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The Structure of Translation-Invariant Spaces on Locally Compact Abelian Groups

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Abstract Let Γ be a closed co-compact subgroup of a second countable locally compact abelian (LCA) group G. In this paper we study translation-invariant (TI) subspaces of $L^2(G)$ by elements of Γ. We characterize such spaces in terms of range functions extending the results from the Euclidean and LCA setting. The main innovation of this paper, which contrasts with earlier works, is that we do not require that Γ be discrete. As a consequence, our characterization of TI-spaces is new even in the classical setting of $G = \mathbb{R}^n$. We also extend the notion of the spectral function in $\mathbb{R}^n$ to the LCA setting. It is shown that spectral functions, initially defined in terms of Γ, do not depend on Γ. Several properties equivalent to the definition of spectral functions are given. In particular, we show that the spectral function scales nicely under the action of epimorphisms of G with compact kernel. Finally, we show that for a large class of LCA groups, the spectral function is given as a pointwise limit.

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1 Introduction

The main goal of this paper is to develop a comprehensive analogue of the theory of shift-invariant (SI) spaces in the setting of locally compact abelian (LCA) groups. Classically, SI spaces are closed subspaces of $L^2(\mathbb{R}^n)$ that are invariant under all integer translations. The theory of shift-invariant subspaces of $L^2(\mathbb{R}^n)$ plays an important role in many areas, most notably in the theory of wavelets, spline systems, Gabor systems, and approximation theory [3–5,7,29,30]. Hence, we are interested in finding out how much of the classical SI space theory extends to the LCA setting.

Let G be a second countable LCA group. The aim of this paper is to investigate the structure of subspaces of $L^2(G)$ that are invariant under translations by elements of a closed co-compact subgroup $\Gamma \subset G$, where co-compact means that the quotient group G/Γ is compact. We characterize such spaces in terms of range functions which originated in the characterization of doubly invariant spaces by Helson [17] and Srinivasan [32]. Our work extends the results from the Euclidean setting $G = \mathbb{R}^n$ by the first author [5] and the LCA setting by Cabrelli and Paternostro [9] and by Kamyabi Gol and Raisi Tousi [23]. The main innovation of this paper, which contrasts with earlier works, is that we do not require that Γ be discrete. To emphasize the fact that Γ need not be discrete we deliberately adopt the name translation-invariant (TI) spaces instead of SI spaces. As a consequence, our characterization of Γ-TI spaces is new even in the classical setting of $G = \mathbb{R}^n$. Moreover, our paper also applies to LCA groups G that do not have uniform lattices (discrete co-compact subgroups of G) such as the group $\mathbb{Q}_p$ of p-adic numbers and its compact subgroup of p-adic integers, where the results of [9,23] are not applicable. In what follows we describe the content of the paper.

In Sect. 2 we introduce a general machinery of multiplicatively-invariant (MI) subspaces of the vector-valued space $L^2(\Omega, \mathcal{H})$, where Ω is a σ-finite measure space and $\mathcal{H}$ is a separable Hilbert space. In Theorem 2.4 we show that MI spaces are characterized in terms of range functions. This result extracts a measure theoretic component behind existing proofs of range function characterizations of SI spaces in various settings. In a similar vein, we provide a decomposition of MI spaces into orthogonal sums of principal MI spaces, each generated by a single vector-valued function in $L^\infty(\Omega, \mathcal{H})$. In Sect. 3 we prove our main result, Theorem 3.8, which characterizes TI subspaces of $L^2(G)$. Since a closed subgroup $\Gamma \subset G$ is assumed to be merely co-compact, this theorem unifies under the same framework the characterization of SI spaces (when Γ is assumed to be discrete) and Wiener's theorem on TI spaces (when $\Gamma = G$). At the same time it yields intermediate results when Γ is a non-discrete closed co-compact proper subgroup of G. The key role in our arguments is played by the fiberization map defined on a Borel section of the quotient of the dual group $\widehat{G}$ with the annihilator of Γ. In the case when Γ is not discrete, Ω has infinite measure which requires some special technical tools such as a continuous variant of the Parseval identity in Lemma 3.5.

In Sect. 4 we define the spectral function for TI spaces on LCA groups which was originally introduced by Rzeszotnik and the first author [7,8] for SI subspaces of $L^2(\mathbb{R}^n)$. We show that that spectral function, initially defined in terms of Γ, actually does not depend on the choice of Γ. This is a generalization of [8, Corollary 2.7]. To

illustrate more properties of the spectral function, we introduce in Sect. 5 the notion of continuous frame sequences for TI spaces. In the case when Γ is discrete, this notion coincides with the classical definition of an SI frame. Theorem 5.1 provides a characterization of continuous frames in terms of the fiberization operator by generalizing the corresponding result on $\mathbb{R}^n$ by the first author [5] and its LCA extension by Cabrelli and Paternostro [9]. We also show the decomposition of TI spaces as an orthogonal sum of principal TI spaces. Several properties are established for spectral functions, such as Theorems 5.4 and 5.5, which generalize the respective results from [7].

In Sect. 6 we investigate how the spectral function behaves under the action of dilations. The results of Rzeszotnik and the first author [7,8] show that the spectral function on $\mathbb{R}^n$ scales nicely when an SI space is dilated by an invertible matrix. We obtain a satisfactory analogue of this result for a general LCA group G by replacing an invertible matrix on $\mathbb{R}^n$ by an epimorphism of G with compact kernel. In Sect. 7 we show that the spectral function can be obtained as a pointwise limit using the Lebesgue Differentiation Theorem. Our result requires the existence of so-called D'-sequences which exist for compactly generated abelian Lie groups. However, it is shown that D'-sequences do not exist for all LCA groups such as the "tubby torus" $\mathbb{T}^{\aleph_0}$. Finally, in the 'Appendix' we give a simplified proof of Theorem (9.12) in the book of Hewitt and the second author [20].

2 General Range Functions and Multiplicatively-Invariant Spaces

In this section (Ω, m) represents a σ-finite measure space, and $\mathcal{H}$ is a separable Hilbert space. Our goal is to establish a general machinery of multiplicatively-invariant subspaces of the vector-valued space $L^2(\Omega, \mathcal{H})$. In a nutshell, our result on multiplicatively-invariant subspaces, Theorem 2.4, extracts a measure theoretic component from the proof of the characterization of shift-invariant spaces on $L^2(\mathbb{R}^n)$ in [5]. In turn, Theorem 2.4 can be used not only to recover the existing results on SI spaces on LCA groups [9,23], but also to obtain our main result on TI spaces in Theorem 3.8. Also our machinery is versatile enough to apply in the study of shift-modulation spaces [6]. Thus, Theorem 2.4 yields a streamlined approach in a variety of settings instead of repeating ad hoc arguments as has been done in the past.

Definition 2.1 A range function is a mapping $J : \Omega \to \{\text{closed subspaces of } \mathcal{H}\}$. For $\omega \in \Omega$, we write $P_J(\omega)$ for the projection of $\mathcal{H}$ onto $J(\omega)$. A range function is measurable if for each $a \in \mathcal{H}$, $\omega \mapsto P_J(\omega)(a)$ from $\Omega \to \mathcal{H}$ is measurable, which is equivalent in our setting to: for $a, b \in \mathcal{H}$, $\omega \mapsto \langle P_J(\omega)(a), b \rangle$ is measurable.

Remark 2.1 J is a measurable range function if and only if $\omega \to \langle P_J(\omega)(\Phi(\omega)), b \rangle$ is measurable for each Φ in $L^2(\Omega, \mathcal{H})$. This can be deduced from Pettis's measurability theorem; see Theorem 2 in [11, Chapter 2]. Indeed, suppose J is a measurable range function. Then, using linearity, $\omega \mapsto \langle P_J(\omega)(\Phi(\omega)), b \rangle$ is measurable if Φ is a simple measurable function, i.e., $\Phi(\omega) = \sum_{k=1}^{n} \mathbf{I}_{A_k}(\omega) a_k$, where A_k are measurable subsets of Ω and $a_k \in \mathcal{H}$. By Pettis's measurability theorem any Φ is a limit a.e. (in Ω) of a sequence (Φ_j) of simple functions. So $\langle P_J(\omega)(\Phi(\omega)), b \rangle = \lim_j \langle P_J(\omega)(\Phi_j(\omega)), b \rangle$ is also measurable.

