, \quad a_I \in \mathbb{C}.
\]

The homogeneous degree of \( P = \sum_I a_I \eta^I \) is defined as

\[
\text{deg}(P) = \max \{ d(I) : a_I \neq 0 \}.
\]

Given \( s \in \Delta \), we denote the space of polynomials of homogeneous degree \( \leq s \) by

\[
P_s = \{ P \in \mathcal{P} : \text{deg}(P) \leq s \}.
\]

We recall that \( \mathcal{P}_s \) is invariant under left and right translations; see [3, Proposition 1.25].

Suppose that \( 0 < p \leq 1, 1 \leq q \leq \infty \) and \( s \in \Delta \). We say that a triplet \( (p, q, s) \) is admissible if \( p < q \) and

\[
s \geq \max \{ s' \in \Delta : s' \leq Q(1/p - 1) \}.
\]

We say that a function \( a \) is a \( (p, q, s) \)-atom, where \( (p, q, s) \) is admissible, if

\[
\text{supp } a \subset B(x_0, r) \quad \text{for some} \quad x_0 \in G, \quad r > 0,
\]

\[
\|a\|_q \leq |B(x_0, r)|^{1-1/p},
\]

\[
\int_G a(x) P(x) \, dx = 0 \quad \text{for all} \quad P \in \mathcal{P}_s.
\]

The atomic Hardy space \( H^q_{p,s} \) is the set of all tempered distributions \( f \) such that \( f = \sum \lambda_i a_i \) (convergence in \( S' \)) such that the \( a_i \) are \( (p,q,s) \)-atoms, \( \lambda_i \geq 0 \), and \( \sum \lambda_i^q < \infty \). \( H^q_{p,s} \) is actually independent of \( q \) and \( s \) ([3, Theorem 3.30]) and so may be denoted simply by \( H^p \).

Let \( B \) denote the collection of all open balls in \( G \). If \( l \geq 0, 1 \leq q \leq \infty \), and \( s \in \Delta \), we define the Campanato space \( C^q_{l,s} \) to be the space of all locally \( L^q \) functions \( a \) on \( G \) so that

\[
\|a\|_{C^q_{l,s}} := \sup_{B \in B} \inf_{P \in \mathcal{P}_s} |B|^{-l} \left( \frac{1}{|B|} \int_B |a(x) - P(x)|^q \, dx \right)^{1/q} < \infty \quad (q < \infty),
\]

\[
\|a\|_{C^{q\infty}_{l,s}} := \sup_{B \in B} \inf_{P \in \mathcal{P}_s} |B|^{-l} \text{ess sup}_{x \in B} |a(x) - P(x)| < \infty \quad (q = \infty).
\]

We identify two elements of \( C^q_{l,s} \) if they are equal almost everywhere. (Note: The space called \( C^q_{l,s} \) here is called \( C^q_{l,s} \) in [3].)

\section{Duals of Hardy spaces}

The main goal of this note is to prove that the dual of the Hardy space \( H^p_{q,s} \) is isomorphic to the Campanato space \( C^{1/p-1}_{q',s} / \mathcal{P}_s \), where \( (p,q,s) \) is an admissible triplet and \( 1/q + 1/q' = 1 \). This result in the setting of Hardy spaces on homogeneous groups was obtained by Folland and Stein [3, Chapter 5]. However, careful examination of the arguments in [3] reveals a gap in the first part of the proof of [3, Theorem 5.3]. The trouble is that uniform boundedness of a functional on atoms does not guarantee that the functional is bounded on \( H^p \); see [2]. Hence, the operator norm of a functional \( L \) on \( H^p \) is given by the supremum of \( |L a| \) over all atoms \( a \), as asserted in [3, Lemma 5.1], only when the functional is known a priori to be continuous. To remedy this situation we will apply a rather subtle approximation argument inspired by [4, Chapter III.5], see also [1, Section 8].
We will need some simple observations about Campanato spaces. First, note that for any $t > 0$, the substitution $s = r/t$ gives

