



# Constructive Proof of the Carpenter's Theorem

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*Abstract.* We give a constructive proof of the carpenter's theorem due to Kadison. Unlike the original proof, our approach also yields the real case of this theorem.

## 1 Kadison's Theorem

In [14] and [15] Kadison gave a complete characterization of the diagonals of orthogonal projections on a Hilbert space  $\mathcal{H}$ .

**Theorem 1.1** (Kadison) *Let  $\{d_i\}_{i \in I}$  be a sequence in  $[0, 1]$ . Define*

$$a = \sum_{d_i < 1/2} d_i \quad \text{and} \quad b = \sum_{d_i \geq 1/2} (1 - d_i).$$

*There exists a projection  $P$  with diagonal  $\{d_i\}$  if and only if one of the following holds:*

- (i)  $a, b < \infty$  and  $a - b \in \mathbb{Z}$ ,
- (ii)  $a = \infty$  or  $b = \infty$ .

The goal of this paper is to give a constructive proof of the sufficiency direction of Kadison's theorem. Kadison [14, 15] referred to the necessity part of Theorem 1.1 as the Pythagorean Theorem and the sufficiency as the carpenter's theorem. Arveson [3] gave a necessary condition on the diagonals of a certain class of normal operators with finite spectrum. When specialized to the case of two point spectrum Arveson's theorem yields the Pythagorean Theorem, *i.e.*, the necessity of (i) or (ii) in Theorem 1.1. Whereas Kadison's original proof is a beautiful direct argument, Arveson's proof uses the Fredholm Index Theory.

In contrast, until very recently there were no proofs of the carpenter's theorem other than the original one by Kadison, although its extension for  $\text{II}_1$  factors was studied by Argerami and Massey [2]. A notable exception is a recent paper by Argerami [1] about which we became aware only after completing this work. In this paper we give an alternative proof of the carpenter's theorem which has two main advantages over the original. First, the original proof does not yield the real case,

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which ours does. Second, our proof is constructive in the sense that it gives a concrete algorithmic process for finding the desired projection. This is distinct from Kadison's original proof, which is mostly existential.

The paper is organized as follows. In Section 2, we state preliminary results, such as a finite rank version of Horn's theorem. These results are then used in Section 3 to show the sufficiency of (i) in Theorem 1.1. The key role in the proof is played by a lemma from [8] that enables modifications of diagonal sequences into more favorable configurations. Section 4 contains the proof of sufficiency of (ii) in Theorem 1.1. To this end we introduce an algorithmic procedure for constructing a projection with prescribed diagonal that is reminiscent of the spectral tetris construction introduced by Casazza *et al.* [10] in their study of tight fusion frames. Finally, in Section 5 we formulate an open problem of characterizing spectral functions of shift-invariant spaces in  $L^2(\mathbb{R}^d)$ , introduced by the first author and Rzesotnik in [9], which was a motivating force behind this paper.

## 2 Preliminary Results

The main goal of this section is to give a constructive proof of Horn's Theorem [18, Theorem 9.B.2], which is the sufficiency part of the Schur–Horn Theorem [13, 21]. We present this proof both for the sake of self-sufficiency of part (i) of the carpenter's theorem and also to cover the more general case of finite rank operators on an infinite dimensional Hilbert space, see also [4, 16, 17]. Moreover, we also give an argument reducing Theorem 1.1 to the countable case.

**Theorem 2.1** (Horn's Theorem) *Let  $\{\lambda_i\}_{i=1}^N$  be a positive nonincreasing sequence, and let  $\{d_i\}_{i=1}^M$  be a nonnegative nonincreasing sequence, where  $M \in \mathbb{N} \cup \{\infty\}$  and  $M \geq N$ . If*

$$(2.1) \quad \sum_{i=1}^n d_i \leq \sum_{i=1}^n \lambda_i \quad \text{for all } n \leq N, \quad \sum_{i=1}^M d_i = \sum_{i=1}^N \lambda_i,$$

*then there is a positive rank  $N$  operator  $S$  on a real  $M$ -dimensional Hilbert space  $\mathcal{H}$  with positive eigenvalues  $\{\lambda_i\}_{i=1}^N$  and diagonal  $\{d_i\}_{i=1}^M$ .*

We need a basic lemma.

**Lemma 2.2** *Let  $M \in \mathbb{N} \cup \{\infty\}$ . If  $\{d_i\}_{i=1}^M$  is a nonzero nonnegative sequence with  $\sum_{i=1}^M d_i = \lambda < \infty$ , then there is a positive rank 1 operator  $S$  on an  $M$ -dimensional Hilbert space  $\mathcal{H}$  with eigenvalue  $\lambda$  and diagonal  $\{d_i\}$ .*

**Proof** Let  $\{e_i\}_{i=1}^M$  be an orthonormal basis for the Hilbert space  $\mathcal{H}$ . For each  $f \in \mathcal{H}$ , define  $S: \mathcal{H} \rightarrow \mathcal{H}$  by  $Sf = \langle f, v \rangle v$ , where  $v = \sum_{i=1}^M \sqrt{d_i} e_i$ . Clearly  $S$  is rank 1, and since  $\|v\|^2 = \lambda$ , the vector  $v$  is an eigenvector with eigenvalue  $\lambda$ . Finally, it is simple to check that  $S$  has the desired diagonal. ■

**Proof of Theorem 2.1** The proof proceeds by induction on  $N$ . The base case  $N = 1$  follows from Lemma 2.2. Suppose that Theorem 2.1 holds for ranks up to  $N - 1$ .

