Baggett’s problem for frame wavelets

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1.1 Introduction

For a function $\psi \in L^2(\mathbb{R})$, we define its affine (or wavelet) system by

$$W(\psi) = \{ \psi_{j,k}(x) = 2^j \psi(2^j x - k) : j, k \in \mathbb{Z} \}$$

If the system is an orthonormal basis of $L^2(\mathbb{R})$, then we call $\psi$ a wavelet. In the more general case when the system forms a frame for $L^2(\mathbb{R})$, we call $\psi$ a frame wavelet, or simply a framelet. If $W(\psi)$ is a tight frame (with constant 1), i.e.,

$$\|f\|^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |(f, \psi_{j,k})|^2 \quad \text{for all } f \in L^2(\mathbb{R}),$$

then $\psi$ is a tight framelet and also called a Parseval wavelet.

One of the fundamental problems in the theory of wavelets is a problem posed by Baggett in 1999. Baggett’s problem asks whether every Parseval wavelet $\psi$ must necessarily come from a generalized multi-resolution analysis (GMRA). The precise meaning of this statement is explained later. Nonetheless, this problem can be reformulated in terms of the space of negative dilates of $\psi$ defined as

$$V(\psi) = \text{span}\{ \psi_{j,k} : j < 0, k \in \mathbb{Z} \}. \quad (1.1)$$

Question 1 (Baggett, 1999). Let $\psi$ be a Parseval wavelet with the space of negative dilates $V = V(\psi)$. Is it true that

$$\bigcap_{j \in \mathbb{Z}} D^j(V) = \{0\}?$$

Despite its simplicity Question 1 is a difficult open problem and only partial results are known. For example, Rzeszotnik and the author proved in [15] that if the dimension function (also called multiplicity function) of $V(\psi)$ is not identically $\infty$, then the answer to Question 1 is affirmative.
Question 1 is not only interesting for its own sake, but it also has several implications for other aspects of the wavelet theory. Rzeszotnik and the author [14] showed that a positive answer to Question 1 would imply that all compactly supported Parseval wavelets come from a MRA, thus generalizing the well-known result of Lemarié-Rieusset [1, 31] for compactly supported (orthonormal) wavelets. Furthermore, the answer to Question 1 would help in understanding the structure of the set of Parseval wavelets which was recently studied by Śkić, Speegle, and Weiss [37].

However, there is some evidence that the answer to Question 1 might be negative. This is because there exists a (non-tight) frame wavelet $\psi$ with a very large space of negative dilates. The first example of such $\psi$ was given by Rzeszotnik and the author in [14]. In fact, $\psi$ has a dual frame wavelet and the space of negative dilates of $\psi$ is the largest possible $V(\psi) = L^2(\mathbb{R})$. Here, we improve this result by showing that one can find such $\psi$ with good smoothness and decay properties, e.g., $\psi$ in the Schwartz class $\mathcal{S}(\mathbb{R})$.

1.2 Preliminaries

Despite the fact that all of our results are motivated by the classical case of dyadic dilations in $\mathbb{R}$ we will adopt a more general setting of an expansive integer-valued matrix, i.e., an $n \times n$ matrix whose eigenvalues have modulus greater than 1. That is, we shall assume that we are given an $n \times n$ expansive matrix $A$ with integer entries, which plays the role of the usual dyadic dilation. The dilation operator $D$ is given by $D\psi(x) = |\det A|^{1/2}\psi(Ax)$ and the translation operator $T_k$ is given by $T_k f(x) = f(x - k), k \in \mathbb{Z}^n$.

We say that a finite family $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is a wavelet if its associated affine system

$$\psi_{j,k} = D^j T_k \psi, \quad j \in \mathbb{Z}, k \in \mathbb{Z}^n, \psi \in \Psi$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. In the more general case, when the affine system is a frame or tight frame (with constant 1), we say that $\Psi$ is a frame wavelet or a Parseval wavelet, resp. Moreover, a frame wavelet $\Psi$ is called semi-orthogonal if

$$D^j W \perp D^{j'} W \quad \text{for all } j \neq j' \in \mathbb{Z}.$$  \hspace{1cm} (1.2)

where

$$W = W(\Psi) = \text{span}\{T_k \psi : k \in \mathbb{Z}^n, \psi \in \Psi\}.$$ 

The support of a function $f$ defined on $\mathbb{R}^n$ is denoted by

$$\text{supp } f = \{x \in \mathbb{R}^n : f(x) \neq 0\}.$$ 

Note that we are not taking the closure, since most of our functions are elements of $L^2(\mathbb{R}^n)$ and hence they are defined a.e. Given a Lebesgue measurable set $K \subset \mathbb{R}^n$, define the space
\[ L^2(K) = \{ f \in L^2(\mathbb{R}^n) : \text{supp} \hat{f} \subset K \}. \]

Here, the Fourier transform is defined by
\[
\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i (x,\xi)} \, dx.
\]

1.2.1 GMRAs

**Definition 1.** A sequence \( \{D^j(V)\}_{j \in \mathbb{Z}} \) of closed subspaces of \( L^2(\mathbb{R}^n) \) is called a generalized multiresolution analysis (GMRA) if

(1) \( T_k V = V \) for all \( k \in \mathbb{Z}^n \),

(2) \( V \subset D(V) \),

(3) \( \bigcup_{j \in \mathbb{Z}} D^j(V) = L^2(\mathbb{R}^n) \),

(4) \( \bigcap_{j \in \mathbb{Z}} D^j(V) = \{0\} \).

In addition, if (5) holds,

(5) \( \exists \varphi \in V \) such that \( \{T_k\varphi\}_{k \in \mathbb{Z}^n} \) is an orthonormal basis of \( V \),

then \( \{D^j(V)\}_{j \in \mathbb{Z}} \) is a multiresolution analysis (MRA).

A GMRA \( \{D^j(V)\}_{j \in \mathbb{Z}} \) is customarily written as \( \{V_j\}_{j \in \mathbb{Z}} \), where \( V_j = D^j(V) \). The space \( V \) is called the core space of the GMRA. Condition (1) means that \( V \) is shift-invariant (SI) and allows us to use the theory of shift-invariant spaces for understanding the connections between the GMRA structure and wavelets or framelets. This is a subject of an extensive study by several authors, e.g. [3, 4, 5, 7, 11, 13, 17, 29, 30].

For a family \( \Psi \subset L^2(\mathbb{R}^n) \) we define its space of negative dilates by

\[
V = V(\Psi) = \text{span}\{\psi_{j,k} : j < 0, k \in \mathbb{Z}^n, \psi \in \Psi\}. \tag{1.3}
\]

We say that a frame wavelet \( \Psi \) is associated with a GMRA, or shortly comes from a GMRA, if its space \( V = V(\Psi) \) satisfies (1)–(4). In addition, if \( V \) satisfies (5), then \( V \) is associated with an MRA.