$$
|\|u_t||_{C_q^{l,s}} = \sup_{x_0 \in G, r > 0} \inf_{P \in P_x} |B(x_0, r)|^{-1} \int_{B(x_0, r)} |t^{-Q}u(x/t) - P(x)|^q dx
$$

$$
= \sup_{x_0 \in G, s > 0} \inf_{P \in P_x} \left( t^{-Q} |B(x_0, s)|^{-1} \int_{B(x_0, s)} |t^{-Q}u(x) - t^Q P(tx)|^q dx \right)^{1/q}
$$

Next, for any $B \in \mathcal{B}$, let $\pi_B : L^1(B) \to \mathcal{P}_s$ be the natural projection defined by

$$
\int_B (\pi_B f(x))Q(x) \, dx = \int_B f(x)Q(x) \, dx \quad \text{for all } f \in L^1(B), \quad Q \in \mathcal{P}_s.
$$

We claim that there is a constant $C = C_s$, independent of $f$ and $B$, such that

$$
\sup_B |\pi_B f(x)| \leq C \frac{1}{|B|} \int_B |f(x)| \, dx.
$$

Indeed, for the fixed ball $B_0 = B(0, 1)$, let $\{Q_I : d(I) \leq s\}$ be an orthonormal basis of $\mathcal{P}_s$ with respect to the $L^2(B_0)$ norm. Then

$$
\pi_{B_0} f = \sum_{d(I) \leq s} \left( \int_{B_0} f(x)Q_I(x) \, dx \right) Q_I,
$$

so the estimate (2.2) holds for $B = B_0$ with $C = |B_0| \sum_{d(I) \leq s} (\sup_{x \in B_0} |Q_I(x)|)^2$. Since $\pi_{B(x_0, r)} f = (\tau_{x_0} \circ D_r \circ \pi_{B_0} \circ D_{1/r} \circ \tau_{x_0}) f$, (2.2) then follows for arbitrary $B = B(x_0, r) \in \mathcal{B}$.

Next, we claim that we can define an equivalent norm on $C_q^{l,s}$ by setting

$$
|\|u|||_{C_q^{l,s}} = \sup_B |B|^{-1} \int_B |u(x) - \pi_B u(x)|^q dx
$$

$$
|\|u|||_{C_q^{l,\infty}} = \sup_B |B|^{-1} \inf_{u \in B} |u(x) - \pi_B u(x)|
$$

Indeed, for any $B \in \mathcal{B}$ and $P \in \mathcal{P}_s$, by the fact that $P = \pi_B P$ and (2.2) we have

$$
\left( \frac{1}{|B|} \int_B |u(x) - \pi_B u(x)|^q dx \right)^{1/q} \leq \left( \frac{1}{|B|} \int_B |u(x) - P(x)|^q dx \right)^{1/q} + \left( \frac{1}{|B|} \int_B |\pi_B (P-u)(x)|^q dx \right)^{1/q} \leq \left( \frac{1}{|B|} \int_B |u(x) - P(x)|^q dx \right)^{1/q} + C \frac{1}{|B|} \int_B |u(x) - P(x)| dx \leq (C + 1) \left( \frac{1}{|B|} \int_B |u(x) - P(x)|^q dx \right)^{1/q}.
$$

Therefore,

$$
|\|u|||_{C_q^{l,s}} \leq |\|u|||_{C_q^{l,\infty}} \leq (C + 1) |\|u|||_{C_q^{l,s}} \quad \text{for all } u \in C_q^{l,s}.
$$