Define

$$m_0 = \max \left\{ m : \sum_{i=m}^M d_i \geq \lambda_N \right\}$$

and

$$(2.2) \quad \eta = \left( \sum_{i=m_0}^M d_i \right) - \lambda_N = \sum_{i=1}^{N-1} \lambda_i - \sum_{i=1}^{m_0-1} d_i.$$

Note that the maximality of  $m_0$  implies that  $m_0 \geq N$ . For each  $n \leq N$  define

$$\delta_n = \sum_{i=1}^n (\lambda_i - d_i) \geq 0.$$

For a certain value  $0 \leq \Delta \leq \eta$ , which will be specified later, define the sequence

$$(2.3) \quad \tilde{d}_i = \begin{cases} d_1 + \Delta & i = 1, \\ d_{m_0} - \Delta & i = m_0, \\ d_i & i \neq 1, m_0. \end{cases}$$

From the maximality of  $m_0$  we have

$$\tilde{d}_{m_0} = d_{m_0} - \Delta \geq d_{m_0} - \eta = \lambda_N - \sum_{i=m_0+1}^M d_i > 0.$$

This shows that  $\{\tilde{d}_i\}$  is a nonnegative sequence. However, note that this sequence might fail to be nonincreasing at the position  $i = m_0$ , which requires extra care in our considerations.

Our next goal is to construct an operator  $\tilde{S}$  with positive eigenvalues  $\{\lambda_i\}_{i=1}^N$ , diagonal  $\{\tilde{d}_i\}_{i=1}^M$  with respect to the orthonormal basis  $\{e_i\}_{i=1}^M$ , and the property that  $\langle \tilde{S}e_1, e_{m_0} \rangle = 0$ . The argument splits into two cases.

**Case 1** Assume there exists  $n \leq \min\{N, m_0 - 1\}$  such that  $\delta_n < \eta$ . Fix  $n_0 \leq \min\{N, m_0 - 1\}$  such that  $\delta_{n_0} \leq \delta_n$  for all  $n \leq \min\{N, m_0 - 1\}$ . Define  $\{\tilde{d}_i\}$  as in (2.3) with  $\Delta = \delta_{n_0}$ .

Note that

$$(2.4) \quad \sum_{i=n_0+1}^M \tilde{d}_i = -\delta_{n_0} + \sum_{i=n_0+1}^M d_i = \sum_{i=1}^M d_i - \sum_{i=1}^{n_0} \lambda_i = \sum_{i=1}^N \lambda_i - \sum_{i=1}^{n_0} \lambda_i = \sum_{i=n_0+1}^N \lambda_i.$$

Since  $m_0 > n_0$  and  $\tilde{d}_{m_0} > 0$ , from (2.4) we see that  $n_0 < N$ .

For  $n \leq n_0$ ,

$$\sum_{i=1}^n \tilde{d}_i = \delta_{n_0} + \sum_{i=1}^n d_i \leq \delta_n + \sum_{i=1}^n d_i = \sum_{i=1}^n \lambda_i$$

with equality when  $n = n_0$ . Since  $n_0 < N$ , by the inductive hypothesis there is a positive rank  $n_0$  operator  $\tilde{S}_1$  with eigenvalues  $\{\lambda_i\}_{i=1}^{n_0}$  and diagonal  $\{\tilde{d}_i\}_{i=1}^{n_0}$  with respect to the basis  $\{e_i\}_{i=1}^{n_0}$ .

Observe that the subsequence  $\{\tilde{d}_i\}_{i=n_0+1}^{N-1}$  coincides with  $\{d_i\}_{i=n_0+1}^{N-1}$  since  $N - 1 < m_0$ . Thus, for any  $n_0 + 1 \leq n \leq N - 1$  we have

$$\sum_{i=n_0+1}^n \tilde{d}_i = \sum_{i=n_0+1}^n d_i \leq \delta_n - \delta_{n_0} + \sum_{i=n_0+1}^n d_i = \sum_{i=n_0+1}^n \lambda_i.$$

Moreover, by (2.4) we have

$$\sum_{i=n_0+1}^N \tilde{d}_i \leq \sum_{i=n_0+1}^M \tilde{d}_i = \sum_{i=n_0+1}^N \lambda_i.$$

Thus,  $\{\lambda_i\}_{i=n_0+1}^N$  and the nonincreasing rearrangement of  $\{\tilde{d}_i\}_{i=n_0+1}^M$  satisfy the inductive hypothesis (2.1). That is, there is a positive rank  $N - n_0$  operator  $\tilde{S}_2$  with eigenvalues  $\{\lambda_i\}_{i=n_0+1}^N$  and diagonal  $\{\tilde{d}_i\}_{i=n_0+1}^M$  with respect to the basis  $\{e_i\}_{i=n_0+1}^M$ . Thus, the operator  $\tilde{S} = \tilde{S}_1 \oplus \tilde{S}_2$  has the desired properties. Indeed, the property that  $\langle \tilde{S}e_1, e_{m_0} \rangle = 0$  follows immediately from the definition of  $\tilde{S}$  and the fact that  $n_0 < m_0$ .

