Existence of frames with prescribed norms and frame operator

Marcin Bownik and John Jasper

Abstract In this chapter we survey several recent results on the existence of frames with prescribed norms and frame operator. These results are equivalent to Schur-Horn type theorems which describe possible diagonals of positive self-adjoint operators with specified spectral properties. The first infinite dimensional result of this type is due to Kadison who characterized diagonals of orthogonal projections. Kadison's theorem automatically gives a characterization of all possible sequences of norms of Parseval frames. We present some generalizations of Kadison's result such as (a) the lower and upper frame bounds are specified, (b) the frame operator has two point spectrum, and (c) the frame operator has a finite spectrum.

Key words: Frame, Frame operator, Diagonals of self-adjoint operators, The Schur-Horn theorem, The Pythagorean theorem, The Carpenter theorem, Spectral theory

Frames and the Schur-Horn Theorem

The concept of frames in Hilbert spaces was originally introduced in the context of nonharmonic Fourier series by Duffin and Schaeffer [18] in the 1950's. The advent of wavelet theory brought a renewed interest in frame theory as is attested by now classical books of Daubechies [16], Meyer [31], and Mallat [30]. For an introduction to frame theory we refer to the book by Christensen [15].

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Definition 1. A sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathscr{H}$ is called a *frame* if there exists $0 < A \leq B < \infty$ such that

$$A||f||^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B||f||^2 \qquad \text{for all } f \in \mathscr{H}. \tag{1}$$

The numbers A and B are called the *frame bounds*. The supremum over all As and infimum over all Bs which satisfy (1) are called the *optimal frame bounds*. If $A = B$, then $\{f_i\}$ is said to be a *tight frame*. In addition, if $A = B = 1$, then $\{f_i\}$ is called a *Parseval frame*. The frame operator is defined by

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

It is well known that S is a self-adjoint operator satisfying $A\mathbf{I} \leq S \leq B\mathbf{I}$.

The construction of frames with desired properties is a vast subject that is central to frame theory. Among the recently studied classes of frames with desired features are Grassmanian frames, equiangular frames, equal norm tight frames, finite frames for sigma-delta quantization, fusion frames, and frames for signal reconstruction without the phase. In particular, the construction of frames with prescribed norms and frame operator has been studied by many authors.

Problem 1. Characterize all possible sequences of norms $\{||f_i||\}_{i\in I}$ of frames $\{f_i\}_{i\in I}$ with prescribed frame operator S.

In the finite dimensional case Casazza and Leon [12, 13] gave explicit and algorithmic construction of tight frames with prescribed norms. Moreover, Casazza, Fickus, Kovačević, Leon, and Tremain [14] characterized norms of finite tight frames in terms of their "fundamental frame inequality" using frame potential methods of Benedetto and Fickus [5]. An alternative approach using projection decomposition was undertaken by Dykema, Freeman, Kornelson, Larson, Ordower, and Weber [19], which yields some necessary and some sufficient conditions for infinite dimensional Hilbert spaces [29]. A significantly refined eigenstep method for constructing finite frames with prescribed spectrum and diagonal was recently introduced by Cahill, Fickus, Mixon, Poteet, and Strawn [11, 20]. These results are described in Section "Finite dimensional frames".

Significant progress in the area became possible thanks to the Schur-Horn theorem as noted by Antezana, Massey, Ruiz, and Stojanoff [1] and Tropp, Dhillon, Heath, and Strohmer [34]. In particular, the authors of [1] established the following connection between Schur-Horn-type theorems and the existence of frames with prescribed norms and frame operator, see [1, Proposition 4.5] and [6, Proposition 2.3].

Theorem 1 (Antezana-Massey-Ruiz-Stojanoff). *Let S be a positive self-adjoint operator on a Hilbert space $\mathscr{H}$. Let $\{d_i\}_{i\in I}$ be a bounded sequence of positive numbers. Then the following are equivalent:*

1. *there exists a frame $\{f_i\}_{i\in I}$ in $\mathscr{H}$ with the frame operator S such that $d_i = ||f_i||^2$ for all $i \in I$,*

2. *there exists a larger Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a self-adjoint operator E acting on $\ell^2(I)$, which is unitarily equivalent with $S \oplus \mathbf{0}$, where $\mathbf{0}$ is the zero operator acting on $\mathscr{K} \ominus \mathscr{H}$, such that its diagonal $\langle Ee_i, e_i\rangle = d_i$ for all $i \in I$.*

Thus, Problem 1 of characterizing sequences $\{\|f_i\|^2\}_{i\in I}$ for all frames $\{f_i\}_{i\in I}$ with frame operator S is subsumed by the Schur-Horn problem:

Problem 2. Characterize diagonals $\{\langle Ee_i, e_i\rangle\}_{i\in I}$ of a self-adjoint operator E, where $\{e_i\}_{i\in I}$ is any orthonormal basis of $\mathscr{H}$.