It turns out that every semi-orthogonal frame wavelet \( \Psi \) comes from a GMRA. That is, the space \( V = V(\Psi) \) satisfies the conditions (1)–(4) and, therefore, \( V \) is a core space of a GMRA. This is an easy consequence of the fact that the spaces \( V \) and \( W \) given by (1.2) and (1.3) satisfy

\[
\bigoplus_{j \in \mathbb{Z}} D^j(W) = L^2(\mathbb{R}^n), \quad V = \bigoplus_{j \leq -1} D^j(W) = \left( \bigoplus_{j \geq 0} D^j(W) \right)^\perp. \tag{1.4}
\]

Conversely, if we want to see when a GMRA gives rise to a wavelet, or a semi-orthogonal frame wavelet, then some knowledge of shift-invariant spaces is useful.
1.2.2 The spectral function of shift-invariant spaces

Every shift-invariant space $V \subset L^2(\mathbb{R}^n)$ has a set of generators $\Phi$, that is, a countable family of functions whose integer shifts form a tight frame (with constant 1) for $V$, see [10, Theorem 3.3]. Although this family is not unique, the function

$$\sigma_V(\xi) = \sum_{\varphi \in \Phi} |\hat{\varphi}(\xi)|^2$$

does not depend (except on a set of null measure) on the choice of the family of generators. We call $\sigma_V$ the spectral function of $V$. This notion was introduced by Rzeszotnik and the author in [13]. The basic property of $\sigma$ is that it is additive on countable orthogonal sums of SI spaces and that $\sigma_{L^2(\mathbb{R}^n)} = 1$. The spectral function also behaves nicely under dilations since $\sigma_{D(V)}(\xi) = \sigma_V((A^T)^{-1}\xi)$. Moreover, if $V$ is generated by a single function $\varphi$ then

$$\sigma_V(\xi) = \begin{cases} |\hat{\varphi}(\xi)|^2 (\sum_{k \in \mathbb{Z}^n} |\hat{\varphi}(\xi + k)|^2)^{-1} & \text{for } \xi \in \text{supp} \hat{\varphi}, \\ 0 & \text{otherwise.} \end{cases}$$

We also mention that there are several other equivalent ways of defining the spectral function among which we note the following formula

$$\sigma_V(\xi) = \lim_{\varepsilon \to 0} \frac{|P_V(1_{(\xi-\varepsilon/2,\xi+\varepsilon/2)^n})|^2}{\varepsilon^n} \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

where $P_V$ denotes the orthogonal projection of $\mathcal{F}(V) = \hat{V}$ onto $L^2(\mathbb{R}^n)$.

The spectral function also allows us to define the dimension function of $V$

$$\dim_V(\xi) = \sum_{k \in \mathbb{Z}^n} \sigma_V(\xi + k).$$

The dimension function (also called the multiplicity function) takes values in $\mathbb{N} \cup \{0, \infty\}$. It is additive on countable orthogonal sums as the spectral function. Moreover, the minimal number of functions needed to generate $V$ is equal to the $L^\infty$ norm of $\dim_V$. In particular, $V$ can be generated by a single function if and only if $\dim_V \leq 1$. Moreover, condition (M5) is equivalent to the equation $\dim_V \equiv 1$. We refer the reader to [10, 13] for the proofs of all these facts.

1.2.3 Semi-orthogonal Parseval wavelets and GMRAs

The dimension function can be applied to connect GMRAs to semi-orthogonal Parseval wavelets. If $V$ is a core space of a GMRA, then the space $W = D(V) \ominus V$ is shift-invariant and has a (possibly infinite) set of generators $\Psi$. From (M2), (M3), and (M4) it follows that

$$L^2(\mathbb{R}^n) = \bigoplus_{j \in \mathbb{Z}} D^j(W),$$
so we conclude that $\Psi$ is a Parseval wavelet possibly of infinite order. That is, $\Psi$ may have infinite number of generators and the affine system generated by the elements of $\Psi$ forms a tight frame for $L^2(\mathbb{R}^n)$. Moreover, $\Psi$ is clearly semi-orthogonal.

Conversely, if $\Psi$ is a semi-orthogonal Parseval wavelet (possibly of infinite order), then the space $V$ of its negative dilates satisfies conditions (M1)–(M4) due to (1.4). Therefore, there is a perfect duality between GMRA structures and semi-orthogonal Parseval wavelets (with possibly infinite number of generators).

Since we are interested in finitely generated frame wavelets, the following result provides the required connection.

**Theorem 1.** Suppose that $\Psi$ is a semi-orthogonal Parseval wavelet with $L$ generators and $V$ is the space of negative dilates of $\Psi$. Then, $\{D^j(V)\}_{j \in \mathbb{Z}}$ is a GMRA such that

$$\dim_V(\xi) < \infty \quad \text{for a.e. } \xi,$$

and

$$\sum_{d \in \mathcal{D}} \dim_V((A^*)^{-1}(\xi + d)) - \dim_V(\xi) \leq L \quad \text{for a.e. } \xi,$$

where $\mathcal{D}$ consists of representatives of distinct cosets of $\mathbb{Z}^n/(A^*\mathbb{Z}^n)$.

Conversely, if $\{D^j(V)\}_{j \in \mathbb{Z}}$ is a GMRA satisfying (1.5) and (1.6), then there exists a a semi-orthogonal Parseval wavelet $\Psi$ (with at most $L$ generators) associated with this GMRA.

Theorem 1 is a variant of the following well-known result of Baggett et al. [4]. For simplicity we state Theorem 2 in a shorter form. Its full form looks analogously as Theorem 1.

**Theorem 2 (Baggett, Medina, Merrill, 1999).** A GMRA gives rise to a wavelet with $L$ generators if and only if the dimension function of its core space $V$ satisfies (1.5) and

$$\sum_{d \in \mathcal{D}} \dim_V((A^*)^{-1}(\xi + d)) - \dim_V(\xi) = L \quad \text{for a.e. } \xi.$$  

Equation (1.7) is often referred as the consistency equation of Baggett. In order to establish Theorem 1 we recall the following fact shown in [13].

**Lemma 1.** If $\Psi$ is a semi-orthogonal Parseval wavelet and $V$ is the space of negative dilates of $\Psi$, then

$$\sigma_V(\xi) = \sum_{\psi \in \Psi} \sum_{j = 1}^{\infty} |\hat{\psi}(A^*)^j\xi|^2.$$  

In particular,

$$\dim_V(\xi) = D_\Psi(\xi) \quad \text{for a.e. } \xi,$$
where

\[ D_\psi(\xi) := \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}^n} \sum_{j=1}^{\infty} |\hat{\psi}((A^*)^j(\xi + k))|^2. \] (8)

The function \( D_\psi \) is often referred to as the wavelet dimension function [1, 2, 16, 27, 35].