We will work with

$$L^2(\Omega, \mathcal{H}) := \{ \text{ measurable } \Phi : \Omega \to \mathcal{H} : \|\Phi\|^2 = \int_\Omega \|\Phi(\omega)\|_{\mathcal{H}}^2 \, dm(\omega) < \infty \}.$$

The inner product on $L^2(\Omega, \mathcal{H})$ is given by

$$\langle \Phi, \Psi \rangle = \int_\Omega \langle \Phi(\omega), \Psi(\omega) \rangle_{\mathcal{H}} \, dm(\omega).$$

The following result is well known; see [5] or [9, Lemma 3.8].

Proposition 2.1 *Let J be a (not necessarily measurable) range function. Then*

$$M_J := \{ \Phi \in L^2(\Omega, \mathcal{H}) : \Phi(\omega) \in J(\omega) \text{ a.e. } \omega \in \Omega \} \tag{2.1}$$

is a closed linear subspace of $L^2(\Omega, \mathcal{H})$.

Let $\mathcal{P}_J$ denote the orthogonal projection of $L^2(\Omega, \mathcal{H})$ onto M_J. The next proposition relates this function to the projections $P_J(\omega)$ of $\mathcal{H}$ onto $J(\omega)$, in the case when J is measurable. Proposition 2.2 is an abstract version of a lemma due to Helson [17, Lecture VI], which was reproven in various special settings in [5,6,9].

Proposition 2.2 *Let J be a measurable range function. Then, for $\Phi \in L^2(\Omega, \mathcal{H})$, we have*

$$\mathcal{P}_J(\Phi)(\omega) = P_J(\omega)(\Phi(\omega)) \text{ a.e. } \omega \in \Omega. \tag{2.2}$$

Proof Define $\mathcal{Q}_J : L^2(\Omega, \mathcal{H}) \to L^2(\Omega, \mathcal{H})$ by

$$\mathcal{Q}_J(\Phi)(\omega) = P_J(\omega)(\Phi(\omega)).$$

It suffices to prove $\mathcal{Q}_J = \mathcal{P}_J$.

By Remark 2.1, $\mathcal{Q}_J(\Phi)$ is measurable for each Φ in $L^2(\Omega, \mathcal{H})$, and clearly $\mathcal{Q}_J$ is linear. Since each $P_J(\omega)$ is a projection, each $P_J(\omega)$ has norm 1, so

$$\|\mathcal{Q}_J(\Phi)\|_2^2 = \int_\Omega \|\mathcal{Q}_J(\Phi)(\omega)\|_{\mathcal{H}}^2 \, dm(\omega) = \int_\Omega \|P_J(\omega)(\Phi(\omega))\|_{\mathcal{H}}^2 \, dm(\omega)$$

$$\leq \int_\Omega \|\Phi(\omega)\|_{\mathcal{H}}^2 \, dm(\omega) = \|\Phi\|_2^2 < \infty.$$

Thus, $\mathcal{Q}_J$ is well defined and it has norm ≤ 1.

Since $\mathcal{Q}_J^2(\Phi)(\omega) = P_J^2(\omega)(\Phi(\omega)) = P_J(\omega)(\Phi(\omega))$, we have $\mathcal{Q}_J^2 = \mathcal{Q}_J$ and similarly $\mathcal{Q}_J^* = \mathcal{Q}_J$, so $\mathcal{Q}_J$ is a projection onto a closed subspace M of $L^2(\Omega, \mathcal{H})$. It suffices to show $M = M_J$, and clearly $M \subset M_J$. If $M \neq M_J$, then there is $\Psi \in M_J$ such that $\Psi \neq 0$ and $\Psi \perp M$. For all Φ in $L^2(\Omega, \mathcal{H})$, we now have

$$0 = \langle \mathcal{Q}_J(\Phi), \Psi \rangle = \langle \Phi, \mathcal{Q}_J(\Psi) \rangle,$$

so $\mathcal{Q}_J(\Psi) = 0$. Thus $P_J(\omega)(\Psi(\omega)) = 0$ a.e. $\omega \in \Omega$. Since $\Psi(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$, we conclude that $0 = P_J(\omega)(\Psi(\omega)) = \Psi(\omega)$ a.e. $\omega \in \Omega$, and this contradicts $\Psi \neq 0$.
□

Proposition 2.3 is another abstract variant of a known result; see [5] or [9, Lemma 3.11].

Proposition 2.3 *If J and K are measurable range functions and $M_J = M_K$, then $J(\omega) = K(\omega)$ a.e. $\omega \in \Omega$.*

Proof The projections $\mathcal{P}_J$ and $\mathcal{P}_K$ are equal, so by (2.2),

$$P_J(\omega)(\Phi(\omega)) = P_K(\omega)(\Phi(\omega)) \text{ a.e. } \omega \in \Omega,$$

for each Φ in $L^2(\Omega, \mathcal{H})$. Let D be a countable dense subset of $\mathcal{H}$, and let Ω' be a subset of Ω of finite measure. For each $a \in D$, we apply this to the function $\Phi(\omega) = a \mathbf{1}_{\Omega'}$. Thus $P_J(\omega)(a) = P_K(\omega)(a)$ a.e. on Ω', for each $a \in D$. Since Ω is σ-finite, we conclude that

for a.e. $\omega \in \Omega$, we have $P_J(\omega)(a) = P_K(\omega)(a)$ for all $a \in D$.

For each such ω, the functions $a \mapsto P_J(\omega)(a)$ and $a \mapsto P_K(\omega)(a)$ are continuous on $\mathcal{H}$ and agree on the dense subset D of $\mathcal{H}$, so $P_J(\omega) = P_K(\omega)$ a.e. $\omega \in \Omega$. Therefore, $J(\omega) = K(\omega)$ a.e. $\omega \in \Omega$.
□

The following concept plays a key role in our development of multiplicatively-invariant spaces.

Definition 2.2 A subset $\mathcal{D}$ of $L^\infty(\Omega)$ is a *determining set* for $L^1(\Omega)$ if

$$\text{for every } f \in L^1(\Omega), \quad \int_\Omega fg \, dm = 0 \quad \text{for all } g \in \mathcal{D} \implies f = 0. \tag{2.3}$$

In particular, any subset $\mathcal{D}$ of $L^\infty(\Omega)$, for which span($\mathcal{D}$) is weak-∗ dense in $L^\infty(\Omega)$, is a determining set for $L^1(\Omega)$.

Example 2.2 (a) Let $\mathcal{D}$ consist of all characteristic functions $\mathbf{1}_A$ for measurable subsets A of Ω having finite measure. Then $\mathcal{D}$ is a determining set for $L^1(\Omega)$.

(b) Let G be a second countable LCA group. Then the set $\widehat{G}$ of characters on G is a determining set for $L^1(G, m_G)$, where m_G is a Haar measure on G. This follows from the uniqueness theorem for Fourier transforms:

$$\int_G \overline{\chi(x)} f(x) \, dm_G(x) = 0 \quad \text{for all } \chi \in \widehat{G} \implies f = 0.$$

Definition 2.3 Let W be a closed linear subspace of $L^2(\Omega, \mathcal{H})$. We say that W is *multiplicatively-invariant* with respect to a determining subset $\mathcal{D}$ for $L^1(\Omega)$ (briefly, $\mathcal{D}$-MI) if

$$\Phi \in W \text{ and } g \in \mathcal{D} \implies g\Phi \in W. \tag{2.4}$$

Note that we may as well assume that the constant function 1 is in $\mathcal{D}$. Note also that $g\Phi$ is the product of a bounded measurable scalar-valued function and a measurable Hilbert-space-valued function on Ω. Thus $\Phi \mapsto g\Phi$ maps $L^2(\Omega, \mathcal{H})$ into itself. The equivalence of (ii) and (iii) below was shown by Helson [18, Theorem 1, p. 7].