**Case 2** Assume  $\eta \leq \delta_n$  for all  $n \leq \min\{N, m_0 - 1\}$ . Define  $\{\tilde{d}_i\}$  as in (2.3) with  $\Delta = \eta$ . For  $n \leq N - 1$  we have

$$\sum_{i=1}^n \tilde{d}_i = \eta + \sum_{i=1}^n d_i \leq \delta_n + \sum_{i=1}^n d_i = \sum_{i=1}^n \lambda_i.$$

We also have by (2.2)

$$\sum_{i=1}^{m_0-1} \tilde{d}_i = \eta + \sum_{i=1}^{m_0-1} d_i = \sum_{i=1}^{N-1} \lambda_i.$$

By the inductive hypothesis there is a positive rank  $N - 1$  operator  $\tilde{S}_1$  with diagonal  $\{\tilde{d}_i\}_{i=1}^{m_0-1}$  and positive eigenvalues  $\{\lambda_i\}_{i=1}^{N-1}$ . Using the equality in (2.1), we have

$$\sum_{i=m_0}^M \tilde{d}_i = -\eta + \sum_{i=m_0}^M d_i = \sum_{i=1}^M d_i - \sum_{i=1}^{N-1} \lambda_i = \lambda_N.$$

By Lemma 2.2 there is a positive rank 1 operator  $\tilde{S}_2$  with diagonal  $\{\tilde{d}_i\}_{i=m_0}^M$  and eigenvalue  $\lambda_N$ . Thus, the operator  $\tilde{S} = \tilde{S}_1 \oplus \tilde{S}_2$  has the desired properties.

Combining the above two cases shows that the desired operator  $\tilde{S}$  exists. Let  $\alpha \in [0, 1]$  be such that  $\alpha(d_1 + \Delta) + (1 - \alpha)(d_{m_0} - \Delta) = d_1$ . Define the unitary operator  $U$  on the orthonormal basis  $\{e_i\}_{i=1}^M$  by

$$U(e_i) = \begin{cases} \sqrt{\alpha}e_1 - \sqrt{1 - \alpha}e_{m_0} & i = 1, \\ \sqrt{1 - \alpha}e_1 + \sqrt{\alpha}e_{m_0} & i = m_0, \\ e_i & \text{otherwise.} \end{cases}$$

A simple calculation shows that  $S = U^* \tilde{S} U$  has diagonal  $\{d_i\}_{i=1}^M$  in the basis  $\{e_i\}_{i=1}^M$ . This completes the proof of Theorem 2.1. ■

The following “moving toward 0-1” lemma first appeared in [8]. Its proof is constructive, as it consists of a finite number of “convex moves” as at the end of the previous proof. Moreover, from the proof in [8] it follows that Lemma 2.3 holds for real Hilbert spaces as well as complex.

**Lemma 2.3** *Let  $\{d_i\}_{i \in I}$  be a sequence in  $[0, 1]$ . Let  $I_0, I_1 \subset I$  be two disjoint finite subsets such that  $\max\{d_i : i \in I_0\} \leq \min\{d_i : i \in I_1\}$ . Let  $\eta_0 \geq 0$  and*

$$\eta_0 \leq \min \left\{ \sum_{i \in I_0} d_i, \sum_{i \in I_1} (1 - d_i) \right\}.$$

(i) *There exists a sequence  $\{\tilde{d}_i\}_{i \in I}$  in  $[0, 1]$  satisfying*

$$\begin{aligned} \tilde{d}_i &= d_i \quad \text{for } i \in I \setminus (I_0 \cup I_1), \\ \tilde{d}_i &\leq d_i \quad i \in I_0, \quad \text{and} \quad \tilde{d}_i \geq d_i, \quad i \in I_1, \\ \eta_0 + \sum_{i \in I_0} \tilde{d}_i &= \sum_{i \in I_0} d_i \quad \text{and} \quad \eta_0 + \sum_{i \in I_1} (1 - \tilde{d}_i) = \sum_{i \in I_1} (1 - d_i). \end{aligned}$$

(ii) *For any self-adjoint operator  $\tilde{E}$  on  $\mathcal{H}$  with diagonal  $\{\tilde{d}_i\}_{i \in I}$ , there exists an operator  $E$  on  $\mathcal{H}$  unitarily equivalent to  $\tilde{E}$  with diagonal  $\{d_i\}_{i \in I}$ .*

We end this section by remarking that the indexing set  $I$  in Theorem 1.1 need not be countable. In [15] the possibility that  $I$  is an uncountable set is addressed in all but the most difficult case where  $\{d_i\}$  and  $\{1 - d_i\}$  are nonsummable [15, Theorem 15]. However, the case when  $I$  is uncountable is a simple extension of the countable case, as we explain below.

**Proof of Reduction of Theorem 1.1 to Countable Case** First, we consider a projection  $P$  with diagonal  $\{d_i\}_{i \in I}$  with respect to some orthonormal basis  $\{e_i\}_{i \in I}$  of a Hilbert space  $\mathcal{H}$ . If  $a$  or  $b$  is infinite, then there is nothing to show, so we may assume  $a, b < \infty$ . Set  $J = \{i \in I : d_i = 0\} \cup \{i \in I : d_i = 1\}$ , and let  $P'$  be the restriction of  $P$  to the subspace  $\mathcal{H}' = \overline{\text{span}}\{e_i\}_{i \in I \setminus J}$ . Since  $e_i$  is an eigenvector for each  $i \in J$ ,  $\mathcal{H}'$  is an invariant subspace  $P'(\mathcal{H}') \subset \mathcal{H}'$ . Hence,  $P'$  is a projection with diagonal  $\{d_i\}_{i \in I \setminus J}$ . The assumption that  $a, b < \infty$  implies  $I \setminus J$  is at most countable. Thus, the countable case of Theorem 1.1 applied to the operator  $P'$  yields  $a - b \in \mathbb{Z}$ . This shows that (ii) is necessary.