Proof (Theorem 1). Suppose that \( \Psi \) is a semi-orthogonal Parseval wavelet with \( L \) generators and the spaces \( W \) and \( V \) are given by (1.2) and (1.3). We already know that \( \{D^j(V)\}_{j \in \mathbb{Z}} \) is a GMRA. By Lemma 1,

\[
\int_{[0,1]^n} \dim_V(\xi) d\xi = \int_{\mathbb{R}^n} \sigma_V(\xi) d\xi = \sum_{\psi \in \Psi} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |\hat{\psi}((A^*)^j(\xi))|^2
\]

\[
= \sum_{\psi \in \Psi} \|\psi\|^2/(|\det A| - 1) \leq L/(|\det A| - 1) < \infty.
\] (9)

Hence, (1.5) holds. Since \( W \oplus V = D(V) \), we have

\[
\sigma_W(\xi) + \sigma_V(\xi) = \sigma_{D(V)}(\xi) = \sigma_V((A^*)^{-1}\xi).
\]

This implies that

\[
\dim_W(\xi) + \dim_V(\xi) = \sum_{d \in D} \dim_V((A^*)^{-1}(\xi + d)) \quad \text{for a.e. } \xi, \] (10)

where \( D \) consists of representatives of distinct cosets of \( \mathbb{Z}^n/(A^*\mathbb{Z}^n) \). Since \( \dim_W(\xi) \leq L \), (1.6) holds.

Conversely, let \( \{D^j(V)\}_{j \in \mathbb{Z}} \) be a GMRA satisfying (1.5) and (1.6). Let \( W = D(V) \cap V \). The consistency equation (1.10) and (1.6) yields

\[
\dim_W(\xi) \leq L \quad \text{for a.e. } \xi.
\]

By [10, Theorem 3.3] this implies that \( W \) has a set \( \Psi \) of \( \leq L \) generators. Since

\[
V = \bigoplus_{j \leq -1} D^j(W),
\]

we infer that \( \Psi \) is a semi-orthogonal Parseval wavelet associated with the GMRA \( \{D^j(V)\}_{j \in \mathbb{Z}} \).

1.3 Baggett’s problem for Parseval wavelets

Baggett posed the following open problem during his talk at Washington University in 1999.
Question 2 (Baggett, 1999). Is every Parseval wavelet Ψ associated with a GMRA?

For the sake of historical accuracy, one should add that Baggett actually attempted to answer affirmatively Question 2 during his momentous lecture. This has sparked the interest of two listeners, Rzeszotnik and the author, who pointed out a missing argument in Baggett’s approach. Despite several attempts in the next few years Question 2 remains unanswered as of now. Nonetheless, in his talk Baggett proved that Questions 1 and 2 are equivalent. Indeed, the following observation is due to Baggett.

Proposition 1 (Baggett, 1999). If Ψ is a Parseval wavelet, then its space of negative dilates V is shift-invariant.

Proof. It is enough to prove that the orthogonal complement $V^\perp$ of V is shift-invariant. It is clear that this complement is given by

$$V^\perp = \{ f \in L^2(\mathbb{R}^n) : ||f||_2^2 = \sum_{\psi \in \Psi} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 \}$$

by the tight frame property. Thus, we can see immediately that the space $V^\perp$ is shift–invariant. □

We remark that the above result also holds if we assume that the framelet Ψ has a canonical dual framelet with the same number of generators, or equivalently, that Ψ has period one in the terminology of Daubechies and Han [23]. However, Proposition 1 in general is false for non-tight framelets and even for framelets which have a dual framelet. These facts were shown by Weber and the author in [17].

Proposition 1 proves that the space of negative dilates of a Parseval wavelet Ψ satisfies condition (M1). The other two conditions, (M2) and (M3), are clearly satisfied leaving only (M4). This crucial obstacle leads naturally to Question 1. Consequently, Questions 1 and 2 are equivalent.

In general, one might want to know what conditions on a shift-invariant space V guarantee that

$$\bigcap_{j \in \mathbb{Z}} D^j(V) = \{0\}. \quad (1.11)$$

A non-trivial result of this type was shown by Rzeszotnik in [36].

Proposition 2 (Rzeszotnik, 2001). Let V be a shift-invariant space. If $\sigma_V \in L^1(\mathbb{R}^n)$, then condition (1.11) holds.

In the case when V is a space of negative dilates we have a stronger result due to Rzeszotnik and the author [15].
Theorem 3 (Bownik, Rzeszotnik, 2006). Let \( \Psi \subset L^2(\mathbb{R}^n) \) be a Parseval wavelet and \( V \) be its space of negative dilates. If

\[
|\{ \xi \in \mathbb{R}^n : \dim_V(\xi) < \infty \}| > 0,
\]

then (1.11) holds and \( \Psi \) generates a GMRA.

While the complete proof of Theorem 3 can be found in [15], we present its outline containing the key idea of semi-orthogonalization appearing later in the proof of Theorem 4. This procedure constructs a semi-orthogonal wavelet which is associated to the same GMRA as a given Parseval wavelet. In practice, it may not even be known whether a Parseval wavelet \( \Psi \), as in Theorem 3, is associated with a GMRA. Nevertheless, one can use the idea of semi-orthogonalization to eventually deduce this property.

Proof. Let \( W = D(V) \oplus V \). Observe that \( W \) is a shift-invariant space generated by \( \{ \psi - P_V \psi \}_{\psi \in \Psi} \), where \( P_V \) is the orthogonal projection on \( V \). Since \( \Psi \) is finite, \( W \) has a finite number of generators. That is, we have \( \dim_W \leq L \) for some \( L \in \mathbb{N} \). The equation \( D(V) = V \oplus W \) implies that

\[
\sum_{d \in D} m(B^{-1} \xi + d) = m(\xi) + \dim_W(\xi) \leq m(\xi) + L,
\]

where \( m = \dim_V \) and \( B = A^* \). To complete the proof we need the following result from [15].

Lemma 2. Suppose that \( m : \mathbb{R}^n \to [0, \infty) \) is \( \mathbb{Z}^n \)-periodic, measurable function such that

\[
\sum_{d \in D} m(\xi + d) \leq m(B\xi) + L \quad \text{for a.e. } \xi \in \mathbb{T}^n,
\]

for some \( L \geq 0 \). Then,

\[
\int_{\mathbb{T}^n} m(\xi)d\xi \leq L/(|\det A| - 1).
\]

To apply Lemma 2 we need to show that \( m \) is finite a.e. This can be done using a simple ergodic argument.