Theorem 2.4 *Suppose that $L^2(\Omega)$ is separable, so that $L^2(\Omega, \mathcal{H})$ is also separable. Let $W \subset L^2(\Omega, \mathcal{H})$ be a closed subspace, and let $\mathcal{D}$ be a determining set for $L^1(\Omega)$. The following are equivalent:*

(i) *W is $\mathcal{D}$-MI,*
(ii) *W is $L^\infty(\Omega)$-MI,*
(iii) *there exists a measurable range function J such that*

$$W = \{\Phi \in L^2(\Omega, \mathcal{H}) : \Phi(\omega) \in J(\omega) \; a.e. \; \omega \in \Omega\}. \tag{2.5}$$

Identifying range functions which are equal almost everywhere, the correspondence between $\mathcal{D}$-MI spaces and measurable range functions is one-to-one and onto.

Moreover, since $L^2(\Omega, \mathcal{H})$ is separable, there are countable subsets $\mathcal{A} \subset L^2(\Omega, \mathcal{H})$ such that W is the smallest closed $\mathcal{D}$-MI space containing $\mathcal{A}$. For any such $\mathcal{A}$, the measurable range function J associated to W satisfies

$$J(\omega) = \overline{\mathrm{span}}\{\Phi(\omega) : \Phi \in \mathcal{A}\}, \quad a.e. \; \omega \in \Omega. \tag{2.6}$$

Proof The implications $(iii) \implies (ii) \implies (i)$ are trivial. Hence, it remains to show $(i) \implies (iii)$. At the same time we prove the "moreover" part of Theorem 2.4. Suppose W is a $\mathcal{D}$-MI subspace of $L^2(\Omega, \mathcal{H})$. Since $L^2(\Omega, \mathcal{H})$ is separable,

$$W = \overline{\mathrm{span}}\{g\Phi : g \in \mathcal{D}, \Phi \in \mathcal{A}\} \tag{2.7}$$

for some countable $\mathcal{A} \subset L^2(\Omega, \mathcal{H})$. Now we define J via (2.6), so J is clearly a range function.

We need the following lemma.

Lemma 2.5 *Let J and W be given by (2.6) and (2.7), respectively. For any Ψ in $L^2(\Omega, \mathcal{H})$, we have*

$$\Psi \perp W \iff \Psi(\omega) \in J(\omega)^\perp \; a.e. \; \omega \in \Omega. \tag{2.8}$$

Thus, $W^\perp = \{\Phi \in L^2(\Omega, \mathcal{H}) : \Phi(\omega) \in J(\omega)^\perp \; a.e. \; \omega \in \Omega\}$.

Proof To show $\implies$ in (2.8), the definition of J implies that it suffices to show

$$\langle \Phi(\omega), \Psi(\omega) \rangle = 0 \; a.e. \; \omega \in \Omega, \quad \Phi \in \mathcal{A}. \tag{2.9}$$

Fix Φ in $\mathcal{A}$, and let $F(\omega) = \langle \Phi(\omega), \Psi(\omega) \rangle$. Since $|F(\omega)| \leq \| \Phi(\omega)\|_{\mathcal{H}} \|\Psi(\omega)\|_{\mathcal{H}}$, we have

$$\int_{\Omega} |F(\omega)| \, dm(\omega) \leq \int_{\Omega} \|\Phi(\omega)\|_{\mathcal{H}} \|\Psi(\omega)\|_{\mathcal{H}} \, dm(\omega)$$
$$\leq \left(\int_{\Omega} \|\Phi(\omega)\|_{\mathcal{H}}^2 \, dm(\omega) \right)^{1/2} \left(\int_{\Omega} \|\Psi(\omega)\|_{\mathcal{H}}^2 \, dm(\omega) \right)^{1/2}$$
$$= \|\Phi\|_2 \|\Psi\|_2 < \infty.$$

Therefore, F is in $L^1(\Omega)$. For $g \in \mathcal{D}$, we have $g\Phi \in W$, so

$$\int_{\Omega} g F \, dm = \int_{\Omega} \langle g(\omega)\Phi(\omega), \Psi(\omega) \rangle \, dm(\omega) = \langle g\Phi, \Psi \rangle = 0.$$

Since $\mathcal{D}$ is a determining set for $L^1(\Omega)$, we have $F(\omega) = 0$ a.e. $\omega \in \Omega$, i.e., (2.9) holds.

For $\Longleftarrow$ in (2.8), it suffices to observe that $\Psi(\omega) \in J(\omega)^{\perp}$ a.e. $\omega \in \Omega$ implies

$$\langle g\Phi, \Psi \rangle = \int_{\Omega} g(\omega) \langle \Phi(\omega), \Psi(\omega) \rangle \, dm(\omega) = 0 \quad \text{for all } g \in \mathcal{D}, \ \Phi \in \mathcal{A}.$$

Thus, (2.5) holds for $W^{\perp}$ and the range function $\omega \mapsto J(\omega)^{\perp}$. $\square$

We return to the proof of $(i) \implies (iii)$ in Theorem 2.4. We first establish (2.5), i.e., $W = M_J$. By assumption, W is a closed subspace of $L^2(\Omega, \mathcal{H})$, and M_J given by (2.1) is a closed subspace of $L^2(\Omega, \mathcal{H})$ by Proposition 2.1. To show that $W \subset M_J$, consider $g \in \mathcal{D}$ and $\Phi \in \mathcal{A}$. Then, $\Phi(\omega)$ is in $J(\omega)$ for almost all $\omega \in \Omega$ by (2.6), so $g(\omega)\Phi(\omega)$ is also in $J(\omega)$ for almost all $\omega \in \Omega$. So $W \subset M_J$. To show the converse implication, on the contrary suppose there exists $\Psi \in M_J \setminus W$, $\Psi \neq 0$. We can choose Ψ orthogonal to W. Then, by Lemma 2.5 and (2.1), we have $\Psi(\omega) \in J(\omega) \cap J(\omega)^{\perp} = \{0\}$ for a.e. $\omega \in \Omega$, contradicting $\Psi \neq 0$. Thus, we conclude that $W = M_J$.

Next we show that the range function J is measurable. Let $\mathbf{I}$ be the identity on $L^2(\Omega, \mathcal{H})$, and let $\mathcal{P}_J$ be the projection of $L^2(\Omega, \mathcal{H})$ onto M_J, as in Proposition 2.2. If Ψ is in $L^2(\Omega, \mathcal{H})$, then $(\mathbf{I} - \mathcal{P}_J)(\Psi)$ is orthogonal to $M_J = W$. Hence, by Lemma 2.5, $((\mathbf{I} - \mathcal{P}_J)(\Psi))(\omega)$ is in $J(\omega)^{\perp}$ a.e. $\omega \in \Omega$. So

$$P_J(\omega)(((\mathbf{I} - \mathcal{P}_J)(\Psi))(\omega)) = 0 \ \ a.e. \ \ \omega \in \Omega,$$

and this implies $P_J(\omega)(\Psi(\omega)) = P_J(\omega)(\mathcal{P}_J(\Psi))(\omega)) = \mathcal{P}_J(\Psi)(\omega)$ a.e. $\omega \in \Omega$. Thus, $\omega \mapsto P_J(\omega)(\Psi(\omega))$ is measurable, and by Remark 2.1, J is measurable.