To show that (i) or (ii) is sufficient, we claim that it is enough to assume that all of  $d_i$ ’s are in  $(0, 1)$ . If we can find a projection  $P$  with only these  $d_i$ ’s, then we take  $\mathbf{I}$  to be the identity and  $\mathbf{0}$  the zero operator on Hilbert spaces with dimensions equal to the cardinalities of the sets  $\{i \in I : d_i = 1\}$  and  $\{i \in I : d_i = 0\}$ , respectively. Then,  $P \oplus \mathbf{I} \oplus \mathbf{0}$  has diagonal  $\{d_i\}$ . Since  $a$  and  $b$  do not change when we restrict to  $(0, 1)$ , we may assume that  $\{d_i\}_{i \in I}$  has uncountably many terms and is contained in  $(0, 1)$ . There is some  $n \in \mathbb{N}$  such that  $J = \{i \in I : 1/n < d_i < 1 - 1/n\}$  has the same

cardinality as  $I$ . Thus, we can partition  $I$  into a collection of countable infinite sets  $\{I_k\}_{k \in K}$  such that  $I_k \cap J$  is infinite for each  $k \in K$ . Each sequence  $\{d_i\}_{i \in I_k}$  contains infinitely many terms bounded away from 0 and 1, thus (ii) holds. Again, by the countable case of Theorem 1.1, for each  $k \in K$  there is a projection  $P_k$  with diagonal  $\{d_i\}_{i \in I_k}$ . Thus,  $\bigoplus_{k \in K} P_k$  is a projection with diagonal  $\{d_i\}_{i \in I}$ . ■

### 3 The Carpenter’s Theorem Part (i)

The goal of this section is to give a proof of the sufficiency of (i) in Theorem 1.1. As a corollary of Theorem 2.1 we have the summable version of the carpenter’s theorem.

**Theorem 3.1** *Let  $M \in \mathbb{N} \cup \{\infty\}$ , and let  $\{d_i\}_{i=1}^M$  be a sequence in  $[0, 1]$ . If  $\sum_{i=1}^M d_i \in \mathbb{N}$ , then there is a projection  $P$  with diagonal  $\{d_i\}$ .*

**Proof** Let  $\{d'_i\}_{i=1}^{M'}$  be the terms of  $\{d_i\}$  in  $(0, 1]$ , listed in nonincreasing order. Set  $N = \sum_{i=1}^M d_i$ , and define  $\lambda_i = 1$  for  $i = 1, \dots, N$ . Since  $d'_i \leq 1$  for all  $i$  we have

$$\sum_{i=1}^n d'_i \leq \sum_{i=1}^n \lambda_i \quad \text{for } n = 1, 2, \dots, N.$$

We also have

$$\sum_{i=1}^{M'} d'_i = N = \sum_{i=1}^N \lambda_i.$$

By Theorem 2.1 there is a rank  $N$  self-adjoint operator  $P'$  with positive eigenvalues  $\{\lambda_i\}_{i=1}^N$  and diagonal  $\{d'_i\}_{i=1}^{M'}$ . Since  $\lambda_i = 1$  for each  $i$ , the operator  $P'$  is a projection. Let  $\mathbf{0}$  be the zero operator on a Hilbert space with dimension equal to  $|\{i: d_i = 0\}|$ . The operator  $P' \oplus \mathbf{0}$  is a projection with diagonal  $\{d_i\}_{i=1}^M$ . ■

**Corollary 3.2** *Let  $M \in \mathbb{N} \cup \{\infty\}$  and let  $\{d_i\}_{i=1}^M$  be a sequence in  $[0, 1]$ . If  $\sum_{i=1}^M (1 - d_i) \in \mathbb{N}$ , then there is a projection  $P$  with diagonal  $\{d_i\}$ .*

**Proof** This follows immediately from the observation that a projection  $P$  has diagonal  $\{d_i\}$  if and only if  $\mathbf{I} - P$  is a projection with diagonal  $\{1 - d_i\}$ . ■

Finally, we can handle the general case (i) of the carpenter’s theorem.

**Theorem 3.3** *Let  $\{d_i\}_{i \in I}$  be a sequence in  $[0, 1]$ . If*

$$(3.1) \quad a = \sum_{d_i < 1/2} d_i < \infty, \quad b = \sum_{d_i \geq 1/2} (1 - d_i) < \infty, \quad \text{and} \quad a - b \in \mathbb{Z},$$

*then there exists a projection  $P$  with diagonal  $\{d_i\}$ .*

**Proof** First, note that if  $\{d_i\}$  or  $\{1 - d_i\}$  is summable, then by (3.1) its sum is in  $\mathbb{N}$ . Thus, we can appeal to Theorem 3.1 or Corollary 3.2, resp., to obtain the desired projection. Hence, we may assume both 0 and 1 are limit points of the sequence  $\{d_i\}$ .