Since the matrix \( B = A^* : \mathbb{R}^n \to \mathbb{R}^n \) preserves the lattice \( \mathbb{Z}^n \), it induces a measure preserving endomorphism \( \hat{B} : \mathbb{T}^n \to \mathbb{T}^n \). Moreover, \( \hat{B} \) is ergodic by [38, Corollary 1.10.1] because \( B \) is expansive. Define the set

\[
E = \{ \xi \in \mathbb{T}^n : m(\xi) < \infty \}.
\]

The condition (1.13) implies that \( \hat{B}^{-1}E \subset E \). Since \( \hat{B} \) is measure preserving we must have \( \hat{B}^{-1}E = E \) (modulo null sets). Finally, by the ergodicity of \( \hat{B} \), we have either \( |E| = 0 \) or \( |E| = 1 \). Combining this with our hypothesis \( |E| > 0 \), proves that \( m(\xi) < \infty \) for a.e. \( \xi \in \mathbb{R}^n \).

Since all the assumptions of Lemma 2 are satisfied for our \( m \), we get that \( m \in L^1(\mathbb{T}^n) \). Equivalently, we have \( \sigma_V \in L^1(\mathbb{R}^n) \). By Proposition 2, (1.11) holds and \( \Psi \) generates a GMRA.
We end this section by mentioning an interesting variant of Baggett’s problem for single-generated Parseval wavelets [37].

Question 3 (Šikić, Speegle, and Weiss, 2007). Let $V$ be the space of negative dilates of a Parseval wavelet $\psi$. Is it true that

$$\psi \notin V.$$ (1.16)

Naturally, an affirmative answer to Question 1 implies a positive answer to Question 3. However, the converse implication is not known. Nonetheless, the following equivalent statements about a Parseval wavelet $\psi$ can be easily shown [37]:

(i) $\psi \in V$,
(ii) $V = DV$,
(iii) $V = L^2(\mathbb{R})$.

Once we relax the assumption that $\psi$ is a Parseval wavelet, then Questions 1 and 3 are distinct. In Theorem 7, we shall exhibit a frame wavelet $\psi$ such that $\psi \notin V$, but (1.11) fails.

1.4 Ramifications of Baggett’s problem

A positive answer to Baggett’s problem influences many other problems involving Parseval wavelets. The reason behind it is a semi-orthogonalization procedure which was introduced by Rzeszotnik and the author in [14].

Theorem 4. Suppose that $\Psi$ is a Parseval wavelet with $L$ generators and its space of negative dilates $V$ satisfies (1.11). Then, there exists a semi-orthogonal Parseval wavelet $\Phi$ with $\leq L$ generators such that its space of negative dilates is also $V$. In other words, both $\Psi$ and $\Phi$ are associated with the same GMRA $\{D^j(V)\}_{j \in \mathbb{Z}}$.

Proof. Let $V$ be the space of negative dilates of $\Psi$. By the hypothesis (1.11), the sequence $\{D^j(V)\}_{j \in \mathbb{Z}}$ is a GMRA. Let $W = D(V) \ominus V$. Observe that $W$ is generated by $L$ functions, namely $\psi - P_V \psi, \psi \in \Psi$, where $P_V$ is the orthogonal projection onto $V$. Therefore, we can find a set $\Phi$ of $\leq L$ generators for $W$. As in the proof of Theorem 1, we have

$$V = \bigoplus_{j \leq -1} D^j(W).$$

Hence, we can infer that $\Phi$ is a semi-orthogonal Parseval wavelet and $V$ is the space of negative dilates of $\Phi$. Therefore, $\Phi$ is associated to the same GMRA as $\Psi$. 
Remark 1. A more explicit semi-orthogonalization procedure for the subclass of MRA Parseval wavelets was introduced recently by Šikić et al. [37]. Suppose that $\psi \in L^2(\mathbb{R})$ is a dyadic Parseval wavelet associated with an MRA. Let $m$ be its generalized low-pass filter [32, 33, 37]. Then, the authors of [37] proved that one can modify the filter $m$ in some minimal way to obtain a new filter corresponding to a semi-orthogonal Parseval wavelet $\phi$ which is associated with the same MRA as $\psi$.

As a corollary of Theorems 1 and 4 we deduce that Parseval wavelets give rise to the same class of GMRAs as semi-orthogonal Parseval wavelets. A priori, this is only true for Parseval wavelets associated with a GMRA which may (or may not) encompass all Parseval wavelets depending on the answer to Question 2.

Corollary 1. Suppose that $\Psi$ is a Parseval wavelet with $L$ generators. Then, either $\{D^j(V)\}_{j \in \mathbb{Z}}$ is a GMRA satisfying (1.5) and (1.6), or $\dim_V \equiv \infty$.

Proof. If $\dim_V$ is not identically $\infty$, then $\{D^j(V)\}_{j \in \mathbb{Z}}$ is a GMRA by Theorem 3. Hence, Theorems 1 and 4 imply that (1.5) and (1.6) hold.

Next, we deduce that an affirmative answer to Baggett’s problem implies that a compactly supported Parseval wavelet comes from an MRA [14].

Theorem 5 (Bownik, Rzeszotnik, 2005). Let $\Psi$ be a Parseval wavelet with $L = |\det A| - 1$ generators such that its space of negative dilates $V$ satisfies condition (1.11). Then, $\Psi$ is associated with an MRA if and only if

$$D\Psi(\xi) = \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}} \sum_{j=1}^{\infty} |\hat{\psi}(\langle A^* \rangle^j(\xi + k))|^2 > 0 \quad \text{a.e.} \quad (1.17)$$

Remark 2. We recall that the restriction on the number of generators $L = |\det A| - 1$ in Theorem 5 is a necessary condition for (orthogonal) wavelet $\Psi$ to be associated with an MRA due to Lemma 1. In the case of Parseval wavelets it is possible to have MRA constructions resulting with bigger number of generators, see [20, 21, 24, 26, 34]. However, Theorem 5 is false if we relax the assumption $L = |\det A| - 1$.

Remark 3. We must emphasize that for general Parseval wavelets $D\Psi$ is not equal to $\dim_V$. This is unlike the case of semi-orthogonal wavelets, where Lemma 1 yields

$$D\Psi \equiv \dim_V. \quad (1.18)$$

Conversely, by the results of Paluszyński et al. [33] the identity (1.18) forces a Parseval wavelet $\Psi$ to be semi-orthogonal, see also [37, Theorem 3.15]. For the sake of accuracy, we should add that this result was shown only for dyadic, single generated, 1-dimensional Parseval wavelets.