In Section "Finite dimensional frames" we discuss the finite dimensional Schur-Horn theorem and its connection to finite dimensional frames. In Section "Infinite dimensional frames" we present the infinite dimensional results. A beautifully simple and complete characterization of Parseval frame norms was given by Kadison [25, 26], which easily extends to tight frames by scaling. The authors [6] have extended this result to the non-tight setting by characterizing frame norms with prescribed optimal frame bounds. The second author [24] has characterized diagonals of self-adjoint operators with three points in the spectrum. This yields a characterization of frame norms whose frame operator has two point spectrum. Finally, the authors [7, 10] have recently extended this result to operators with finite spectrum.

Finite dimensional frames

The classical Schur-Horn theorem [22, 33] characterizes diagonals of self-adjoint (Hermitian) matrices with given eigenvalues. It can be stated as follows, where $\mathscr{H}_N$ is N-dimensional Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, i.e., $\mathscr{H}_N = \mathbb{R}^N$ or $\mathbb{C}^N$.

Theorem 2 (Schur-Horn). *Let $\{\lambda_i\}_{i=1}^N$ and $\{d_i\}_{i=1}^N$ be real sequences in nonincreasing order. There exists a self-adjoint operator $E : \mathscr{H}_N \to \mathscr{H}_N$ with eigenvalues $\{\lambda_i\}$ and diagonal $\{d_i\}$ if and only if*

$$\sum_{i=1}^N \lambda_i = \sum_{i=1}^N d_i \quad \text{and} \quad \sum_{i=1}^n d_i \le \sum_{i=1}^n \lambda_i \quad \text{for all } 1 \le n \le N. \tag{2}$$

The necessity of (2) is due to Schur [33] and the sufficiency of (2) is due to Horn [22]. It should be noted that (2) can be stated by the equivalent convexity condition

$$(d_1, \dots, d_N) \in \operatorname{conv}\{(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)}) : \sigma \in S_N\}, \tag{3}$$

where S_N is a permutation group on N elements.

Using Theorem 1 we obtain a complete solution to Problem 1 for finite frames.

Theorem 3. *Let S be a positive self-adjoint invertible $M \times M$ matrix with eigenvalues $\{\lambda_i\}_{i=1}^M$ in nonincreasing order. Let $\{d_i\}_{i=1}^N$ be a nonnegative nonincreasing sequence. There exists a frame $\{f_i\}_{i=1}^N$ for $\mathscr{H}_M$ with frame operator S and $\|f_i\|^2 = d_i$ for each $i = 1, \dots, N$ if and only if*

$$\sum_{i=1}^N d_i = \sum_{i=1}^M \lambda_i \quad \text{and} \quad \sum_{i=1}^k d_i \le \sum_{i=1}^k \lambda_i \quad \text{for all } 1 \le k \le M.$$

Though this completely answers the question of existence of a frame with prescribed norms and frame operator, it does not give a construction of the desired frame. Indeed, the early proofs of the Schur-Horn theorem were existential, and there have been several recent papers [12, 13, 17, 20] on algorithms for the construction of the matrices in Theorem 2. Therefore, Theorem 3 is not the final word on constructing frames with a given frame operator and set of lengths.

Example 1. Let $\{e_1, e_2\}$ be an orthonormal basis for $\mathbb{C}^2$. Consider the frames $\{f_i\}_{i=1}^4$ and $\{g_i\}_{i=1}^4$ given by

$$f_1 = e_1,\ f_2 = e_2,\ f_3 = e_1,\ f_4 = e_2$$

and

$$g_1 = e_1,\ g_2 = e_2,\ g_3 = 2^{-1/2}(e_1 + e_2),\ g_4 = 2^{-1/2}(e_1 - e_2).$$

A simple calculation shows that each is a frame with frame operator $2\mathbf{I}$ and the norms of the frame vectors are all 1. However, these frames are fundamentally different. Indeed, $\{g_i\}$ is not unitarily equivalent (or even isomorphic) to any reordering of $\{f_i\}$.

Example 1 shows that for a given positive invertible operator S and a sequence of lengths $\{d_i\}$ there may be many different frames with frame operator S and lengths $\{d_i\}$. To understand the set of all frames with a given frame operator and set of lengths authors of [11] consider the problem of constructing every such frame.

For a given vector f in a Hilbert space $\mathscr{H}$, let ff^* denote the rank one operator given by

$$ff^*(g) = \langle g, f \rangle f \qquad \text{for all } g \in \mathscr{H}.$$

We will use the following standard result from linear algebra [23, Propositions 4.3.4 and 4.3.10].