Finally, to show that the correspondence between measurable range functions and $\mathcal{D}$-MI subspaces of $L^2(\Omega, \mathcal{H})$ is one-to-one, we invoke Proposition 2.3. $\square$

Remark 2.3 In light of Theorem 2.4, any space W satisfying either (i), (ii), or (iii) is said to be multiplicatively-invariant (MI). Lemma 2.5 shows that properties (i)–(iii) in Theorem 2.4 are equivalent to the same properties with W replaced by $W^{\perp}$ (and $J(\omega)$ replaced by $J(\omega)^{\perp}$). In particular, W is MI $\Longleftrightarrow$ $W^{\perp}$ is MI.

Definition 2.4 The dimension function $\dim_W : \Omega \to \{0, 1, 2, \ldots, \dim \mathcal{H}\}$ is defined by $\dim_W(\omega) = \dim(J(\omega))$ for $\omega \in \Omega$.

Suppose we have two MI spaces W_1 and W_2 with corresponding range functions J_1 and J_2. Then using Lemma 2.5 one can show that

$$W_1 \perp W_2 \iff J_1(\omega) \perp J_2(\omega) \qquad \text{for a.e. } \omega \in \Omega.$$

Consequently, for any collection of measurable range functions J_n, $n \in \mathbb{N}$, and the corresponding MI spaces $M_{J_n}, n \in \mathbb{N}$, we have

$$J(\omega) = \bigoplus_{n \in \mathbb{N}} J_n(\omega) \quad \text{for a.e. } \omega \in \Omega \iff M_J = \bigoplus_{n \in \mathbb{N}} M_{J_n}. \qquad (2.10)$$

We have the following decomposition result which is an abstract version of [5, Theorem 3.3]; see also [18, Theorem 2, p. 8].

Theorem 2.6 *Suppose that $W \subset L^2(\Omega, \mathcal{H})$ is an MI space. Then there exist MI spaces W_n, $n \in \mathbb{N}$, such that:*

(i) *W can be decomposed as an orthogonal sum*

$$W = \bigoplus_{n \in \mathbb{N}} W_n, \qquad (2.11)$$

(ii) *the dimension function satisfies $\dim_{W_n}(\omega) \leq 1$ for a.e. $\omega \in \Omega$, and*

$$\dim_{W_n}(\omega) = 1 \iff \dim_W(\omega) \geq n, \quad \text{for } n \in \mathbb{N}. \qquad (2.12)$$

(iii) *there exist functions $\Phi_n \in L^\infty(\Omega, \mathcal{H})$ such that $||\Phi_n(\omega)|| \in \{0, 1\}$, and the range function J_n of W_n is given by $J_n(\omega) = \text{span } \Phi_n(\omega)$.*

Proof Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. For any MI space V we shall construct a function $\Phi(V) \in L^\infty(\Omega, \mathcal{H})$ using the following procedure. Let J_V be the range function corresponding to V given by Theorem 2.4(iii), and let $P_V(\omega)$ be the orthogonal projection of $\mathcal{H}$ onto $J_V(\omega)$. If $V = \{0\}$, then $\Phi(V) = 0$. Otherwise, define $A_k = \{\omega \in \Omega : P_V(\omega)e_k \neq 0\}$ for $k \in \mathbb{N}$. Define sets $B_k, k \in \mathbb{N}$, inductively by $B_1 = A_1$, $B_k = A_k \setminus \bigcup_{j=1}^{k-1} A_j$. Finally, define $\Phi(V) \in L^\infty(\Omega, \mathcal{H})$ so that

$$\Phi(V)(\omega) = P_V(\omega)e_k / ||P_V(\omega)e_k|| \text{ for } \omega \in B_k, \quad \text{and}$$
$$\Phi(V)(\omega) = 0 \text{ for } \omega \in \Omega \setminus \bigcup_{k=1}^{\infty} B_k.$$

Clearly, for a.e. $\omega \in \Omega$ we have

$$\Phi(V)(\omega) \in J_V(\omega) \quad \text{and} \quad ||\Phi(V)(\omega)|| = \begin{cases} 1 & \text{when } \dim_V(\omega) \geq 1, \\ 0 & \text{when } \dim_V(\omega) = 0. \end{cases}$$

In addition, suppose that for some $k_0 \in \mathbb{N}$ we have $m(A_k) = 0$ for all $1 \leq k < k_0$. Equivalently,

$$J_V(\omega) \subset \overline{\text{span}}\{e_k : k \geq k_0\} \qquad \text{for a.e. } \omega \in \Omega.$$

Consider the MI space

$$\tilde{V} = \{\Psi \in V : \Psi(\omega) \perp \Phi(V)(\omega) \quad \text{for a.e. } \omega \in \Omega\}. \qquad (2.13)$$

Then the range function $J_{\tilde{V}}$ of $\tilde{V}$ satisfies

$$J_{\tilde{V}}(\omega) = J_V(\omega) \ominus \text{span}\,\Phi(V)(\omega) \subset \overline{\text{span}}\{e_k : k \geq k_0 + 1\} \qquad \text{for a.e. } \omega \in \Omega. \tag{2.14}$$

Indeed, if $\Psi \in \tilde{V}$, then for a.e. $\omega \in A_{k_0}$

$$\langle \Psi(\omega), e_{k_0} \rangle = \langle \Psi(\omega), P_V(\omega)e_{k_0} \rangle = ||P_V(\omega)e_{k_0}||\langle \Psi(\omega), \Phi(V)(\omega) \rangle = 0.$$

Since the above equality also holds for $\omega \notin A_{k_0}$, we have $J_{\tilde{V}}(\omega) \perp e_{k_0}$ for a.e. $\omega \in \Omega$. This shows (2.14).

Define a sequence of nested MI spaces $\{V_n\}_{n \in \mathbb{N}}$ and a sequence of functions $\{\Phi_n\}_{n \in \mathbb{N}}$ in $L^\infty(\Omega, \mathcal{H})$ by the following inductive procedure. Let $V_1 = W$ and $\Phi_1 = \Phi(V_1)$. Once $\Phi_1, \ldots, \Phi_n$ and $V_1, \ldots, V_n$ are constructed define $\Phi_{n+1} = \Phi(V_{n+1})$, where

$$V_{n+1} = \{\Psi \in V_n : \Psi(\omega) \perp \Phi_n(\omega) \quad \text{for a.e. } \omega \in \Omega\}. \qquad (2.15)$$

For any $n \in \mathbb{N}$ we have $V_n = V_{n+1} \oplus W_n$, where W_n is a MI space with the range function $J_n(\omega) = \text{span}\,\Phi_n(\omega)$. Hence, to establish (2.11) we need to show that $\bigcap_{n \in \mathbb{N}} V_n = \{0\}$. Indeed, by (2.13), (2.14), and (2.15), the range function of V_n satisfies

$$J_{V_n}(\omega) \subset \overline{\text{span}}\{e_k : k \geq n\} \qquad \text{for a.e. } \omega \in \Omega.$$

This shows $\bigcap_{n \in \mathbb{N}} V_n = \{0\}$, so that (i) and (iii) hold.

By (2.10) and (2.11) we have that $J(\omega) = \bigoplus_{n \in \mathbb{N}} J_n(\omega)$ for a.e. $\omega \in \Omega$. Hence, $\dim_W = \sum_{n \in \mathbb{N}} \dim_{W_n}$. Combining this with the fact that $1 \geq \dim_{W_n} \geq \dim_{W_{n+1}}$ for all $n \in \mathbb{N}$, we obtain (ii). □

3 Translation-Invariant Spaces on LCA Groups

In this section we establish a characterization of TI subspaces of $L^2(G)$ under the action of a closed co-compact subgroup $\Gamma \subset G$. We deliberately adopt the name TI instead of the commonly used shift-invariant (SI) subspaces since the subgroup Γ does not have to be discrete as in the existing works [3,5,9,23]. This is the main innovation of our approach which has substantial consequences. On the one hand, Theorem 3.8 generalizes the characterization of SI spaces on LCA groups [9,23] including the familiar $\mathbb{R}^n$ setting [5]. On the other hand, Theorem 3.8 encompasses a

range of results so that Wiener's theorem on TI subspaces and the characterization of shift-invariant spaces are two extremes of the same result. Some of the intermediate results are new even in the case of $L^2(\mathbb{R}^n)$; see Example 3.3.