Next, we claim that it is enough to prove the theorem under the assumption that  $d_i \in (0, 1)$  for all  $i$ . Indeed, if  $P$  is a projection with diagonal  $\{d_i\}_{d_i \in (0,1)}$ ,  $\mathbf{I}$  is the identity operator on a space of dimension  $|\{i : d_i = 1\}|$ , and  $\mathbf{0}$  is the zero operator on a space of dimension  $|\{i : d_i = 0\}|$ , then  $P \oplus \mathbf{I} \oplus \mathbf{0}$  is a projection with diagonal  $\{d_i\}_{i \in I}$ .

Define  $J_0 = \{i \in I : d_i < 1/2\}$  and  $J_1 = \{i \in I : d_i \geq 1/2\}$ . Choose  $i_1 \in J_1$  such that  $d_{i_1} \leq d_i$  for all  $i \in J_1$ . Choose  $J'_0 \subseteq J_0$  such that  $J_0 \setminus J'_0$  is finite and  $\sum_{i \in J'_0} d_i < 1 - d_{i_1}$ . Let  $i_2 \in J_1$  be such that  $d_{i_2} > d_{i_1}$  and  $d_{i_2} + \sum_{i \in J'_0} d_i \geq 1$ . Set

$$(3.2) \quad \eta_0 = \sum_{i \in J'_0} d_i - (1 - d_{i_2}) < \sum_{i \in J'_0} d_i < 1 - d_{i_1}.$$

Let  $I_0 \subset J'_0$  be a finite set such that

$$(3.3) \quad \sum_{i \in I_0} d_i > \eta_0.$$

By (3.2) and (3.3), we can apply Lemma 2.3 to finite subsets  $I_0$  and  $I_1 = \{i_1\}$  to obtain a sequence  $\{\tilde{d}_i\}_{i \in I}$  coinciding with  $\{d_i\}_{i \in I}$  outside of  $I_0 \cup I_1$  and such that

$$\sum_{i \in I_0} \tilde{d}_i = \sum_{i \in I_0} d_i - \eta_0 \quad \text{and} \quad 1 - \tilde{d}_{i_1} = 1 - d_{i_1} - \eta_0.$$

Note that

$$\sum_{i \in J'_0 \cup \{i_2\}} \tilde{d}_i = d_{i_2} + \sum_{i \in J'_0 \setminus I_0} d_i + \sum_{i \in I_0} \tilde{d}_i = d_{i_2} + \sum_{i \in J'_0 \setminus I_0} d_i + \sum_{i \in I_0} d_i - \eta_0 = 1.$$

Thus, by Theorem 3.1 there is a projection  $P_1$  with diagonal  $\{\tilde{d}_i\}_{i \in J'_0 \cup \{i_2\}}$ . Next, we note that

$$\begin{aligned} \sum_{i \in I \setminus (J'_0 \cup \{i_2\})} (1 - \tilde{d}_i) &= \sum_{i \in J_0 \setminus J'_0} (1 - \tilde{d}_i) + \sum_{i \in J_1 \setminus \{i_2\}} (1 - \tilde{d}_i) \\ &= |J_0 \setminus J'_0| - \sum_{i \in J_0 \setminus J'_0} d_i + \sum_{i \in J_1 \setminus \{i_2\}} (1 - d_i) - \eta_0 \\ &= |J_0 \setminus J'_0| - \sum_{i \in J_0} d_i + \sum_{i \in J_1} (1 - d_i) \\ &= |J_0 \setminus J'_0| - a + b \in \mathbb{N}. \end{aligned}$$

By Corollary 3.2 there is a projection  $P_2$  with diagonal  $\{\tilde{d}_i\}_{i \in I \setminus (J'_0 \cup \{i_2\})}$ .

The projection  $P_1 \oplus P_2$  has diagonal  $\{\tilde{d}_i\}_{i \in I}$ . By Lemma 2.3 (ii) there is an operator  $P$  with diagonal  $\{d_i\}_{i \in I}$  which is unitarily equivalent to  $P_1 \oplus P_2$ . Thus,  $P$  is the required projection. ■

In [14, Remark 8] Kadison asked whether it is possible to construct projections with specified diagonal so that all its entries are real and nonnegative. While the answer is positive for rank one, in general it is negative for higher rank projections.

**Example 3.4** Consider any sequence  $\{d_i\}_{i=1}^3$  of numbers in  $(0, 1)$  such that  $d_1 + d_2 + d_3 = 2$ . By Theorem 3.1 there exists a projection  $P$  on  $\mathbb{R}^3$  with such diagonal. However, some entries of  $P$  must be negative. Indeed,  $\mathbf{I} - P$  is rank one projection. Hence,  $(\mathbf{I} - P)x = \langle x, \nu \rangle \nu$  for some unit vector  $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ . That is,  $(i, j)$  entry of  $\mathbf{I} - P$  equals  $\nu_i \nu_j$ . In particular,  $(\nu_i)^2 = 1 - d_i > 0$  for each  $i$ . This implies that for some  $i \neq j$ , the off-diagonal entry  $(i, j)$  of  $\mathbf{I} - P$  must be positive. Consequently,  $(i, j)$  entry of  $P$  is negative.

#### 4 The Algorithm and the Carpenter's Theorem Part (ii)

In this section we introduce an algorithmic technique for finding a projection with prescribed diagonal. The main result of this section is Theorem 4.3. Given a non-summable sequence  $\{d_i\}$  with all terms in  $[0, 1/2]$ , except possibly one term in  $(1/2, 1)$ , Theorem 4.3 produces an orthogonal projection with the diagonal  $\{d_i\}$ . Applying this result countably many times allows us to deal with all possible diagonal sequences in part (ii) of the carpenter's theorem.