Despite that (1.18) may fail we have that for any Parseval wavelet $\Psi$
see [14]. Indeed, by Proposition 1, V is a shift-invariant space generated by the functions
\[ \{ D^{-j} \psi : \psi \in \Psi, j = 1, 2, \ldots \}. \]
This, combined with an equivalent definition of the dimension function of shift-invariant spaces in terms of its range function, see [8, 10], yields
\[ \dim_V(\xi) = \dim \text{span} \{ \hat{\psi}( (A^*)^j (\xi + k))_{k \in \mathbb{Z}^n} : \psi \in \Psi, j = 1, 2, \ldots \}, \]
which shows (1.19).

Proof (Theorem 5). First, suppose that \( \Psi \) is associated with an MRA, i.e., its space of negative dilates satisfies \( \dim_V \equiv 1 \). By (1.18) we have that \( \supp D_\psi = \mathbb{R}^n \) and thus (1.17) holds.

Conversely, assume (1.17). We need to show that (M5) is satisfied, or equivalently that \( \dim_V \equiv 1 \). Let \( \Phi \) be the semi-orthogonal Parseval wavelet obtained from \( \Psi \) by Theorem 4. By Lemma 1 and the estimate (1.9) with \( \Phi \) taking place of \( \Psi \), we have
\[ \int_{[0,1]^n} \dim_V(\xi) d\xi = \sum_{\varphi \in \Phi} \| \hat{\varphi} \|^2/(|\det A| - 1) \leq L/(|\det A| - 1) = 1. \]
On the other hand, (1.17) and (1.18) imply that \( \dim_V(\xi) > 0 \) for a.e. \( \xi \). Since \( \dim_V \) is integer-valued we have that \( \dim_V \equiv 1 \), which concludes the proof of
Theorem 5. \( \square \)

As a corollary of Theorem 5 we have the following extension of a result of Lemarié-Rieusset [31] to Parseval wavelets.

**Corollary 2 (Bownik, Rzeszotnik, 2005).** Suppose that a Parseval wavelet \( \Psi \) satisfies the assumptions of Theorem 5 and at least one generator of \( \Psi \) is compactly supported. Then, \( \Psi \) is associated with an MRA.

Combining Corollary 2 with Theorem 3, we have the following corollary.

**Corollary 3.** Suppose that a Parseval wavelet \( \Psi \) has \( L = |\det A| - 1 \) generators and at least one of them is compactly supported. If the space \( V \) of negative dilates of \( \Psi \) satisfies (1.12), then \( \Psi \) comes from an MRA.

### 1.5 Frame wavelets with large spaces of negative dilates

In this section we prove that the assumption in Question 1 on \( \psi \) being a Parseval wavelet is necessary. This result is due to Rzeszotnik and the author [14] who constructed an example of a dyadic framelet \( \psi \in L^2(\mathbb{R}) \), such that its space of negative dilates \( V \) is the largest possible, i.e., \( V = L^2(\mathbb{R}) \). Furthermore, such a framelet can have frame bounds arbitrarily close to 1 and it has a dual framelet. Here, we shall improve the example in [14] by showing that such a framelet can also have good smoothness and decay properties.
Theorem 6. For any \( \delta > 0 \), there exists a frame wavelet \( \psi \in L^2(\mathbb{R}) \) such that:

(i) \( \hat{\psi} \) is \( C^\infty \) and all its derivatives have exponential decay,
(ii) the frame bounds of \( \mathcal{W}(\psi) \) are 1 and \( 1 + \delta \),
(iii) the space of negative dilates of \( \psi \) is equal to \( L^2(\mathbb{R}) \),
(iv) \( \psi \) has a dual frame wavelet.

While the proof of Theorem 6 follows the general construction method of [14], there are also some significant changes due to the additional smoothness requirement on \( \psi \). In the proof of Theorem 6 we will use the following two standard results. Lemma 3 gives a sufficient condition for an affine system to be a Bessel sequence. Its proof can be found in [28, Theorem 13.0.1]. Lemma 4 is a basic perturbation result for frames which can be found in [19, Corollary 15.1.5].

Lemma 3. Suppose that \( \psi \in L^2(\mathbb{R}) \) is such that \( \hat{\psi} \in L^\infty(\mathbb{R}) \) and

\[
\hat{\psi}(\xi) = O(|\xi|^\delta) \quad \text{as} \quad \xi \to 0, \tag{1.20}
\]

\[
\hat{\psi}(\xi) = O(|\xi|^{-1/2-\delta}) \quad \text{as} \quad |\xi| \to \infty, \tag{1.21}
\]

for some \( \delta > 0 \). Then the affine system \( \mathcal{W}(\psi) \) is a Bessel sequence.

Lemma 4. Suppose that \( \mathcal{H} \) is a Hilbert space, \( \{f_j\} \subset \mathcal{H} \) is a frame with constants \( C_1 \) and \( C_2 \),

\[
C_1 ||f||^2 \leq \sum_j |(f, f_j)|^2 \leq C_2 ||f||^2 \quad \text{for all} \quad f \in \mathcal{H},
\]

and \( \{g_j\} \subset \mathcal{H} \) is a Bessel sequence with constant \( C_0 \),

\[
\sum_j |(f, g_j)|^2 \leq C_0 ||f||^2 \quad \text{for all} \quad f \in \mathcal{H}.
\]

If \( C_0 < C_1 \), then \( \{f_j + g_j\} \) is a frame with constants \( ((C_1)^{1/2} - (C_0)^{1/2})^2 \) and \( ((C_2)^{1/2} + (C_0)^{1/2})^2 \).

We will also need the following fact about the scale averaging of periodic functions. Lemma 5 can be considered as a special case of a result due to Bui and Laugesen [18, Lemma 9] which also holds for functions in \( L^p_{\text{loc}}(\mathbb{R}^n) \) and fairly general dilation matrices. This result is very close in spirit to the classical results of Banach-Saks and Szlenk asserting that weak convergence in \( L^p \) implies norm convergence of arithmetic means. Since we impose weaker assumptions on \( \Psi \) than in [18], we present the proof of Lemma 5 for completeness.
Lemma 5. Suppose $\Psi \in L^2(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In other words, $\Psi$ is a 1-periodic function in $L^2_{\text{loc}}(\mathbb{R})$. Let $\Psi_j(x) = \Psi(2^j x)$, and $c = \int_0^1 \Psi$. Then, for any strictly increasing sequence $(I_j)_{j \in \mathbb{N}} \subset \mathbb{N}$,

$$\lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^J \Psi_{I_j} = c \quad \text{in } L^2_{\text{loc}}(\mathbb{R}).$$

(1.23)

Proof. Without loss of generality we can assume that $c = \int_0^1 \Psi = 0$. Otherwise, it suffices to apply (1.23) for a function $\Psi - c$. For the purpose of Lemma 5, let $||\Psi|| = (\int_0^1 |\Psi|^2)^{1/2}$ be the norm in $L^2(\mathbb{T})$ with the corresponding scalar product $\langle \cdot , \cdot \rangle$.