3.1 Standing Hypotheses

Our standing hypotheses on an LCA group G, with character group $\widehat{G}$, are any/all of the following equivalent conditions:

(i) G is second countable, i.e., has a countable base.
(ii) G is σ-compact and metrizable,
(iii) $\widehat{G}$ is σ-compact and metrizable,
(iv) $\widehat{G}$ is second countable,
(v) G is topologically isomorphic with $\mathbb{R}^n \times G_0$, where G_0 contains a compact metrizable open subgroup H_0 such that G_0/H_0 is countable,
(vi) $L^2(G)$ and $L^2(\widehat{G})$ are separable Hilbert spaces. Of course, these Hilbert spaces are isometrically isomorphic, by Plancherel's theorem [19, Theorem (31.18)], provided the Haar measures on G and $\widehat{G}$ are chosen properly.

Unless otherwise specified, throughout this paper, Γ will denote a closed co-compact subgroup of G, i.e., G/Γ is compact. Haar measure on G will be denoted by $m = m_G$. For topological groups G_1 and G_2, we write $G_1 \cong G_2$ if the groups are topologically isomorphic. Other LCA terminology will be as in [19,20,28].

Remark 3.1 Our standing hypotheses on G agree exactly with those in Cabrelli and Paternostro [9], as well as in R. A. Kamyabi Gol and R. Raisi Tousi [22–24]. In contrast to their work, we shall not require Γ to be a discrete subgroup (in addition to being co-compact). In other words, Γ does not have to be a uniform lattice.

Example 3.2 (a) Every second countable LCA group G has at least one co-compact closed subgroup, namely $\Gamma = G$. For some groups this is the only choice. For example, the group of p-adic numbers $\mathbb{Q}_p$ has no proper co-compact closed subgroups; in fact, except for the trivial subgroup $\{0\}$, all the proper closed subgroups $H \subset \mathbb{Q}_p$ are compact and open, so G/H is infinite and discrete.
(b) Obviously all closed subgroups of compact groups are co-compact. For a torsion-free compact abelian group, such as the group Δ_p of p-adic integers, the only discrete subgroup is $\{0\}$, though there are numerous closed co-compact subgroups. See [20, Theorem (25.8)] for a characterization of such groups. The product $\mathbb{Q}_p \times \Delta_p$ is an example of a noncompact group with a nontrivial co-compact closed subgroup but no discrete subgroups.

Definition 3.1 Let Γ^* be the annihilator of Γ in $\widehat{G}$ defined by

$$\Gamma^* = \{\chi \in \widehat{G} : \chi(\gamma) = 1 \quad \text{for all } \gamma \in \Gamma\}.$$

Let $\Omega \subset \widehat{G}$ be a Borel section of $\widehat{G}/\Gamma^*$, also known as a *fundamental domain* of $\widehat{G}/\Gamma^*$, whose existence is guaranteed by [26, Lemma 1.1] or [14]. Then, $\widehat{G} = \Omega \oplus \Gamma^*$, i.e.,

every $\chi \in \widehat{G}$ has the form $\omega + \kappa$ for some $\omega \in \Omega$ and $\kappa \in \Gamma^*$, and this representation is unique.

By the duality theorem [31, Lemma 2.1.3], Γ^* is topologically isomorphic to the dual of G/Γ, i.e., $\Gamma^* \cong \widehat{(G/\Gamma)}$. Consequently, Γ is co-compact in $G \iff \Gamma^*$ is a discrete subgroup of $\widehat{G}$. Thus, by our assumption on Γ, Γ^* will always be discrete. Observe also that the group $\widehat{G}/\Gamma^* \cong \widehat{\Gamma}$ need not be compact, so the fundamental domain Ω need not be pre-compact. Since G is σ-compact, Γ^* is countable, so $\widehat{G} = \Gamma^* \oplus \Omega$ is a countable union of translates of Ω. Therefore, Ω has positive, possibly infinite, measure. In the special case of a uniform lattice $\Gamma \subset G$ considered in [9,23, 25], Ω can be chosen to be pre-compact and Ω necessarily has finite measure.

We always choose the Haar measure on Γ^* so that $m_{\Gamma^*}(\{0\}) = 1$. With this agreement, we can choose the Haar measure on $\widehat{G}/\Gamma^*$ so that the following version of [19, Theorem (28.54)] holds.

Theorem 3.1 *For ϕ in $L^1(\widehat{G})$,*

$$\int_{\widehat{G}} \phi \, dm_{\widehat{G}} = \int_{\widehat{G}/\Gamma^*} \sum_{\kappa \in \Gamma^*} \phi(\chi + \kappa) \, dm_{\widehat{G}/\Gamma^*}(\chi + \Gamma^*), \qquad (3.1)$$

where $\sum_{\kappa \in \Gamma^*} \phi(\chi + \kappa)$ converges absolutely for $m_{\widehat{G}}$-a.e. χ.

Proposition 3.2 *The function $\omega \mapsto \Gamma^* + \omega$ is a measure-preserving mapping from $(\Omega, m_{\widehat{G}})$ onto $(\widehat{G}/\Gamma^*, m_{\widehat{G}/\Gamma^*})$. The group Γ is discrete if and only if $m_{\widehat{G}}(\Omega)$ is finite. For a function ϕ on Ω, we define*

$$\phi'(\Gamma^* + \omega) = \phi(\omega) \quad for \quad \Gamma^* + \omega \in \widehat{G}/\Gamma^*, \omega \in \Omega. \qquad (3.2)$$

Then $\phi \mapsto \phi'$ is an isometry of $L^p(\Omega)$ onto $L^p(\widehat{G}/\Gamma^*)$ for $1 \le p \le \infty$.

Proof The first sentence is verified by applying Theorem 3.1 to characteristic functions of measurable subsets of Ω having finite measure. The second claim holds because Γ is discrete if and only if its character group $\widehat{G}/\Gamma^*$ is compact, and this holds if and only if $m_{\widehat{G}/\Gamma^*}(\widehat{G}/\Gamma^*)$ is finite.

The last claim follows from the fact that $\phi \mapsto \phi'$ preserves linearity, pointwise multiplication, and the taking of absolute values. $\square$

For $\gamma \in G$, the corresponding character on $\widehat{G}$ is written X_γ, where $X_\chi(\chi) = \chi(\gamma)$ for $\chi \in \widehat{G}$. For $\gamma \in \Gamma$, we write X^*_γ for the corresponding character on $\widehat{G}/\Gamma^*$ defined by $X^*_\gamma(\Gamma^* + \chi) = X_\gamma(\chi)$ for $\chi \in \widehat{G}$.

Corollary 3.3 *For ϕ in $L^1(\Omega, m_{\widehat{G}})$, we have*

$$\int_\Omega \phi(\omega) dm_{\widehat{G}}(\omega) = \int_{\widehat{G}/\Gamma^*} \phi' dm_{\widehat{G}/\Gamma^*}. \qquad (3.3)$$

In particular, for any $\gamma \in \Gamma$, we have

$$\int_{\Omega} X_{\gamma}(\omega)\phi(\omega)dm_{\widehat{G}}(\omega) = \int_{\widehat{G}/\Gamma^*} X_{\gamma}^* \phi' dm_{\widehat{G}/\Gamma^*}. \tag{3.4}$$

Proof Equation (3.4) follows from (3.3), because $(X_{\gamma}|_{\Omega}\phi)' = X_{\gamma}^*\phi'$; indeed

$$(X_{\gamma}|_{\Omega}\phi)'(\Gamma^* + \omega) = (X_{\gamma}|_{\Omega}\phi)(\omega) = X_{\gamma}(\omega)\phi(\omega) = X_{\gamma}^*(\Gamma^* + \omega)\phi'(\Gamma^* + \omega)$$

for $\omega \in \Omega$. □

The next corollary is Proposition 2.16 in [9].