The procedure of Theorem 4.3 is reminiscent to spectral tetris construction of tight frames introduced by Casazza *et al.* in [10] and further investigated in [11]. In fact, the infinite matrix constructed in the proof of Theorem 4.3 consists of column vectors forming a Parseval frame with squared norms prescribed by the sequence  $\{d_i\}$ . However, our construction was discovered independently with a totally different aim than that of [10].

**Lemma 4.1** *Let  $\sigma, d_1, d_2 \in [0, 1]$ . If  $\max\{d_1, d_2\} \leq \sigma$  and  $\sigma \leq d_1 + d_2$ , then there exists a number  $a \in [0, 1]$  such that the matrix*

$$(4.1) \quad \begin{bmatrix} a & \sigma - a \\ d_1 - a & d_2 - \sigma + a \end{bmatrix}$$

has entries in  $[0, 1]$  and

$$(4.2) \quad a(d_1 - a) = (\sigma - a)(d_2 - \sigma + a).$$

Moreover, if  $d_1 + d_2 < 2\sigma$ , then  $a$  is unique and given by

$$(4.3) \quad a = \frac{\sigma(\sigma - d_2)}{2\sigma - d_1 - d_2}.$$

**Proof** First, assume  $\max\{d_1, d_2\} \leq \sigma$  and  $\sigma \leq d_1 + d_2$ . If  $d_1 = d_2 = \sigma$ , then any  $a \in [0, \sigma]$  will satisfy (4.2) and the matrix (4.1) will have entries in  $[0, 1]$ . Thus, we may additionally assume  $d_1 + d_2 < 2\sigma$ , and hence  $\sigma > 0$ . Since the quadratic terms in (4.2) cancel out, the equation is linear and the unique solution is given by (4.3).

It remains to show that the entries of the matrix in (4.1) are in  $[0, 1]$ . It is clear that  $a \geq 0$ . Next, we calculate

$$\sigma - a = \sigma \left( 1 - \frac{\sigma - d_2}{2\sigma - d_1 - d_2} \right) = \frac{\sigma(\sigma - d_1)}{2\sigma - d_1 - d_2},$$

which implies that  $\sigma - a \geq 0$ . Since  $\sigma \leq 1$  we clearly have  $a, \sigma - a \in [0, 1]$ . It remains to prove that the second row of (4.1) has nonnegative entries. Since  $d_1 + d_2 \in [\sigma, 2\sigma)$ , we have

$$(d_1 - a) + (d_2 - \sigma + a) = d_1 + d_2 - \sigma \in [0, \sigma).$$

If one of  $d_1 - a$  or  $d_2 - \sigma + a$  is negative, then the other must be positive. From (4.2) we see that  $a = \sigma - a = 0$ . This contradicts the assumption that  $\sigma > 0$ . Thus, both  $d_1 - a$  and  $d_2 - \sigma + a$  are nonnegative. ■

**Lemma 4.2** *Let  $\{d_i\}_{i \in \mathbb{N}}$  be a sequence such that  $d_1 \in [0, 1)$ ,  $d_i \in [0, \frac{1}{2}]$  for  $i \geq 2$  and  $\sum_{i=1}^{\infty} d_i = \infty$ . There is a bijection  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  such that for each  $n \in \mathbb{N}$  we have*

$$(4.4) \quad d_{\pi(k_n-1)} \geq d_{\pi(k_n)} \quad \text{where } k_n := \min \left\{ k \in \mathbb{N} : \sum_{i=1}^k d_{\pi(i)} \geq n \right\}.$$

**Proof** For  $n \in \mathbb{N}$  define

$$m_n := \min \left\{ k \in \mathbb{N} : \sum_{i=1}^k d_i \geq n \right\}.$$

Define a bijection

$$\pi_n : \{m_{n-1} + 1, \dots, m_n\} \rightarrow \{m_{n-1} + 1, \dots, m_n\}$$

such that  $\{d_{\pi(i)}\}_{i=m_{n-1}+1}^{m_n}$  is in nonincreasing order with the convention that  $m_0 = 0$ . Finally, define a bijection  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  by

$$\pi(i) = \pi_n(i) \quad \text{if } m_{n-1} < i \leq m_n, n \in \mathbb{N}.$$

We claim that

$$(4.5) \quad m_{n-1} + 2 \leq k_n \leq m_n \quad \text{for all } n \in \mathbb{N}.$$

Indeed, by the minimality of  $m_{n-1}$  we have for  $n \geq 2$ ,

$$\sum_{i=1}^{m_{n-1}+1} d_{\pi(i)} = \sum_{i=1}^{m_{n-1}} d_i + d_{\pi(m_{n-1}+1)} < (n - 1/2) + 1/2 = n.$$

The above holds also holds trivially for  $n = 1$ . Thus,  $k_n > m_{n-1} + 1$  for all  $n \in \mathbb{N}$ . On the other hand, we have

$$\sum_{i=1}^{m_n} d_{\pi(i)} = \sum_{i=1}^{m_n} d_i \geq n.$$

This yields  $k_n \leq m_n$  and, thus, (4.5) is shown. By (4.5) we have  $m_{n-1} + 1 \leq k_n - 1 < k_n \leq m_n$ . Since  $\{d_{\pi(i)}\}_{i=m_{n-1}+1}^{m_n}$  is nonincreasing, this yields (4.4). ■