The key ingredients in the proof of the duality theorem are some approximation results for Campanato spaces. To begin with, define the space

$$
\Theta_q^s = \{ f \in L^q(G) : \text{supp } f \text{ is compact and } \int_G f(x)P(x) \, dx = 0 \text{ for } P \in \mathcal{P}_s \}.
$$
Lemma 2.1 Suppose \( u \in C^l_{q',s} \), where \( l \geq 0, 1 \leq q' \leq \infty \), and \( s = 0, 1, \ldots \). Fix a nonnegative function \( \varphi \in C^\infty \) with compact support and \( \int \varphi = 1 \), and let \( \varphi_r(x) = r^{-Q} \varphi(x/r) \). Then
\[
\int_G f(x)(u * \varphi_r)(x) \, dx \longrightarrow \int_G f(x)u(x) \, dx \quad \text{as} \quad r \to 0 \quad \text{for all} \quad f \in \Theta^q_2, \tag{2.7}
\]
and
\[
||u * \varphi_r||_{C^l_{q',s}} \leq ||u||_{C^l_{q',s}} \quad \text{for all} \quad r > 0. \tag{2.8}
\]

Proof. If \( q' < \infty \), (2.7) holds since \( u * \varphi_r \to u \) in \( L^q_{\text{loc}}(G) \) as \( r \to 0 \). If \( q' = \infty \), \( u * \varphi_r \) is uniformly bounded on compact sets and converges a.e. to \( u(x) \) by [3, Theorem 2.6], so (2.7) holds by the dominated convergence theorem. Next, given \( B \in \mathcal{B} \) and \( r > 0 \), define a function \( P_r \) by
\[
P_r(x) = \int_G \pi_{y-1,B}u(y^{-1}x) \varphi_r(y) \, dy.
\]
Since we can write \( \pi_{y-1,B}u(y^{-1}x) = \sum_{d(f) \leq c_0(y)}q^f(y^{-1}x) \) and the coefficients \( c_0(y) \) are continuous functions of \( y \), \( P_r \) is a polynomial of homogeneous degree \( \leq s \). By the Minkowski inequality,
\[
\left( \frac{1}{|B|} \int_B \left| (u * \varphi_r)(x) - P_r(x) \right|^q \, dx \right)^{1/q} \leq \left( \frac{1}{|B|} \int_B \left( u(y^{-1}x) - \pi_{y-1,B}u(y^{-1}x) \right) \varphi_r(y) \, dy \right)^{1/q'} \leq \left( \frac{1}{|y^{-1}B|} \int_{y^{-1}B} |u(z) - \pi_{y-1,B}u(z)|^{q'} \, dz \right)^{1/q'} \leq ||u||_{C^l_{q',s}} |B|^l.
\]
This proves (2.8). \( \square \)

Lemma 2.2 Let \( \psi \in C^\infty \) be such that \( supp \psi \subset B(0,1), 0 \leq \psi(x) \leq 1 \), and \( \psi(x) = 1 \) for \( x \in B(0,1/2) \). There exist \( C > 0 \) and \( \delta \in \Delta \) with \( \delta \geq 2 \) such that
\[
||(u - \pi_{B_0}u)\psi||_{C^l_{q',s}} \leq C ||u||_{C^l_{q',s}} \quad \text{for all} \quad u \in C^l_{q',s}, \tag{2.9}
\]
where \( B_0 = B(0, \gamma(2\gamma + 1)) \).