Proposition 1. *Let S be an $M \times M$ self-adjoint matrix with eigenvalue list $\lambda_1 \ge \dots \ge \lambda_M$. If $f \in \mathbb{C}^M$ with $\gamma = \|f\|^2$ and $\mu_1 \ge \dots \ge \mu_M$ is the eigenvalue list of $S + ff^*$, then*

$$\mu_1 \ge \lambda_1 \ge \mu_2 \ge \lambda_2 \ge \dots \ge \mu_M \ge \lambda_M \tag{4}$$

and

$$\sum_{i=1}^M \mu_i = \gamma + \sum_{i=1}^M \lambda_i. \tag{5}$$

Conversely, for any sequence $\{\mu_i\}_{i=1}^M$ *and* γ *satisfying (4) and (5), there exists a vector* $f \in \mathbb{C}^M$ *with* $\|f\|^2 = \gamma$ *such that* $S + ff^*$ *has eigenvalue list* $\{\mu_i\}_{i=1}^M$.

Two sequences $\{\lambda_i\}_{i=1}^M$ and $\{\mu_i\}_{i=1}^M$ satisfying (4) are said to *interlace*, in symbols $\{\lambda_i\}_{i=1}^M \sqsubseteq \{\mu_i\}_{i=1}^M$. For a frame $\{f_i\}_{i=1}^N$ define the *jth partial frame operator*

$$S_j = \sum_{i=1}^{j} f_i f_i^*.$$

By applying Proposition 1 to the partial frame operators, starting with the zero operator, we obtain the following result [11, Theorem 2].

Theorem 4 (Cahill-Fickus-Mixon-Poteet-Strawn). *Let S be a positive self-adjoint operator with eigenvalue list* $\lambda_1 \geq \ldots \geq \lambda_M > 0$. *Let* $\{d_i\}_{i=1}^N$ *be a nonnegative sequence.*

(i) If there is a sequence of nonincreasing nonnegative sequences $\{\{\lambda_{i,j}\}_{i=1}^M\}_{j=0}^N$ *such that*

$$\lambda_{i,0} = 0 \text{ and } \lambda_{i,N} = \lambda_i \qquad \text{for all } i = 1,\ldots,M, \tag{6}$$

$$\{\lambda_{i,j}\}_{i=1}^M \sqsubseteq \{\lambda_{i,j+1}\}_{i=1}^M, \tag{7}$$

$$\sum_{i=1}^{M} \lambda_{i,j} = \sum_{i=1}^{j} d_i \qquad \text{for each } j = 0,\ldots,N-1, \tag{8}$$

then there exists a frame $\{f_i\}_{i=1}^N$ *with* $\|f_i\|^2 = d_i$ *such that* $\{\lambda_{i,j}\}_{i=1}^M$ *is the eigenvalue sequence of the jth partial frame operator.*

(ii) Conversely, if there exists a frame $\{f_i\}_{i=1}^N$ *with frame operator S,* $\|f_i\|^2 = d_i$ *for* $i = 1,\ldots,N$, *and* $\{\lambda_{i,j}\}_{i=1}^M$ *is the eigenvalue sequence of the jth partial frame operator, then* $\{\{\lambda_{i,j}\}_{i=1}^M\}_{j=0}^N$ *satisfies (6), (7), and (8).*

A doubly indexed sequence $\{\{\lambda_{i,j}\}_{i=1}^M\}_{j=0}^N$ satisfying (6), (7), and (8) is called a sequence of *eigensteps*. In Example 1 we saw that a given frame operator and set of lengths does not determine a frame. In light of Theorem 4 one might think that for a given sequence of eigensteps $\{\{\lambda_{i,j}\}_{i=1}^M\}_{j=0}^N$ there is a unique (up to unitarily equivalence) frame such that $\{\lambda_{i,j}\}_{i=1}^M$ is the eigenvalue sequence of the jth partial frame operator. The following simple example shows that this is not true.

Example 2. Let $d_i = 2$ for $i = 1,2,3$. Let $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 1$. A sequence of eigensteps is given by the following table

j	0	1	2	3
$\lambda_{3,j}$	0	0	0	1
$\lambda_{2,j}$	0	0	2	2
$\lambda_{1,j}$	0	2	2	3

We may choose $f_1 \in \mathbb{C}^3$ to be any vector with $\|f_1\|^2 = 2$. Next, we may choose f_2 to be any vector with $\|f_2\|^2 = 2$ in $\operatorname{span}\{f_1\}^\perp$. Finally, let $f_3 = x + y$ where

$x \in \text{span}\{f_1, f_2\}$ with $\|x\|^2 = \frac{1}{2}$, and $y \in \text{span}\{f_1, f_2\}^\perp$ with $\|y\|^2 = \frac{3}{2}$. A simple calculation shows that $\{\lambda_{i,j}\}_{i=1}^3$ is the eigenvalue sequence of the jth partial frame operator of $\{f_i\}_{i=1}^3$.