**Theorem 4.3** *Let  $\{d_i\}_{i \in I}$  be a sequence such that  $d_{i_0} \in [0, 1)$  for some  $i_0 \in I$ ,  $d_i \in [0, \frac{1}{2}]$  for all  $i \neq i_0$ , and  $\sum_{i \in I} d_i = \infty$ . There exists an orthogonal projection  $P$  with diagonal  $\{d_i\}_{i \in I}$ .*

**Proof** Since  $I$  is a countable set and  $\sum_{i \in I} d_i = \infty$  we may assume without loss of generality that  $I = \mathbb{N}$  and  $i_0 = 1$ . By Lemma 4.2, there is a bijection  $\pi: \mathbb{N} \rightarrow \mathbb{N}$  such that (4.4) holds.

For each  $n \in \mathbb{N}$  set

$$(4.6) \quad \sigma_n = n - \sum_{i=1}^{k_n-2} d_{\pi(i)}.$$

From the definition of  $k_n$  we see that

$$\sigma_n = n - \sum_{i=1}^{k_n} d_{\pi(i)} + d_{\pi(k_n-1)} + d_{\pi(k_n)} \leq d_{\pi(k_n-1)} + d_{\pi(k_n)}.$$

From the minimality of  $k_n$  and (4.4) we see that

$$\sigma_n = n - \sum_{i=1}^{k_n-1} d_{\pi(i)} + d_{\pi(k_n-1)} \geq d_{\pi(k_n-1)} \geq d_{\pi(k_n)},$$

which implies that

$$\sigma_n \geq \max\{d_{\pi(k_n-1)}, d_{\pi(k_n)}\}.$$

By Lemma 4.1, for each  $n$  there exists  $a_n \in [0, 1]$  such that the matrix

$$\begin{bmatrix} a_n & \sigma_n - a_n \\ d_{\pi(k_n-1)} - a_n & d_{\pi(k_n)} - \sigma_n + a_n \end{bmatrix}$$

has nonnegative entries and

$$(4.7) \quad a_n(d_{\pi(k_n-1)} - a_n) = (\sigma_n - a_n)(d_{\pi(k_n)} - \sigma_n + a_n).$$

Let  $\{e_i\}_{i \in \mathbb{N}}$  be an orthonormal basis for a Hilbert space  $\mathcal{H}$ . Set

$$v_1 = \sum_{i=1}^{k_1-2} d_{\pi(i)}^{1/2} e_i + a_1^{1/2} e_{k_1-1} - (\sigma_1 - a_1)^{1/2} e_{k_1},$$

and for  $n \geq 2$  define

$$\begin{aligned} v_n &= (d_{\pi(k_{n-1}-1)} - a_{n-1})^{1/2} e_{k_{n-1}-1} + (d_{\pi(k_{n-1})} - \sigma_{n-1} + a_{n-1})^{1/2} e_{k_{n-1}} \\ &\quad + \sum_{i=k_{n-1}+1}^{k_n-2} d_{\pi(i)}^{1/2} e_i + a_n^{1/2} e_{k_n-1} - (\sigma_n - a_n)^{1/2} e_{k_n}. \end{aligned}$$



**Proof** Set  $I_0 = \{i : d_i \leq 1/2\}$  and  $I_1 = \{i : d_i > 1/2\}$ . Our hypothesis (4.8) implies that

$$(4.9) \quad a' = \sum_{i \in I_0} d_i = \infty \quad \text{or} \quad b = \infty.$$

**Case 1** Assume that  $a' = \infty$ . We can partition  $I$  into countably many sets  $\{J_n\}_{n \in \mathbb{N}}$  such that each  $J_n$  contains at most one element in  $I_1$  and  $\sum_{i \in J_n} d_i = \infty$  for all  $n \in \mathbb{N}$ . This is possible since  $I_0$  satisfies (4.9). By Theorem 4.3, for each  $n \in \mathbb{N}$  there is a projection  $P_n$  with diagonal  $\{d_i\}_{i \in J_n}$ . Thus, the projection  $P = \bigoplus_{n \in \mathbb{N}} P_n$  has the desired diagonal  $\{d_i\}_{i \in I}$ . This completes the proof of Case 1.

**Case 2** Assume that  $b = \infty$ . Note that

$$b = \sum_{1-d_i \leq 1/2} (1-d_i).$$

Thus, by Case 1 there is a projection  $P'$  with diagonal  $\{1-d_i\}$ . Hence,  $P = \mathbf{I} - P'$  is a projection with diagonal  $\{d_i\}$ . ■

## 5 A Selector Problem

Kadison's Theorem 1.1 is closely connected with an open problem of characterizing all spectral functions of shift-invariant spaces. Shift-invariant (SI) spaces are closed subspaces of  $L^2(\mathbb{R}^d)$  that are invariant under all shifts, *i.e.*, integer translations. That is, a closed subspace  $V \subset L^2(\mathbb{R}^d)$  is SI if  $T_k(V) = V$  for all  $k \in \mathbb{Z}^d$ , where  $T_k f(x) = f(x-k)$  is the translation operator. The theory of shift-invariant spaces plays an important role in many areas, most notably in the theory of wavelets, spline systems, Gabor systems, and approximation theory [5–7, 19, 20]. The study of analogous spaces for  $L^2(\mathbb{T}, \mathcal{H})$  with values in a separable Hilbert space  $\mathcal{H}$  in terms of the range function, often called doubly-invariant spaces, is quite classical and goes back to Helson [12].