Proof. Suppose \( u \in C^l_{q',s} \) with \( ||u||_{C^l_{q',s}} \leq 1 \). For brevity, we only consider the case \( q' < \infty \); the case \( q' = \infty \) uses a similar argument. Let \( U = u - \pi_{B_0}u \). Since \( supp \psi \subset B_0 \),
\[
\int_G |U(x)\psi(x)|^q \, dx \leq \int_{B_0} |U(x)|^q \, dx \leq |B_0|^{q' + 1} < \infty. \tag{2.10}
\]
Therefore, if \( B = B(x_0, r) \in \mathcal{B} \) with \( r \geq 1 \), then
\[
|B|^{-1} \left( \frac{1}{|B|} \int_B |U(x)\psi(x)|^{q'} \, dx \right)^{1/q'} \leq \left( \frac{|B_0|}{|B|} \right)^{l + 1/q'} \leq C < \infty.
\]
Hence, to show (2.9) it is enough to estimate the integral of \( U\psi \) over balls \( B = B(x_0, r) \) with \( 0 < r < 1 \). Moreover, we can assume that \( B \cap B(0,1) \neq \varnothing \), since otherwise \( U\psi \) is 0 on \( B \). Consequently, we are limited to balls \( B \subset B_0 \). Let \( P_1 = \pi_{B}U \). By (2.2) and (2.10),
\[
\left( \frac{1}{|B|} \int_B |P_1(x)|^{q'} \, dx \right)^{1/q'} \leq C \left( \frac{1}{|B|} \int_B |U(x)|^{q'} \, dx \right)^{1/q'} \leq C_1 |B|^{-1/q'}.
\]
Let \( P_2(x) \) be the left Taylor polynomial of \( \psi \) at \( x_0 \) of homogenous degree \( s_0 \) (i.e., the polynomial whose left-invariant derivatives at the origin of homogeneous degree \( \leq s_0 \) agree with the corresponding derivatives of \( f \) at \( x_0 \)), where \( s_0 \in \Delta \) is chosen to satisfy \( s_0 \geq q(l + 1/q') \). By the Taylor inequality ([3, Theorem 1.37] and the remark following it), the remainder satisfies
\[
\left| \psi(x) - P_2(x^{-1}0) \right| \leq C_2|x^{-1}0|^{s_0} \quad \text{for} \quad x \in B \subset B(x_0, 1),
\]
with \( C_2 \) independent of \( x_0 \). Finally, let \( P(x) = P_1(x)P_2(x^{-1}0) \), which is a polynomial of homogeneous degree at most \( \tilde{s} = s + s_0 \). By (2.11),
\[
\left( \int_B |U(x)\psi(x) - P(x)|^q \, dx \right)^{1/q} \\
\leq \left( \int_B \left| \psi(x) - P_2(x^{-1}0) \right|^q \, dx \right)^{1/q} + \left( \int_B |P_1(x)|^q \, dx \right)^{1/q} \leq \left( \int_B \left| \psi(x) - P_2(x^{-1}0) \right|^q \, dx \right)^{1/q} + \sup_{x \in B} \left| \psi(x) - P_2(x^{-1}0) \right| \left( \int_B |P_1(x)|^q \, dx \right)^{1/q} \\
\leq |B|^{1/q'} + C_1C_2r^{r_0}.
\]
But \( r^{r_0} = C_3|B|^{s_0} = C_3|B|^{l+1/q} \), so (2.9) is proved with \( \tilde{s} = s + s_0 \).

**Lemma 2.3** Suppose \( u \in C_{q', s}^l \), where \( l \geq 0, 1 \leq q' \leq \infty \), and \( s = 0, 1, \ldots \). There exist \( \tilde{s} \geq s \), a constant \( C > 0 \) independent of \( u \), and a sequence of test functions \( \{u_k\}_{k \in \mathbb{N}} \subset S \) so that
\[
\|u_k\|_{C_{q', s}^l} \leq C \|u\|_{C_{q', s}^l} \quad \text{for all} \quad k \in \mathbb{N},
\]
\[
\lim_{k \to \infty} \int_{\mathbb{G}} f(x)u_k(x) \, dx = \int_{\mathbb{G}} f(x)u(x) \, dx \quad \text{for all} \quad f \in \Theta^q_{q'}, \quad 1/q + 1/q' = 1.
\]

**Proof.** First suppose \( u \in C_{q', s}^l \cap C^\infty \). Let \( \tilde{u}_k = D_{2-k}u \) and \( u_k = D_{2^k}((\tilde{u}_k - \pi_{B_0}\tilde{u}_k)\psi) \), where \( B_0 \) is as in Lemma 2.2. By (2.1) and (2.9),
\[
\|((\tilde{u}_k - \pi_{B_0}\tilde{u}_k)\psi)\|_{C_{q', s}^l} \leq C \|u_k\|_{C_{q', s}^l} \leq C 2^{kQ(l+1)}\|u\|_{C_{q', s}^l}.
\]