Corollary 3.4 *If Γ is discrete, then $m_{\widehat{G}}(\Omega)$ is finite, and $\{X_{\gamma}|_{\Omega}\}_{\gamma \in \Gamma}$ is an orthogonal basis in $L^2(\Omega)$.*

Proof In this case, $\widehat{G}/\Gamma^*$ is a compact group, and $\{X_{\gamma}^*\}_{\gamma \in \Gamma}$ is an orthogonal basis for $L^2(\widehat{G}/\Gamma^*)$. Apply (3.4) with $\gamma = \gamma_1$ and $\phi = \overline{X_{\gamma_2}|_{\Omega}}$ to obtain

$$\int_{\Omega} X_{\gamma_1}(\omega)\overline{X_{\gamma_2}(\omega)}dm_{\widehat{G}}(\omega) = \int_{\widehat{G}/\Gamma^*} X_{\gamma_1}^* \overline{X_{\gamma_2}^*}dm_{\widehat{G}/\Gamma^*},$$

from which the corollary follows. □

When Γ is not discrete, we have the following substitute for Corollary 3.4.

Lemma 3.5 *For ϕ in $L^1(\Omega, m_{\widehat{G}})$, we have*

$$\int_{\Gamma} \left| \int_{\Omega} X_{\gamma}(\omega)\phi(\omega)dm_{\widehat{G}}(\omega) \right|^2 dm_{\Gamma}(\gamma) = \int_{\Omega} |\phi(\omega)|^2 dm_{\widehat{G}}(\omega); \tag{3.5}$$

both sides of the equation can be infinite.

Proof Since the character group of $\widehat{G}/\Gamma^*$ is Γ, for ϕ in $L^1(\Omega, m_{\widehat{G}})$, we have

$$\widehat{\phi'}(\gamma) = \int_{\widehat{G}/\Gamma^*} \overline{X_{\gamma}^*}\phi' dm_{\widehat{G}/\Gamma^*} \quad \text{for} \quad \gamma \in \Gamma.$$

Thus by (3.4),

$$\left| \int_{\Omega} X_{\gamma}(\omega)\phi(\omega)dm_{\widehat{G}}(\omega) \right|^2 = \left| \int_{\widehat{G}/\Gamma^*} X_{\gamma}^* \phi' dm_{\widehat{G}/\Gamma^*} \right|^2 = |\widehat{\phi'}(-\gamma)|^2 \quad \text{for} \quad \gamma \in \Gamma,$$

and the left-hand-side of (3.5) is $\int_{\Gamma} |\widehat{\phi'}|^2 \, dm_{\Gamma}$. Hence to verify equation (3.5), it suffices to verify

$$\int_{\Gamma} |\widehat{\phi'}|^2 \, dm_{\Gamma} = \int_{\Omega} |\phi(\omega)|^2 dm_{\widehat{G}}(\omega). \tag{3.6}$$

If $\phi \in L^2(\Omega, m_{\widehat{G}})$, then $\phi' \in L^2(\widehat{G}/\Gamma^*, m_{\widehat{G}/\Gamma^*})$ by Proposition 3.2. By the Plancherel Theorem [19, Theorem (31.18)] and Corollary 3.3, we have

$$\int_\Gamma |\widehat{\phi'}|^2 \, dm_\Gamma = \int_{\widehat{G}/\Gamma^*} |\phi'|^2 \, dm_{\widehat{G}/\Gamma^*} = \int_\Omega |\phi(\omega)|^2 \, dm_{\widehat{G}}(\omega),$$

so that (3.6) holds.

However, if $\phi \notin L^2(\Omega, m_{\widehat{G}})$, then $\phi' \notin L^2(\widehat{G}/\Gamma^*, m_{\widehat{G}/\Gamma^*})$ by Proposition 3.2. Then, [19, (31.44a)] implies that $\widehat{\phi'} \notin L^2(\Gamma, m_\Gamma)$. This shows that both sides of (3.6) are equal to ∞. □

Corollary 3.6 *The set of functions $\{X_\gamma|_\Omega : \gamma \in \Gamma\}$ constitutes a determining set for $L^1(\Omega)$.*

Proof This result is essentially noted in Example 2.2. Consider $\phi \in L^1(\Omega)$ and assume that $\int_\Omega X_\gamma \phi \, dm_{\widehat{G}} = 0$ for all $\gamma \in \Gamma$. Then by (3.5), we have $\int_\Omega |\phi|^2 \, dm_{\widehat{G}} = 0$ and so $\phi = 0$. □

The following definition formalizes the concept of a TI subspace of $L^2(G)$.

Definition 3.2 Suppose that $\Gamma \subset G$ is a closed co-compact subgroup of G. Let $V \subset L^2(G)$ be a closed subspace. We say that V is TI under Γ, in short Γ-TI, if $f \in V$ implies $T_\gamma f \in V$ for all $\gamma \in \Gamma$, where $T_y f(x) = f(x - y)$ for $x, y \in G$. Given a countable subset $\mathcal{A}$ of $L^2(G)$, we define a Γ-TI space generated by $\mathcal{A}$ as

$$S^\Gamma(\mathcal{A}) := \overline{\text{span}}\{T_\gamma \phi : \phi \in \mathcal{A}, \gamma \in \Gamma\} \subset L^2(G).$$

The following result is a well-known consequence of the Plancherel Theorem. In the context of LCA groups, it is shown by Cabrelli and Paternostro in [9].

Proposition 3.7 *The fiberization mapping $\mathcal{T} : L^2(G) \to L^2(\Omega, \ell^2(\Gamma^*))$ defined by*

$$\mathcal{T} f(\omega) = \{\widehat{f}(\omega + \kappa)\}_{\kappa \in \Gamma^*},$$

is an isometric isomorphism. Also, for every $\omega \in \Omega$ and $f \in L^2(G)$, we have

$$\mathcal{T}(T_{-\gamma} f)(\omega) = X_\gamma(\omega) \cdot (\mathcal{T} f)(\omega) \quad \text{for all } \gamma \in \Gamma. \tag{3.7}$$

Proof The fact that $\mathcal{T}$ is an isometric isomorphism was shown in [9, Proposition 3.3]. Indeed, the additional assumption made in [9], that Γ is discrete, was not used in the proof.

To verify (3.7), observe that $\widehat{T_{-\gamma} f}(\chi) = \chi(\gamma)\widehat{f}(\chi)$ for $\chi \in \widehat{G}$, so that $\widehat{T_{-\gamma} f}(\chi) = X_\gamma(\chi)\widehat{f}(\chi)$. Therefore, for $\omega \in \Omega$, we have

$$\mathcal{T}(T_{-\gamma} f)(\omega) = \{\widehat{T_{-\gamma} f}(\omega + \kappa)\}_{\kappa \in \Gamma^*} = \{X_\gamma(\omega + \kappa)\widehat{f}(\omega + \kappa)\}_{\kappa \in \Gamma^*} = X_\gamma(\omega)\mathcal{T}(f)(\omega);$$

recall that $X_\gamma(\kappa) = 1$ for $\gamma \in \Gamma$ and $\kappa \in \Gamma^*$. Thus equation (3.7) holds. □

Theorem 3.8 below, which is the main result of this section, characterizes TI spaces. It is reminiscent of a characterization of shift-invariant (SI) spaces which dates back to Helson [17] and Srinivasan [32]. The proof of this fact in the setting of SI spaces on $\mathbb{R}^n$ and on a LCA group G can be found in [5, Proposition 1.5], [9, Theorem 3.10], and [23, Theorem 3.1], respectively. Note that [9] and [23] require the assumption that Γ is a uniform lattice, i.e., Γ is a discrete co-compact subgroup of G; see also [23, Theorem 3.1]. The fact that Γ is not assumed to be discrete is the first innovation of Theorem 3.8. The second innovation is the utilization of multiplicatively-invariant (MI) spaces studied in Sect. 2.