In [11, Theorem 2], Cahill, Fickus, Mixon, Poteet, and Strawn give an algorithm to prove Theorem 4(ii), that is, an algorithm to produce a frame from a given sequence of eigensteps. At step j the vector f_j is chosen and added to the frame so that the partial frame operator has the desired spectrum. As we see in Example 2, at each step there will be a set of choices for the next frame vector. Crucially, their algorithm identifies *all* choices for the jth frame vector, and thus the algorithm constructs all finite frames. In [20, Theorem 5], Fickus, Mixon, Poteet, and Strawn give an explicit algorithm for producing a sequence of eigensteps which they dubbed *Top Kill* since it iteratively "kills" as much as possible from the top portion of staircases starting with the final eigenvalue sequence $\{\lambda_{i,N}\}_{i=1}^M$ and proceeding backward to the zero sequence $\{\lambda_{i,0}\}_{i=1}^M$. Finally, in [20, Theorem 2] they gave an explicit description of all possible sequences of eigensteps as in Theorem 4(i). For details we refer to [11] and [20].

Infinite dimensional frames

The infinite dimensional case of Problem 1 was considered by Kornelson and Larson in [29]. Their result gives a sufficient condition for the sequence of frame norms in terms of the essential norm of the frame operator

$$\|S\|_{ess} = \inf\{\|S+K\| : K \text{ a compact operator}\}.$$

We observe that in light of Kadison's Theorem 6, [29, Proposition 7 and Corollaries 8 and 9] are incorrect as stated, see Example 3. However, this does not affect the correctness of [29, Theorem 6] which can be stated as follows.

Theorem 5 (Kornelson-Larson). *Let S be a positive invertible operator on a Hilbert space $\mathscr{H}$. Suppose that $\{d_i\}_{i\in I}$ is a sequence in $[0,\infty)$ such that*

$$\sum_{i\in I} d_i = \infty \qquad \text{and} \qquad \sup_{i\in I} d_i < ||S||_{ess}.$$

Then, there exists a frame $\{f_i\}_{i\in I} \subset \mathscr{H}$ with $d_i = ||f_i||^2$ for all $i \in I$ such that its frame operator is S.

Antezana, Massey, Ruiz, and Stojanoff have refined Theorem 5 by giving a necessary condition [1, Theorem 5.1] and some sufficient conditions [1, Theorem 5.4] for the sequence of frame norms with a given frame operator. Thus, these results give a partial answer to Problem 1.

Kadison's Pythagorean Theorem

The first complete characterization of frame norms was achieved by Kadison for Parseval frames, i.e., when the frame operator $S = \mathbf{I}$, which easily extends to tight frames by scaling. In his influential work [25, 26] Kadison discovered a characterization of diagonals of orthogonal projections acting on $\mathscr{H}$.

Theorem 6 (Kadison). *Let $\{d_i\}_{i\in I}$ be a sequence in $[0,1]$ and $\alpha \in (0,1)$. Define*

$$C(\alpha) = \sum_{d_i<\alpha} d_i, \qquad D(\alpha) = \sum_{d_i\geq\alpha} (1-d_i).$$

Then the following are equivalent:

1. *there exists an orthogonal projection of $\ell^2(I)$ onto a subspace $\mathscr{H}$ with diagonal $\{d_i\}_{i\in I}$,*
2. *there exists a Parseval frame $\{f_i\}_{i\in I}$ on a Hilbert space $\mathscr{H}$ such that $||f_i||^2 = d_i$ for all $i \in I$,*
3. *we have $C(\alpha) = \infty$ or $D(\alpha) = \infty$, or*

$$C(\alpha) < \infty \quad \textit{and} \quad D(\alpha) < \infty \quad \textit{and} \quad C(\alpha) - D(\alpha) \in \mathbb{Z}. \tag{9}$$

One can easily show that if (9) is satisfied for some $\alpha \in (0,1)$, then (9) holds for all $\alpha \in (0,1)$. Hence, Kadison's Theorem is often formulated for a specific choice of $\alpha = 1/2$. Alternative proofs of Theorem 6 were provided by the authors [8] and Argerami [2].

Example 3. For a given $\eta \in [0,1]$ define a sequence $\{d_i\}_{i\in\mathbb{Z}}$ by

$$\left(\dots, \frac{1}{2^n}, \dots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \eta, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots, \frac{2^n-1}{2^n}, \dots\right). \tag{10}$$

Since this sequence is symmetric around $1/2$ with the exception of the entry η, we have $C(1/2) - D(1/2) \equiv \eta \mod 1$. Hence, by Theorem 6, $\{d_i\}_{i\in\mathbb{Z}}$ is a diagonal of projection if and only if $\eta = 0$ or $\eta = 1$.