Therefore (2.12) holds, since
\[
\|u_k\|_{C_{q', s}^l} = 2^{-kQ(l+1)}\|((\tilde{u}_k - \pi_{B_0}\tilde{u}_k)\psi)\|_{C_{q', s}^l} \leq C \|u\|_{C_{q', s}^l}.
\]

Moreover,
\[
u_k(x) = u(x) - (D_{2^k} \circ \pi_{B_0} \circ D_{2-k})u(x) = u(x) - \pi_{B(0, 2^{k-1})}u(x) \quad \text{for} \quad x \in B(0, 2^k).
\]

Thus (2.13) also holds.

To end the proof we remove the assumption that \( u \in C^\infty \). Given \( u \in C_{q', s}^l \), define the sequence \( \{u_k\}_{k \in \mathbb{N}} \subset S \) by \( u_k = D_{2^k}((\tilde{u}_k - \pi_{B_0}\tilde{u}_k)\psi) \), where \( \tilde{u}_k = D_{2-k}(u \ast \varphi_k) \) with \( \varphi_k \) as in Lemma 2.1. Combining (2.8) and (2.14) yields (2.12), whereas (2.7) and (2.15) yield (2.13), completing the proof of Lemma 2.3.

**Lemma 2.4** Suppose that \( (p, q, s) \) is admissible and \( f \in \Theta^q_1 \), where \( \Theta^q_1 \) is given by (2.6). Suppose \( u \in C_{q', s}^l \), \( 1/q + 1/q' = 1 \), \( l = 1/p - 1 \). There exists \( \tilde{s} \geq s \) such that if \( f \) is decomposed into \( f = \sum_{i=1}^\infty \lambda_i a_i \), where \( \sum_{i=1}^\infty |\lambda_i|^p < \infty \) and the \( a_i \)’s are \( (p, q, \tilde{s}) \)-atoms, then
\[
\int f u = \sum_{i=1}^\infty \lambda_i \int a_i u.
\]
Proof. Let $a$ be a $(p, q, s)$-atom supported on a ball $B \in \mathcal{B}$ and $u \in C^{l}_{q', s}$. Since $\int u a = \int (u - P)a$ for all $P \in \mathcal{P}_{s}$ then by the standard calculation we have

$$\left| \int u a \right| = \inf_{P \in \mathcal{P}_{s}} \int (u - P)a \leq \left( \int_{B} |u|^{q} \right)^{1/q} \left( \inf_{P \in \mathcal{P}_{s}} \int |u - P|^{q'} \right)^{1/q'} \leq |B|^{1/q - 1/p} |B|^{(l + 1)/q'} \left( \frac{1}{|B|} \inf_{P \in \mathcal{P}_{s}} \int |u - P|^{q'} \right)^{1/q'} \leq \|u\|_{C^{l}_{q', s}}. \tag{2.17}$$

Next, suppose that $f \in \Theta_{q}^{l}$ is decomposed into $f = \sum_{i=1}^{\infty} \lambda_{i} a_{i}$, where $\sum_{i=1}^{\infty} |\lambda_{i}|^{p} < \infty$ and the $a_{i}$'s are $(p, q, s)$-atoms, where $s \geq s$ is the same as in Lemma 2.3. Suppose also that $u \in C^{l}_{q', s}$, $1/q + 1/q' = 1$, $l = 1/p - 1$. Let $(u_{k})_{k \in \mathbb{N}} \subset \mathcal{S}$ be the sequence guaranteed by Lemma 2.3. For every $k \in \mathbb{N}$ we have

$$\int f u_{k} = \sum_{i=1}^{\infty} \lambda_{i} \int a_{i} u_{k}, \tag{2.18}$$

since convergence in $H^{p}$ implies convergence in $\mathcal{S}'$ by [3, Proposition 2.15]. By (2.13)

$$\lim_{k \to \infty} \int_{G} a_{i}(x) u_{k}(x) dx = \int_{G} a_{i}(x) u(x) dx \quad \text{for all} \quad i \in \mathbb{N}. \tag{2.19}$$