We claim that the sequence $(\Psi_j)$ converges to $c$ weakly in $L^2(\mathbb{T})$. Indeed, let $f$ be 1-periodic and continuous. Take any $\varepsilon > 0$ and choose $j \in \mathbb{N}$ such that $|x - y| \leq 2^{-j} \implies |f(x) - f(y)| \leq \varepsilon$. Since

$$\int_{2^{j+1}/2^j}^{2^j} \Psi(2^j x) dx = 0$$

we have

$$\left| \int_0^1 \Psi_j f \right| = \left| \sum_{k=0}^{2^j-1} \int_{2^j(k+1)/2^j}^{2^j(k+1)/2^j} \Psi(2^j x)(f(x) - f(k/2^j)) dx \right| \leq \varepsilon \int_0^1 |\Psi|.$$

A standard approximation argument using Luzin’s theorem and $||\Psi_j|| = ||\Psi||$, shows the claim. In particular, we have that

$$d_j := ||\langle \Psi, \Psi_j \rangle|| \to 0 \quad \text{as } j \to \infty. \quad (1.24)$$

For any $j \leq k \in \mathbb{N}$, the change of variables and 1-periodicity of $\Psi$ yields

$$|\langle \Psi_j, \Psi_k \rangle| = |\langle \Psi, \Psi_{k-j} \rangle| = d_{k-j}.$$

Thus, we have the estimate

$$||\Psi_{I_1} + \ldots + \Psi_{I_j}||^2 = \left| \sum_{j=1}^J \sum_{k=1}^J \langle \Psi_{I_j}, \Psi_{I_k} \rangle \right| \leq \sum_{j=1}^J \sum_{k=1}^J d_{|I_j-I_k|} \leq 2J \sum_{j=0}^{J-1} d_j.$$

Here, we used $d_{|I_j-I_k|} \leq d^*_{|I_j-I_k|}$, where $d^*_j = \sup\{d_k : k \geq j\}$ is a decreasing sequence dominating $(d_j)$. Hence, by (1.24)

$$\left\| \frac{\Psi_{I_1} + \ldots + \Psi_{I_j}}{J} \right\|^2 \leq \frac{2J}{J} \sum_{j=0}^{J-1} d^*_j \to 0 \quad \text{as } J \to \infty.$$

This shows (1.23) and completes the proof of Lemma 5.
Proof (Theorem 6). Define the sets $Z_1, Z_2$ by

$$Z_1 = \bigcup_{k \in \mathbb{Z}} (k + (-1/4, 1/4)),$$
$$Z_2 = \mathbb{R} \setminus Z_1.$$

Suppose that $\psi^0 = \psi^1 + \psi^2$, where $\psi^1 \in L^2(Z_1)$ and $\psi^2 \in L^2(Z_2)$. As usual, define

$$W_j^l = \text{span}\{\psi_{j,k}^l : k \in \mathbb{Z}\} \quad \text{for } l = 0, 1, 2.$$

Lemma 6.

$$W_j^0 = W_j^1 \oplus W_j^2 \quad \text{for } j \in \mathbb{Z}. \quad (1.25)$$

Proof. It suffices to show (1.25) for $j = 0$. Take any $f \in W_0^1$ and $g \in W_0^2$. By the results in [8, 10] we have

$$W_0^j = \{f \in L^2 : \hat{f}(\xi) = m(\xi)\hat{\psi}^j(\xi), \quad m \text{ is measurable and 1-periodic}\}. \quad (1.26)$$

Since supp $\hat{f} \subset Z_1$, supp $\hat{g} \subset Z_2$ we have $f \perp g$. Thus, $W_0^1 \perp W_0^2$. Finally, it suffices to prove $W_0^1 \oplus W_0^2 \subset W_0^0$, since the converse inclusion is trivial. Take any $f \in W_0^1 \oplus W_0^2$. By (1.26) there are 1-periodic measurable functions $m_1$ and $m_2$ such that

$$\hat{f}(\xi) = m_1(\xi)\hat{\psi}^1(\xi) + m_2(\xi)\hat{\psi}^2(\xi) = m_1(\xi)1_{Z_1}(\xi)\hat{\psi}^0(\xi) + m_2(\xi)1_{Z_2}(\xi)\hat{\psi}^0(\xi). \quad (1.27)$$

Since the sets $Z_1$ and $Z_2$ are invariant under integer shifts, $m = m_11_{Z_1} + m_21_{Z_2}$ is 1-periodic. Hence, by (1.26) and (1.27) $f \in W_0^0$, which shows $W_0^0 = W_0^1 \oplus W_0^2$.

It now remains to choose $\psi^1$ and $\psi^2$ appropriately. The idea is that negative dilates of $\psi^1$ will generate functions whose Fourier transform is supported near the origin, whereas the negative dilates of $\psi^2$ will exhaust all functions which are supported away from the origin (in the Fourier domain).

Let $\psi^1$ be a Parseval wavelet such that $\hat{\psi}^1$ is $C^\infty$ and

$$\text{supp} \hat{\psi}^1 = (-1/4, -1/16) \cup (1/16, 1/4).$$

Such a frame wavelet can be constructed by a standard method, for example see [12]. Indeed, it suffices to take the convolution of $1_{(-3/16, -1/8)} \cup (1/8, 3/16)$ with a non-negative smooth bump function supported on $(-1/16, 1/16)$ and normalize the result to obtain the Calderón condition

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}^1(2^j \xi)|^2 = 1 \quad \text{for } \xi \in \mathbb{R} \setminus \{0\}.$$

Note that $\psi^1 \in L^2(Z_1)$ and by (1.26), $W_0^1 = L^2((-1/4, -1/16) \cup (1/16, 1/4))$. Hence,
\[ W_j^1 = L^2((-2j^{-2}, -2j^{-4}) \cup (2j^{-4}, 2j^{-2})) \quad \text{for any } j \in \mathbb{Z}, \]

and therefore, the space of negative dilates of \( \psi^1 \) is

\[
V^1 = \operatorname{span} \bigcup_{j < 0} W_j^1 = \tilde{L}^2 \left( \bigcup_{j = -\infty}^{-1} (-2j^{-2}, -2j^{-4}) \cup (2j^{-4}, 2j^{-2}) \right) = \tilde{L}^2(-1/8, 1/8). \tag{1.28}
\]

The function \( \psi^2 \) should be regarded as a perturbation term of \( \psi^0 = \psi^1 + \psi^2 \).