Characterization of frame norms from frame bounds

The non-tight extension of Theorem 6 was considered by the authors [6] who characterized all possible sequences of norms of a frame with prescribed optimal bounds $A < B$. The special tight case $A = B$ follows immediately from Kadison's Pythagorean Theorem 6 by scaling. Theorem 7 can be also thought as an infinite dimensional Schur-Horn theorem for a class of self-adjoint operators with prescribed lower and upper bounds and with zero in the spectrum.

Theorem 7. *Let $0 < A < B < \infty$ and $\{d_i\}$ be a nonsummable sequence in $[0,B]$. Define the numbers*

$$C(A) = \sum_{d_i < A} d_i \quad \text{and} \quad D(A) = \sum_{d_i \geq A} \big(B - d_i\big). \tag{11}$$

Then the following are equivalent:

1. *there exists a positive operator E on $\ell^2(I)$ with the spectrum satisfying $\{A,B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A,B]$ and the diagonal $\{d_i\}_{i \in I}$,*
2. *there exists a frame $\{f_i\}_{i \in I}$ on some Hilbert space $\mathscr{H}$ with optimal frame bounds A and B, and $d_i = ||f_i||^2$ for all $i \in I$,*
3. *one of the following holds:*

   a. $C(A) = \infty$,
   b. $D(A) = \infty$,
   c. $C(A), D(A) < \infty$ and there exists $n \in \mathbb{N}_0$ such that

$$nA \leq C(A) \leq A + B(n-1) + D(A). \tag{12}$$

Note that the assumption of $\{d_i\}$ being nonsummable in Theorem 7 is not a true limitation. Indeed, the summable case requires more restrictive conditions which are omitted here. One should also emphasize that the non-tight case is not a mere generalization of the tight case $A = B$, since it is qualitatively different from the tight case. Indeed, by setting $A = B$ in Theorem 7 we do not get the correct necessary and sufficient condition (9) previously discovered by Kadison.

Example 4. For a given nonsummable sequence $\{d_i\}_{i \in \mathbb{N}}$ in $[0,1]$ we define

$$\mathscr{A} = \{A \in (0,1] : \exists \text{ frame } \{f_i\}_{i \in \mathbb{N}} \text{ with } ||f_i||^2 = d_i \\ \text{that has optimal frame bounds } A \text{ and } 1\}.$$

Without loss of generality we can assume that $\sup d_i = 1$. Indeed, if $\sup d_i < 1$, then by Theorem 7 we have always $\mathscr{A} = (0,1]$, and this case is not interesting.

In [6, Theorem 7.1] the authors have shown the set $\mathscr{A} \cup \{0,1\}$ is always closed. In general, determining the set $\mathscr{A}$ is not an easy task since it boils down to checking condition (12) for all possible values of $0 < A < 1$ by computing countably many infinite series (11). If $\{d_i\}_{i \in \mathbb{N}}$ happens to be a geometric series this task actually reduces to checking a finite number of conditions, see [6]. Indeed, for a given $\beta \in (0,1)$ define a sequence $\{d_i\}_{i \in \mathbb{N}}$ by

$$d_i = 1 - \beta^i \qquad \text{for } i = 1, 2, \ldots$$

Using Theorem 7 the authors have shown [6] that

$$\mathscr{A} = [(1-2\beta)/(1-\beta), 1-\beta] \qquad \text{for } 0 < \beta < 1/2.$$

Theorem 7 has an analogue for operators without the zero in the spectrum. This result is much easier to show and it leads to a characterization of norms of Riesz bases with prescribed bounds.

Theorem 8. *Let $0 < A \le B < \infty$ and $\{d_i\}_{i\in I}$ be a sequence in $[A,B]$. Then the following are equivalent:*

1. *there exists a positive operator E on $\ell^2(I)$ with the spectrum satisfying $\{A,B\} \subseteq \sigma(E) \subseteq [A,B]$ and the diagonal $\{d_i\}_{i\in I}$,*
2. *there exists a Riesz basis $\{f_i\}_{i\in I}$ with optimal bounds A and B and $d_i = ||f_i||^2$ for all $i \in I$,*
3. *we have*
$$C(A), D(A) \ge B - A. \tag{13}$$

Characterization of frame norms with finite spectrum frame operator

Another extension of Kadison's result [25, 26] was obtained by the second author [24] who characterized the set of diagonals of operators with three points in the spectrum.