By (2.12) and (2.17) we have $|\int u a_{i}| \leq \|u\|_{C^{l}_{q', s}} \leq C \|u\|_{C^{l}_{q', s'}}$. Since $\sum_{i=1}^{\infty} |\lambda_{i}| \leq (\sum_{i=1}^{\infty} |\lambda_{i}|^{p})^{1/p} < \infty$ we can take the limit as $k \to \infty$ in (2.18) by the dominated convergence theorem applied to counting measure on $\mathbb{N}$. This shows (2.16).

At last we are in a position to prove the duality theorem.

**Theorem 2.5** Suppose $(p, q, s)$ is admissible. Then

$$(H^{p}_{q, s})^{\ast} \cong C^{l}_{q', s}/\mathcal{P}_{s}, \quad \text{where} \quad 1/q + 1/q' = 1, \quad l = 1/p - 1. \tag{2.18}$$

More precisely, if $u \in C^{l}_{q', s}$ and $f$ is a finite linear combination of $(p, q, s)$-atoms, let $L_{u} f = \int u f$. Then $L_{u}$ extends continuously to $H^{p}_{q, s}$ and every $L \in (H^{p}_{q, s})^{\ast}$ is of this form. Moreover,

$$\|u\|_{C^{l}_{q', s}} = \|L_{u}\|_{(H^{p}_{q', s})^{\ast}} \quad \text{for all} \quad u \in C^{l}_{q', s}. \tag{2.19}$$

Proof. The fact that any bounded functional $L$ on $H^{p}_{q, s}$ must be of the form $L = L_{u}$ for some $u \in C^{l}_{q', s}$ was already shown in [3].

Conversely, suppose $u \in C^{l}_{q', s}$. Our goal is to demonstrate that the functional $L_{u} f = \int u f$ defined initially for $f \in \Theta_{q}^{l}$, where $\Theta_{q}^{l}$ is given by (2.6), extends to a bounded functional on $H^{p}_{q, s}$ and $\|L_{u}\|_{(H^{p}_{q, s})^{\ast}} \leq \|u\|_{C^{l}_{q', s}}$. We emphasize again that boundedness of $L_{u}$ on atoms (2.17) alone does not guarantee boundedness on the entire space.

Suppose that $f \in \Theta_{q}^{l}$. By [3, Theorem 3.30] we can find an atomic decomposition of $f = \sum_{i=1}^{\infty} \lambda_{i} a_{i}$, where

$$\left( \sum_{i=1}^{\infty} |\lambda_{i}|^{p} \right)^{1/p} \leq 2 \|f\|_{H^{p}_{q, s}} \leq C \|f\|_{H^{p}_{q, s}},$$

and the $a_{i}$'s are $(p, q, s)$-atoms. By (2.17) and Lemma 2.4

$$|L_{u} f| \leq \sum_{i=1}^{\infty} |\lambda_{i}| |L_{u} a_{i}| \leq \|u\|_{C^{l}_{q', s}} \left( \sum_{i=1}^{\infty} |\lambda_{i}|^{p} \right)^{1/p} \leq C \|u\|_{C^{l}_{q', s}} \|f\|_{H^{p}_{q, s}}.$$
Therefore, \( L_u \) extends uniquely to a bounded functional on \( H^p_{q,s} \). Next, we recall that the norm of a bounded functional on \( H^p_{q,s} \) is always achieved on atoms; see [3, Lemma 5.1], which holds under the assumption of continuity. Therefore, (2.17) implies \( \|L_u\|_{(H^p_{q,s})^*} \leq \|u\|_{C_q^r,s} \), which finishes the proof of Theorem 2.5. 

References