We are now ready to describe the construction procedure of \( \psi^2 \).

Let \( \{ \varphi_m : m \in \mathbb{N} \} \) be some enumeration of the “truncated” Gabor system

\[
\{ 1_{(k,k+1)} e^{2\pi i j \xi} : j \in \mathbb{Z}, k \in \mathbb{Z}, k \neq -1, 0 \}.
\]

Clearly, \( \{ \varphi_m : m \in \mathbb{N} \} \) is an orthonormal basis of \( L^2((-\infty, -1) \cup (1, \infty)) \). For any \( m \in \mathbb{N} \), let \( k_m \in \mathbb{Z} \) denote the left endpoint of the support of \( \varphi_m \), i.e.,

\[
\operatorname{supp} \varphi_m = (k_m, k_m + 1).
\]

Let \( \Psi \) be a 1-periodic function such that

\[
\Psi \in C^\infty, \quad \operatorname{supp} \Psi \subset \mathbb{Z}_2, \quad \int_0^1 \Psi = 1. \tag{1.29}
\]

Let \( (m_p)_{p \in \mathbb{N}} \) be a sequence of natural numbers such that each natural number occurs infinitely many times. We construct by induction a sequence of functions \( \{ \phi_p : p \in \mathbb{N} \} \) and a sequence of natural numbers \( (l_p)_{p \in \mathbb{N}} \).

Let \( \phi_1 = D^{-l_1}(\varphi_{m_1})\Psi \) and \( l_1 = 1 \). Suppose we have constructed functions \( \phi_1, \ldots, \phi_p \) and integers \( l_1, \ldots, l_p \) up to some \( p \in \mathbb{N} \). Define \( l_{p+1} \) to be the smallest integer such that

\[
\operatorname{supp} \phi_1 \cup \ldots \cup \operatorname{supp} \phi_p \subset (-2^{l_{p+1}}, 2^{l_{p+1}}), \tag{1.30}
\]

and

\[
\phi_{p+1} = D^{-l_{p+1}}(\varphi_{m_{p+1}})\Psi. \tag{1.31}
\]

It is easy to see that the sequence \( (l_p)_{p \in \mathbb{N}} \) is increasing and the supports of \( \phi_p \)'s are included in pairwise disjoint open intervals. Finally, define \( \tilde{\psi}^2 \in \tilde{L}^2(\mathbb{Z}_2) \) by

\[
\tilde{\psi}^2(\xi) = \sum_{p \in \mathbb{N}} c_p \phi_p(\xi) = \sum_{p \in \mathbb{N}} c_p D^{-l_p}(\varphi_{m_p})\Psi, \tag{1.32}
\]

for some sufficiently fast decaying sequence \( (c_p)_{p \in \mathbb{N}} \) of positive numbers. More precisely, we can choose \( c_p \)'s such that \( 0 < c_{p+1} < c_p/(p + 1) \) for all \( p \in \mathbb{N} \) and all derivatives of \( \tilde{\psi}^2 \) have exponential decay. This will guarantee that \( \psi^0 = \psi^1 + \psi^2 \) satisfies property (i) of Theorem 6. In particular, by Lemma 3, the affine system generated by \( \psi^2 \) is a Bessel sequence.

Our next goal is to show the following key fact.
Lemma 7. Suppose that $\psi^2$ given by (1.32) is constructed as above. Let $V^2$ be the space of negative dilates of $\psi^2$ and $P$ be the orthogonal projection on $L^2((-\infty, -1) \cup (1, \infty))$, i.e.,

$$\widehat{(Pf)}(\xi) = \hat{f}(\xi)1_{(-\infty, -1) \cup (1, \infty)} \quad \text{for } f \in L^2(\mathbb{R}).$$

Then, $P(V^2)$ is dense in $L^2((-\infty, -1) \cup (1, \infty)).$

Proof. Since

$$\tilde{V}^2 := \text{span}\{\psi^2_{\ell p, 0} : p \in \mathbb{N}\} \subset V^2$$

it suffices to show that $P(\tilde{V}^2)$ is dense in $L^2((-\infty, -1) \cup (1, \infty))$. Hence, we need to show that each basis element $\varphi_m, m \in \mathbb{N},$ of $L^2((-\infty, -1) \cup (1, \infty))$ belongs to the closure of $\mathcal{F}(P(\tilde{V}^2))$. Given $r \in \mathbb{N},$

$$\psi^2_{\ell r, 0} = D^r(\psi^2) = \sum_{p \in \mathbb{N}} c_p D^r(\phi_p).$$

By (1.30) and (1.31), supp $D^r(\phi_p) \subset (-1, 1)$ for $p < r$, and we have

$$(P(\psi^2_{\ell r, 0})) = \sum_{p \geq r} c_p D^r(\phi_p) = \sum_{p \geq r} c_p D^r(\varphi_{m_p}) \Psi_r = c_r \Psi_r \left[ \varphi_{m_r} + \sum_{p > r} \frac{c_p}{c_r} D^r(\varphi_{m_p}) \right].$$

Since $c_{r+1}/c_r < 1/(r + 1),$

$$\left| \sum_{p > r} \frac{c_p}{c_r} D^r(\varphi_{m_p}) \right| \leq \sum_{p \geq r} \frac{1}{(r + 1)(r + 2) \ldots p} ||D^r(\varphi_{m_p})|| < 2/r,$$

we conclude that $\Psi_r(\varphi_{m_r} + \eta_r)$ belongs to $\mathcal{F}(P(\tilde{V}^2))$ for some $\eta_r \in L^2$ with $||\eta_r|| < 2/r$. For a fixed $m \in \mathbb{N}$, let $R = \{r \in \mathbb{N} : m_r = m\}$. By our construction $R = \{r_1, r_2, \ldots\}$ is infinite. By Lemma 5

$$\Psi_{r_1} + \ldots + \Psi_{r_j} \rightarrow 1 \quad \text{as } J \rightarrow \infty \quad \text{in } L^2(k_m, k_m + 1).$$

Hence, as $J \rightarrow \infty$

$$\frac{\Psi_{r_1}(\varphi_{m_r} + \eta_{r_1}) + \ldots + \Psi_{r_j}(\varphi_{m_r} + \eta_{r_j})}{J} = \varphi_m \frac{\Psi_{r_1} + \ldots + \Psi_{r_j}}{J} \Psi_{r_1} \eta_{r_1} + \ldots + \Psi_{r_j} \eta_{r_j} \rightarrow \varphi_m \quad \text{in } L^2(\mathbb{R}),$$

since $||\Psi||_\infty = ||\Psi||_\infty < \infty$ and
Therefore, \( \varphi_m \) belongs to the closure of \( \mathcal{F}(P(V^2)) \). Since \( m \in \mathbb{N} \) is arbitrary and \( \{\varphi_m : m \in \mathbb{N}\} \) is an orthonormal basis of \( L^2((-\infty, -1) \cup (1, \infty)) \), this completes the proof of Lemma 7.