Theorem 9. *Let $0 < A < B < \infty$ and $\{d_i\}_{i\in I}$ be a sequence in $[0,B]$ with $\sum d_i = \sum(B-d_i) = \infty$. Define $C(A)$ and $D(A)$ as in (11). Then the following are equivalent:*

1. *there exists a self-adjoint operator E with diagonal $\{d_i\}_{i\in I}$ and spectrum $\sigma(E) = \{0,A,B\}$,*
2. *there exists a frame $\{f_i\}_{i\in I}$ on some Hilbert space $\mathscr{H}$ with $d_i = ||f_i||^2$ for all $i \in I$ such that its frame operator S has spectrum $\sigma(S) = \{A,B\}$,*
3. *one of the following holds:*
   a. *$C(A) = \infty$,*
   b. *$D(A) = \infty$,*
   c. *$C(A), D(A) < \infty$ and there exists $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that*
$$C(A) - D(A) = NA + kB \quad \text{and} \quad C(A) \ge (N+k)A.$$

By combining Theorems 6 and 9 we can show the existence of frames with at most two point spectrum for every nonsummable sequence of frame norms.

Theorem 10. *Let $\{d_i\}_{i\in I}$ be a sequence in $[0,B]$ such that $\sum d_i = \sum(B-d_i) = \infty$, where $B > 0$. Then, there exists a frame $\{f_i\}_{i\in I}$ with $||f_i||^2 = d_i$ for all $i \in I$ such that its frame operator S satisfies either:*

1. *$S = B\mathbf{I}$, i.e., $\{f_i\}_{i\in I}$ is a tight frame with frame bounds $A = B$, or*
2. *S has spectrum $\sigma(S) = \{A,B\}$ for some $0 < A < B$.*

Proof. If $C(B/2) = \infty$ or $D(B/2) = \infty$, then by Kadison's Theorem 6, there exists a tight frame $\{f_i\}_{i\in I}$ with frame bounds $A = B$. The same conclusion holds when $C(B/2) < \infty$ and $D(B/2) < \infty$, and $C(B/2) - D(B/2) \in B\mathbb{Z}$. Hence, without loss of generality we can assume that $C(B/2) < \infty$, $D(B/2) < \infty$, and

$$C(B/2) - D(B/2) \equiv \eta \mod B \qquad \text{for some } 0 < \eta < B.$$

Using Theorem 9 the authors have shown in [9, Theorem 3.4] that there exists a self-adjoint operator E on $\ell^2(I)$ with diagonal $\{d_i\}_{i\in I}$ and spectrum $\sigma(S) = \{0, A, B\}$, where $A = \eta$. Applying Theorem 1 yields conclusion 2 of Theorem 10.

Finally, the authors [7] showed the following characterization of diagonals of self-adjoint operators with finite spectrum. A variant of Theorem 11 with prescribed multiplicities, which has a more complicated statement and proof, was shown by the authors in [10]. In light of Theorem 1, the results in [7, 10] answer Problem 1 in the case when frame operator S has finite spectrum.

Theorem 11. *Let $\{A_j\}_{j=0}^{n+1}$ be an increasing sequence of real numbers such that $A_0 = 0$ and $A_{n+1} = B$, $n \in \mathbb{N}$. Let $\{d_i\}_{i\in I}$ be a sequence in $[0,B]$ with $\sum d_i = \sum(B - d_i) = \infty$. For each $\alpha \in (0,B)$, define*

$$C(\alpha) = \sum_{d_i<\alpha} d_i \quad \text{and} \quad D(\alpha) = \sum_{d_i\geq\alpha} (B - d_i). \tag{14}$$

Then the following are equivalent:

1. *there exists a self-adjoint operator E with diagonal $\{d_i\}_{i\in I}$ and $\sigma(E) = \{A_0, A_1, \dots, A_{n+1}\}$,*
2. *there exists a frame $\{f_i\}_{i\in I}$ on some Hilbert space $\mathscr{H}$ with $d_i = ||f_i||^2$ for all $i \in I$ such that its frame operator S has spectrum $\sigma(S) = \{A_1, \dots, A_{n+1}\}$,*
3. *one of the following holds:*

   a. *$C(B/2) = \infty$,*
   b. *$D(B/2) = \infty$,*
   c. *$C(B/2) < \infty$ and $D(B/2) < \infty$ (and thus $C(\alpha), D(\alpha) < \infty$ for all $\alpha \in (0,B)$), and there exist $N_1, \dots, N_n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that*

$$C(B/2) - D(B/2) = \sum_{j=1}^{n} A_j N_j + kB, \tag{15}$$

and for all $r = 1, \dots, n$,

$$(B - A_r)C(A_r) + A_r D(A_r) \geq (B - A_r)\sum_{j=1}^{r} A_j N_j + A_r \sum_{j=r+1}^{n} (B - A_j)N_j. \tag{16}$$

We remark that the assumption $\sum d_i = \sum(B - d_i) = \infty$ is not a true limitation of Theorems 9 and 11. Indeed, the summable case $\sum d_i < \infty$, or its symmetric variant $\sum(B - d_i) < \infty$, leads to a finite rank Schur-Horn theorem. This case requires a different set of conditions which are closely related to the classical Schur-Horn majorization. Finally, the assumption $A_0 = 0$ is made only for simplicity; the general case follows immediately by a translation argument.