Our main result has far reaching consequences. In particular, we can deduce a range of results including Wiener's characterization of TI subspaces and the characterization of shift-invariant spaces as two extremes of the same result. These intermediate results are new even in the case of $\mathbb{R}^n$; see Example 3.3 below. On a technical level, the lack of discreteness of Γ translates into non-compactness of $\widehat{G}/\Gamma^*$, which makes the Borel section Ω have infinite measure. As we will see, this extra difficulty can be circumnavigated.

Theorem 3.8 *Let $V \subset L^2(G)$ be a closed subspace, and $\Gamma \subset G$ be a closed co-compact subgroup. Then, the following are equivalent:*

(i) *V is Γ-TI,*
(ii) *$\mathcal{T}(V)$ is $L^\infty(\Omega)$-MI,*
(iii) *there exists a measurable range function $J : \Omega \to \{$closed subspaces of $\ell^2(\Gamma^*)\}$ such that*

$$V = \{f \in L^2(G) : \mathcal{T}f(\omega) \in J(\omega) \ a.e. \ \omega \in \Omega\}. \tag{3.8}$$

Identifying range functions which are equal almost everywhere, the correspondence between Γ-TI spaces and measurable range functions is one-to-one and onto.

Moreover, if $V = S^\Gamma(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$, then the measurable range function J associated to V is given by

$$J(\omega) = \overline{\text{span}}\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}, \quad a.e. \ \omega \in \Omega. \tag{3.9}$$

Proof By Corollary 3.6, $\mathcal{D} = \{X_\gamma|_\Omega : \gamma \in \Gamma\}$ is a determining set for $L^1(\Omega)$. Therefore, Proposition 3.7 implies that V is a Γ-TI subspace of $L^2(G)$ if and only if $W = \mathcal{T}(V)$ is a $\mathcal{D}$-MI subspace of $L^2(\Omega, \ell^2(\Gamma^*))$. Thus, Theorem 2.4 implies that V is a Γ-TI space if and only if there is a range function $J : \Omega \to \{$closed subspaces of $\ell^2(\Gamma^*)\}$ such that

$$\mathcal{T}(V) = W = \{\Phi \in L^2(\Omega, \ell^2(\Gamma^*)) : \Phi(\omega) \in J(\omega) \text{ for a.e. } \omega \in \Omega\}.$$

Applying $\mathcal{T}^{-1}$ and Proposition 3.7 yields (3.8):

$$V = \mathcal{T}^{-1}(W) = \{f \in L^2(G) : \mathcal{T}f(\omega) \in J(\omega) \text{ for a.e. } \omega \in \Omega\}.$$

Now suppose that $V = \overline{\text{span}}\{T_\gamma\phi : \phi \in \mathcal{A}, \gamma \in \Gamma\}$ for some countable subset $\mathcal{A}$ of $L^2(G)$. Then $\mathcal{B} = \mathcal{T}(\mathcal{A})$ is a countable subset of $L^2(\Omega, \ell^2(\Gamma^*))$, and using Proposition 3.7 we see that

$$W = \mathcal{T}(V) = \overline{\text{span}}\{X_\gamma(\omega)\Phi(\omega) : \gamma \in \Gamma, \Phi \in \mathcal{B}\}$$
$$= \overline{\text{span}}\{g(\omega)\Phi(\omega) : g \in \mathcal{D}, \Phi \in \mathcal{B}\}.$$

By Theorem 2.4, the measurable range function J associated to W is

$$J(\omega) = \overline{\text{span}}\{\Phi(\omega) : \Phi \in \mathcal{B}\} = \overline{\text{span}}\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\} \text{ for a.e. } \omega \in \Omega.$$

Finally, the one-to-one correspondence between Γ-TI spaces and measurable range functions J follows from the same assertion in Theorem 2.4. □

As an immediate consequence of Theorem 3.8 we have the characterization of TI subspaces of $L^2(G)$ which is often attributed to Wiener [36, Theorem I]. See also [17,32] and [19, Theorem (31.39)].

Corollary 3.9 *Let $V \subset L^2(G)$ be a closed subspace. Then, V is invariant under all translations (G-TI) if and only if there exists a measurable set $E \subset \widehat{G}$ such that*

$$V = \{f \in L^2(G) : \text{supp } \hat{f} \subset E\}. \tag{3.10}$$

Proof In the case $\Gamma = G$, the fiberization map $\mathcal{T}$ coincides with the Fourier transform, and a range function J can be identified with $\mathbf{1}_E$ for some $E \subset \widehat{G}$, since $\ell^2(\Gamma^*)$ is 1-dimensional. Thus, (3.8) can be rewritten as (3.10). □

Example 3.3 Note that in the Euclidean setting $G = \mathbb{R}^n$, any closed subgroup $\Gamma \subset \mathbb{R}^n$ is co-compact $\iff \Gamma = P(\mathbb{Z}^k \times \mathbb{R}^{n-k})$ for some $P \in GL_n(\mathbb{R})$ and $0 \le k \le n$. In other words, any such Γ is a direct sum of a rank k lattice and an $(n-k)$ dimensional subspace. In particular, if $\Gamma = \mathbb{Z}^n$, then Theorem 3.8 is simply [5, Proposition 1.5]. On the other hand, if we take $\Gamma = \mathbb{R}^n$, then Corollary 3.9 yields Wiener's characterization of TI subspaces of $L^2(\mathbb{R}^n)$. However, the most interesting situation happens in the intermediate case when $1 \le k \le n-1$.

Without loss of generality, by a change of coordinates we may assume that $\Gamma = \mathbb{Z}^k \times \mathbb{R}^{n-k}$. Then $\Gamma^* = \mathbb{Z}^k \times \{0\}^{n-k} \cong \mathbb{Z}^k$, and we can choose $\Omega = [-1/2, 1/2)^k \times \mathbb{R}^{n-k}$. The fiberization mapping $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\Omega, \ell^2(\mathbb{Z}^k))$ is given by

$$\mathcal{T}f(\xi) = (\hat{f}(\xi_1 + \gamma, \xi_2))_{\gamma \in \mathbb{Z}^k}, \quad \text{for } \xi = (\xi_1, \xi_2) \in [-1/2, 1/2)^k \times \mathbb{R}^{n-k}.$$

Here, the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$. A range function is a mapping

$$J : [-1/2, 1/2)^k \times \mathbb{R}^{n-k} \to \{\text{closed subspaces of } \ell^2(\mathbb{Z}^k)\}.$$

By Theorem 3.8, any $\mathbb{Z}^k \times \mathbb{R}^{n-k}$-TI space $V \subset L^2(\mathbb{R}^n)$ is of the form

$$V = \{f \in L^2(\mathbb{R}^n) : \mathcal{T}f(\xi_1, \xi_2) \in J(\xi_1, \xi_2) \text{ a.e. } (\xi_1, \xi_2) \in [-1/2, 1/2)^k \times \mathbb{R}^{n-k}\}$$

for some measurable range function J as above.

Example 3.4 Suppose that $G = \mathbb{Z}^n$. Any co-compact subgroup $\Gamma \subset \mathbb{Z}^n$ is of the form $\Gamma = P\mathbb{Z}^n$ for some $n \times n$ invertible matrix P with integer entries. Then $\widehat{G} = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and

$$\Gamma^* = \{\xi \in \mathbb{T}^n : \langle \xi, \gamma \rangle \in \mathbb{Z} \quad \text{for all } \gamma \in \Gamma\}$$

is a finite subgroup of $\mathbb{T}^n$ with $p = |\det P|$ elements. Indeed, Γ^* is isomorphic with $(P^T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n = \{d_1, \ldots, d_p\}$. Thus, we can choose $\Omega = (P^T)^{-1}([-1/2, 1/2)^n)$. The fiberization mapping $\mathcal{T} : \ell^2(\mathbb{Z}^n) \to L^2(\Omega, \mathbb{C}^p)$ is given by

$$\mathcal{T}a(\xi) = (\hat{a}(\xi + d_1), \ldots, \hat{a}(\xi + d_p)) \quad \text{for } \xi \in \Omega.$$

Here the Fourier transform $\hat{} : \ell^2(\mathbb{Z}^n) \to L^2(\mathbb{T}^n)$ is given by

$$\hat{a}(\xi) = \sum_{\gamma \in \mathbb{Z}^n} a_\gamma e^{-2\pi i \langle \gamma, \xi \rangle} \quad \text{for } a = (a_\gamma)_{\gamma \in \mathbb{Z}^n}.$$

A range function is a mapping $J : \Omega \to \{\text{subspaces of } \mathbb{C}^p\}$. Then Theorem 3.8 immediately recovers [6, Theorem 2.1].