**Lemma 8.** Suppose that \( V^2 \) is the same as in Lemma 7. Let \( P_j \) be the orthogonal projection onto \( L^2((-\infty, -2^j) \cup (2^j, \infty)) \), i.e.,

\[
\widehat{(P_j f)}(\xi) = \hat{f}(\xi)1_{(-\infty, -2^j) \cup (2^j, \infty)} \quad \text{for } f \in L^2(\mathbb{R}).
\]

Then, \( P_j(V^2) \) is dense in \( \dot{L}^2((-\infty, -2^j) \cup (2^j, \infty)) \) for any \( j \in \mathbb{Z} \).

**Proof.** Since the case \( j \geq 0 \) follows immediately from Lemma 7, we may assume that \( j < 0 \). A straightforward calculation shows that \( P_j = D^jPD^{-j} \). Take any \( f \in L^2((-\infty, -2^j) \cup (2^j, \infty)) \). Since \( D^{-j}f \in L^2((-\infty, -1) \cup (1, \infty)) \), by Lemma 7 there exists a sequence \( \{f_k : k \in \mathbb{N}\} \subset V^2 \) such that \( P_0f_k \to D^{-j}f \) as \( k \to \infty \). Hence, \( P_jD^jf_k \to f \) as \( k \to \infty \). Since \( D^jf_k \in V^2 \) for \( j \leq 0 \), this shows Lemma 8.

We are now ready to conclude the proof of Theorem 6. Let \( V^0 \) be the space of negative dilates of \( \psi^0 \). By (1.25),

\[
V^0 = \text{span} \left( \bigcup_{j<0} W^0_j \right) = \text{span} \left( \bigcup_{j<0} (W^1_j \cup W^2_j) \right) = \text{span}(V^1 \cup V^2)。
\]

Therefore, by (1.28) and by Lemma 8

\[
P_{-1}(V^2) = \dot{L}^2((-\infty, -1/8) \cup (1/8, \infty))
\]

we have that \( V^0 \) is dense in \( L^2(\mathbb{R}) \). Since \( V^0 \) is closed it must be equal to \( L^2(\mathbb{R}) \).

It remains to show that one can also find a framelet with this property.

Recall that \( \psi^0 = \psi^1 + \psi^2 \), where \( \psi^1 \) is a Parseval wavelet and \( \psi^2 \) generates a Bessel affine system. Therefore, by Lemma 4, there exists \( \varepsilon > 0 \) such that \( \psi' = \psi^1 + \varepsilon \psi^2 \) is a framelet with frame bounds \( 1 - \delta/3 \) and \( 1 + \delta/3 \). Moreover, since \( \varepsilon \psi^2 \) is also of the form (1.32), the space of negative dilates of \( \psi' \) is also \( L^2(\mathbb{R}) \). Therefore, \( \psi = (1 - \delta/3)^{-1/2} \psi' \) is a framelet with constants \( 1 \) and \( 1 + \delta \) whose space of negative dilates is \( L^2(\mathbb{R}) \). In fact, a more delicate argument shows that the lower frame bound of \( \psi' \) is \( \geq 1 \) and the last normalization step is not necessary.

Finally, to show that \( \psi \) has a dual frame wavelet we employ the well-known characterizing equations [9, 25, 27]. We recall that functions \( \phi, \psi \in L^2(\mathbb{R}) \) whose respective affine systems are Bessel sequences form a pair of dual framelets if and only if

\[
\sum_{j \in \mathbb{Z}} \overline{\hat{\phi}(2^j \xi)} \hat{\psi}(2^j \xi) = 1 \quad \text{a.e. } \xi,
\]
Thus, using \( \text{supp} \hat{\psi}^i \subset \mathbb{Z}_i, \ i = 1, 2 \), one can show that \( \phi = (1 - \delta/3)^{1/2}\psi^1 \) is a dual framelet to \( \psi = (1 - \delta/3)^{-1/2}(\psi^1 + \varepsilon \psi^2) \). This completes the proof of Theorem 6.

We finish this section by showing that the affirmative answer to Question 3 does not imply a positive answer to Question 1 for general frame wavelets \( \psi \).

**Theorem 7.** For any \( \delta > 0 \), there exists a frame wavelet \( \psi \in L^2(\mathbb{R}) \) such that:

(i) \( \hat{\psi} \) is \( C^\infty \) and all its derivatives have exponential decay,
(ii) the frame bounds of \( \mathcal{W}(\psi) \) are 1 and \( 1 + \delta \),
(iii) the space \( V \) of negative dilates of \( \psi \) satisfies \( \bigcap_{j \in \mathbb{Z}} D^j(V) \neq \{0\} \),
(iv) \( \psi \notin V \),
(v) \( \psi \) has a dual frame wavelet.

**Proof.** Let \( \psi_1 \) and \( \psi_2 \) be the same as in the proof of Theorem 6. Then, a frame wavelet constructed by Theorem 6 is of the form \( \psi' = c_1 \psi^1 + c_2 \psi^2 \) for some constants \( c_1, c_2 \).

Define a function \( \psi = c_1 \psi^1 + c_2 \psi_+ \), where \( \psi_+ \) is given by \( \hat{\psi}_+ = \hat{\psi}1_{(0,\infty)} \). We claim that \( \psi \) satisfies all properties of Theorem 7. Indeed, (i) is trivial. The property (ii) follows from the same perturbation argument as in Theorem 6. Likewise, the same argument as in Theorem 6 shows that the space of negative dilates \( V \) satisfies \( L^2(0,\infty) \subset V \). This is mainly due to the decomposition

\[
L^2(\mathbb{R}) = H_+^2(\mathbb{R}) \oplus H_-^2(\mathbb{R}), \quad \text{where} \quad H_+^2(\mathbb{R}) = \dot{L}^2(-\infty,0), \quad H_-^2(\mathbb{R}) = \dot{L}^2(0,\infty),
\]

and the fact that Hardy spaces \( H_+^2(\mathbb{R}) \) and \( H_-^2(\mathbb{R}) \) are invariant under the action of \( D \) and \( T_k \). On the other hand, it is clear that \( V \neq L^2(\mathbb{R}) \) and hence (iv) holds. Finally, (v) is shown exactly in the same way as in Theorem 6 with \( \phi = (c_1)^{-1}\psi_1 \) being a dual framelet to \( \psi \).

**References**