We shall illustrate Theorem 11 by considering the problem of describing possible frame operators with prescribed frame norms. This can be thought as the converse to Problem 1. A rigorous formulation of this problem and a solution was shown by the authors in [10, Theorem 8.2]. Here we shall concentrate only on the spectral variant of this problem studied in [9].

Definition 2. Suppose that $\{d_i\}_{i\in I}$ is a sequence in $[0,1]$. For a given $n \in \mathbb{N}$ and a sequence $\{d_i\}_{i\in I}$ in $[0,1]$ we consider the set

$$\begin{aligned}\mathscr{A}_n(\{d_i\}) = \big\{(A_1,\ldots,A_n) \in (0,1)^n : \forall_{j\neq k}\, A_j \neq A_k\ \exists \text{ frame } \{f_i\}_{i\in I} \text{ s. t.}\\ \forall_{i\in I}\, d_i = ||f_i||^2 \text{ and spectrum of frame operator } \sigma(S) = \{A_1,\ldots,A_n,1\}\big\}.\end{aligned}$$

Subsequently we shall assume that $\sum d_i = \sum(1-d_i) = \infty$, so that we can apply Theorem 11. The authors have shown in [9, Theorem 3.8] that the sets $\mathscr{A}_n(\{d_i\})$ are nonempty for each $n \geq 2$ provided that infinitely many d_is satisfy $d_i \in (0,1)$. However, the set $\mathscr{A}_1(\{d_i\})$ might be empty as illustrated by Example 5. Moreover, the second author [24, Theorem 7.1] has shown that

$$\mathscr{A}_1(\{d_i\}) = \begin{cases}(0,1) & \text{if } C(1/2) = \infty \text{ or } D(1/2) = \infty,\\ \text{a finite set} & \text{if } C(1/2), D(1/2) < \infty.\end{cases}$$

Example 5. Let $\beta \in (0,1/2)$ and define the sequence $\{d_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ by

$$d_i = \begin{cases}1-\beta^i & i > 0\\ \beta^{-i} & i < 0.\end{cases}$$

Using Theorem 9 the second author has shown in [24] that

$$\mathscr{A}_1(\{d_i\}) = \begin{cases}\{\frac{1}{3},\frac{1}{2},\frac{2}{3}\} & \frac{-1+\sqrt{13}}{6} \leq \beta < 1/2,\\ \{\frac{1}{2}\} & 1/3 \leq \beta < \frac{-1+\sqrt{13}}{6},\\ \varnothing & 0 < \beta < 1/3.\end{cases}$$

The following result [9, Corollary 3.13] describes the spectral sets $\mathscr{A}_2(\{d_i\})$.

Theorem 12. *Let $\{d_i\}_{i\in I}$ be a sequence in $[0,1]$. If $C(1/2) = \infty$ or $D(1/2) = \infty$, then*

$$\mathscr{A}_2(\{d_i\}) = (0,1)^2 \setminus \Delta, \qquad \text{where } \Delta = \{(x,x) : x \in (0,1)\}.$$

Otherwise, if $C(1/2), D(1/2) < \infty$, then the set $\mathscr{A}_2(\{d_i\})$ is nonempty and it consists of a countable union of line segments. Moreover, one end point of each of these line segments must lie in the boundary of the unit square.

Example 6. For a given $\eta \in [0,1]$ consider the sequence $\{d_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ from Example 3. Using the characterization from Theorem 11 and numerical calculations performed with *Mathematica*, Figures 1, 2, 3, and 4 depict the set $\mathscr{A}_2(\{d_i\})$ for different values of the parameter η.

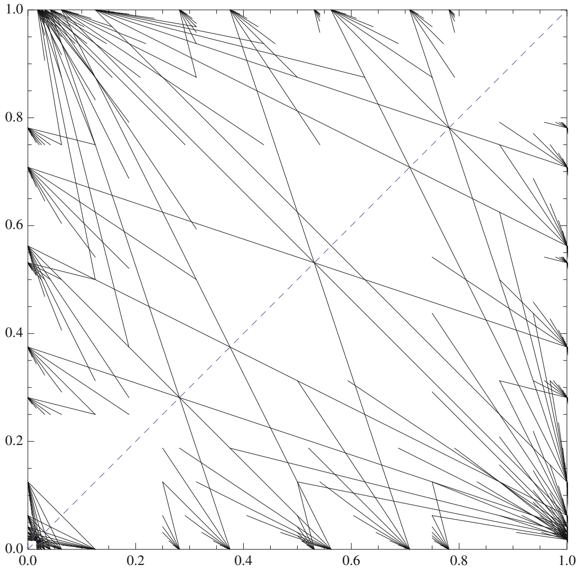

Fig. 1 The set $\mathscr{A}_2(\{d_i\})$ for the sequence (10) with $\eta = \frac{1}{8}$.