In the context of SI spaces a *range function* is any mapping

$$J: \mathbb{T}^d \rightarrow \{\text{closed subspaces of } \ell^2(\mathbb{Z}^d)\},$$

where  $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$  is identified with its fundamental domain  $[-1/2, 1/2)^d$ . We say that  $J$  is *measurable* if the associated orthogonal projections  $P_J(\xi): \ell^2(\mathbb{Z}^d) \rightarrow J(\xi)$  are operator measurable, *i.e.*,  $\xi \mapsto P_J(\xi)v$  is measurable for any  $v \in \ell^2(\mathbb{Z}^d)$ . We follow the convention that identifies range functions if they are equal a.e. A fundamental result due to Helson [12, Theorem 8, p. 59] gives a one-to-one correspondence between SI spaces  $V$  and measurable range functions  $J$ , see also [7, Proposition 1.5]. Among several equivalent ways of introducing the spectral function of an SI space, the most relevant definition uses a range function.

**Definition 5.1** The spectral function of an SI space  $V$  is a measurable mapping  $\sigma_V: \mathbb{R}^d \rightarrow [0, 1]$  given by

$$(5.1) \quad \sigma_V(\xi + k) = \|P_J(\xi)e_k\|^2 = \langle P_J(\xi)e_k, e_k \rangle \quad \text{for } \xi \in \mathbb{T}^d, k \in \mathbb{Z}^d,$$

where  $\{e_k\}_{k \in \mathbb{Z}^d}$  denotes the standard basis of  $\ell^2(\mathbb{Z}^d)$  and  $\mathbb{T}^d = [-1/2, 1/2)^d$ . In other words,  $\{\sigma_V(\xi + k)\}_{k \in \mathbb{Z}^d}$  is a diagonal of a projection  $P_J(\xi)$ .

Note that  $\sigma_V(\xi)$  is well defined for a.e.  $\xi \in \mathbb{R}^d$ , since  $\{k + \mathbb{T}^d : k \in \mathbb{Z}^d\}$  is a partition of  $\mathbb{R}^d$ . As an immediate consequence of Theorem 1.1 we have the following result.

**Theorem 5.2** Suppose that  $V \subset L^2(\mathbb{R}^d)$  is an SI space. Let  $\sigma = \sigma_V: \mathbb{R}^d \rightarrow [0, 1]$  be its spectral function. For  $\xi \in \mathbb{T}^d$  define

$$a(\xi) = \sum_{k \in \mathbb{Z}^d, \sigma(\xi+k) < 1/2} \sigma(\xi + k) \quad \text{and} \quad b(\xi) = \sum_{k \in \mathbb{Z}^d, \sigma(\xi+k) \geq 1/2} (1 - \sigma(\xi + k)).$$

Then for a.e.  $\xi \in \mathbb{R}^d$  we either have

- (i)  $a(\xi), b(\xi) < \infty$  and  $a(\xi) - b(\xi) \in \mathbb{Z}$ , or
- (ii)  $a(\xi) = \infty$  or  $b(\xi) = \infty$ .

It is an open problem whether the converse to Theorem 5.2 holds.

**Problem 1** Suppose that a measurable function  $\sigma: \mathbb{R}^d \rightarrow [0, 1]$  satisfies either (i) or (ii) for a.e.  $\xi \in \mathbb{R}^d$ . Does there exist a SI space  $V \subset L^2(\mathbb{R}^d)$  such that its spectral function  $\sigma_V$  is  $\sigma$ ?

The sufficiency part of Theorem 1.1, i.e., the carpenter’s theorem, suggests a positive answer to this problem. That is, for a.e.  $\xi$  it yields a projection  $P_J(\xi)$  whose diagonal satisfies (5.1). However, it does not guarantee *a priori* that the corresponding range function  $J$  is measurable. This naturally leads to the following selector problem.

**Problem 2** Let  $X$  be a finite (or  $\sigma$ -finite) measure space and let  $I$  be a countable index set. Let  $\sigma: X \times I \rightarrow [0, 1]$  be a measurable function. For  $\xi \in X$  define

$$a(\xi) = \sum_{i \in I, \sigma(\xi, i) < 1/2} \sigma(\xi, i) \quad \text{and} \quad b(\xi) = \sum_{i \in I, \sigma(\xi, i) \geq 1/2} (1 - \sigma(\xi, i)).$$

Suppose that for a.e.  $\xi \in X$  we either have

- (i)  $a(\xi), b(\xi) < \infty$  and  $a(\xi) - b(\xi) \in \mathbb{Z}$ , or
- (ii)  $a(\xi) = \infty$  or  $b(\xi) = \infty$ .

Does there exist a measurable range function  $J: X \rightarrow \{\text{closed subspaces of } \ell^2(I)\}$  such that the corresponding orthogonal projections  $P_J(\xi)$  have diagonal  $\{\sigma(\xi, i)\}_{i \in I}$  for a.e.  $\xi \in X$ ?

In other words, Problem 2 asks whether it is possible to find a measurable selector of projections in Theorem 1.1. The constructive proof of the carpenter’s theorem given in this paper might be a first step toward resolving this problem. However, Problem 2 remains open.

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