Other Schur-Horn-type theorems

We shall finish this chapter by mentioning other important developments in extending the Schur-Horn Theorem 2 to infinite dimensional Hilbert spaces. Neumann [32] gave an infinite dimensional version of the Schur-Horn theorem phrased in terms of ℓ^∞ closure of the convexity condition (3). Neumann's result can be considered an initial, albeit somewhat crude, solution to Problem 2. The first fully satisfactory progress was achieved by Kadison [25, 26] who solved Problem 2 for orthogonal projections. The work by Gohberg and Markus [21] and Arveson and Kadison [4] extended the Schur-Horn Theorem 2 to positive trace class operators. This has been further extended to compact positive operators by Kaftal and Weiss [28]. These results are stated in terms of majorization inequalities as in (2). Other notable progress includes the work of Arveson [3] on diagonals of normal operators with finite spectrum. For a detailed survey of recent progress on infinite Schur-Horn majorization theorems and their connections to operator ideals we refer to the paper of Kaftal and Weiss [27].

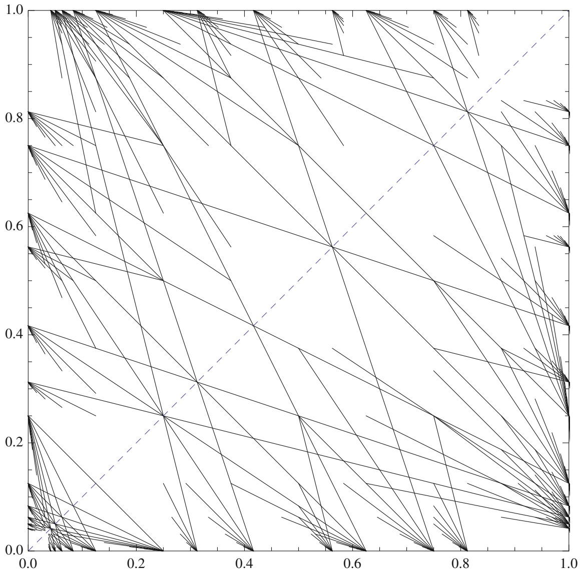

Fig. 2 The set $\mathscr{A}_2(\{d_i\})$ for the sequence (10) with $\eta = \frac{1}{4}$.

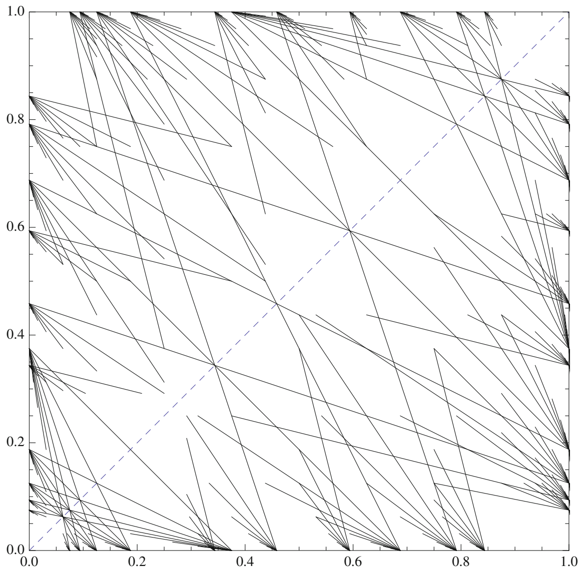

Fig. 3 The set $\mathscr{A}_2(\{d_i\})$ for the sequence (10) with $\eta = \frac{3}{8}$.

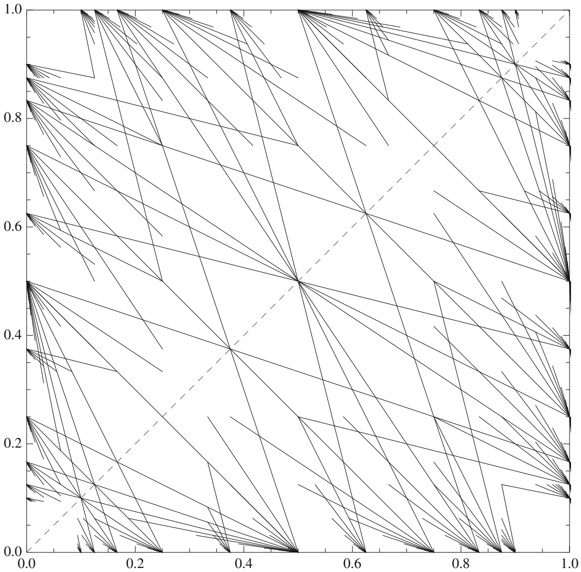

Fig. 4 The set $\mathscr{A}_2(\{d_i\})$ for the sequence (10) with $\eta = \frac{1}{2}$.

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