Example 3.5 Let G be a compact LCA group. Then any closed subgroup $\Gamma \subset G$ is allowed and we have a complete characterization of all TI spaces. In particular, note that if we take $\Gamma = \{0\}$, then $\Gamma^* = \widehat{G}$. Thus, the range function J is defined on the single-element fundamental domain $\Omega = \{0\}$. Thus, $J(0)$ can be an arbitrary (closed) subspace of $\ell^2(\Gamma^*)$, which corresponds to an arbitrary (closed) subspace of $L^2(G)$ under the Fourier transform, since there is no translation invariance assumed a priori.

Theorem 3.8 can be also phrased in terms of direct integrals; see [15, Section 7.4]. That is, V is Γ-TI if and only if $\mathcal{T}(V) = M_J$ for some measurable range function J, where M_J is identified with a direct integral

$$M_J = \{\Phi \in L^2(\Omega, \ell^2(\Gamma^*)) : \Phi(\omega) \in J(\omega) \text{ a.e. } \omega\} = \int_\Omega^\oplus J(\omega)d\omega. \qquad (3.11)$$

Indeed, $\{J(\omega)\}_{\omega \in \Omega}$ is a measurable field of Hilbert spaces together with vector fields $\{f_\kappa\}_{\kappa \in \Gamma^*}$ given by $f_\kappa(\omega) = P_J(\omega)(e_\kappa)$, where $\{e_\kappa\}_{\kappa \in \Gamma^*}$ is the standard basis of $\ell^2(\Gamma^*)$. Then Theorem 2.6 or the structure theorem on measurable vector fields [15, Proposition 7.27] yields vector fields forming an orthonormal basis of $J(\xi)$. This result can be used to establish (3.11); we leave details to the reader.

4 The Spectral Function

In this section we define the spectral function for TI spaces on second countable LCA groups studied in Sect. 3. A limited introduction to spectral functions on LCA groups is given in [24]. We shall employ Theorem 3.8 for the initial definition of the spectral function, following [7, Definition 2.1]. The main result of this section states that the

spectral function does not depend on the choice of a closed co-compact subgroup Γ, which generalizes the result of the first author and Rzeszotnik [8].

Definition 4.1 Suppose $V \subset L^2(G)$ is a Γ-TI vector subspace, with range function $J : \Omega \to \{$closed subspaces of $\ell^2(\Gamma^*)\}$. For $\omega \in \Omega$, let $P_J(\omega)$ be the projection of $\ell^2(\Gamma^*)$ onto $J(\omega)$. The *dimension function* of V is the mapping $\dim_V^\Gamma : \widehat{G} \to \{0, 1, 2, 3, \ldots, \infty\}$ given by

$$\dim_V^\Gamma(\omega + \kappa) = \dim J(\omega) \quad \text{for } \omega \in \Omega \text{ and } \kappa \in \Gamma^*. \tag{4.1}$$

The *spectral function* of V is the mapping $\sigma_V^\Gamma : \widehat{G} \to [0, 1]$ given by

$$\sigma_V^\Gamma(\omega + \kappa) = \|P_J(\omega)e_\kappa\|^2 \quad \text{for } \omega \in \Omega \text{ and } \kappa \in \Gamma^*, \tag{4.2}$$

where $\{e_\kappa\}_{\kappa \in \Gamma^*}$ denotes the standard basis of $\ell^2(\Gamma^*)$. Compare (4.1) with Definition 2.4.

This is well defined, since $\widehat{G} = \Omega \oplus \Gamma^*$. We use the superscript Γ on σ_V to emphasize that the definition depends on Γ. However, we will see later in Theorem 4.2 that the functions σ_V^Γ do not depend on Γ, at which point we will drop the superscripts.

Following [7, equation (2.2)] one can easily show that the dimension function is a Γ^* periodization of the spectral function, i.e., we have

$$\dim_V^\Gamma(\omega) = \sum_{\kappa \in \Gamma^*} \sigma_V^\Gamma(\omega + \kappa) \quad \text{for a.e. } \omega \in \Omega. \tag{4.3}$$

Indeed,

$$\sum_{\kappa \in \Gamma^*} \sigma_V^\Gamma(\omega + \kappa) = \sum_{\kappa \in \Gamma^*} \|P_J(\omega)e_\kappa\|^2 = \sum_{\kappa \in \Gamma^*} \langle P_J(\omega)e_\kappa, e_\kappa \rangle = \text{trace}(P_J(\omega))$$
$$= \dim(\text{range}(P_J(\omega))) = \dim_V^\Gamma(\omega).$$

The following theorem plays a key role in developing properties of the spectral function; see [7, Lemma 2.8] and [8, Lemma 2.3].

Theorem 4.1 *Suppose V is a Γ-TI subspace of $L^2(G)$ and that K is a measurable subset of $\widehat{G}$ with finite measure such that*

$$m_{\widehat{G}}(K \cap (\kappa + K)) = 0 \quad \text{for all } \kappa \in \Gamma^* \setminus \{0\}.$$

Let P_V be the orthogonal projection of $L^2(G)$ onto V. Then

$$\|P_V(\check{\mathbf{1}}_K)\|^2 = \int_K \sigma_V^\Gamma(\chi) dm_{\widehat{G}}(\chi). \tag{4.4}$$

Proof Let $f = \check{\mathbf{1}}_K$. Then for a.e. $\omega \in \Omega$, we have $\mathcal{T}f(\omega) = e_\kappa$ if there is a $\kappa \in \Gamma^*$ such that $\omega + \kappa \in K$; otherwise 0. This is where the hypothesis on K is used; it guarantees that for a.e. $\omega \in \Omega$, there is never more than one $\kappa \in \Gamma^*$ such that $\omega + \kappa \in K$.

By Theorem 3.8 and Proposition 2.2 for any $g \in L^2(G)$ we have

$$\mathcal{T}(P_V g)(\omega) = P_J(\omega)(\mathcal{T}g(\omega)) \qquad \text{for a.e. } \omega \in \Omega.$$

Hence,

$$
\begin{aligned}
\|P_V(f)\|^2 = \|\mathcal{T}P_V(f)\|^2 &= \int_\Omega \|P_J(\omega)(\mathcal{T}f(\omega))\|^2 dm_{\widehat{G}}(\omega) \\
&= \sum_{\kappa \in \Gamma^*} \int_\Omega \|P_J(\omega)(e_\kappa)\|^2 \mathbf{1}_K(\omega + \kappa) dm_{\widehat{G}}(\omega) \\
&= \sum_{\kappa \in \Gamma^*} \int_{\widehat{G}} \mathbf{1}_\Omega(\omega)\sigma_V^\Gamma(\omega + \kappa)\mathbf{1}_K(\omega + \kappa) dm_{\widehat{G}}(\omega) \\
&= \sum_{\kappa \in \Gamma^*} \int_{\widehat{G}} \mathbf{1}_{\kappa+\Omega}(\omega + \kappa)\sigma_V^\Gamma(\omega + \kappa)\mathbf{1}_K(\omega + \kappa) dm_{\widehat{G}}(\omega) \\
&= \sum_{\kappa \in \Gamma^*} \int_{\widehat{G}} \mathbf{1}_{\kappa+\Omega}(\omega)\sigma_V^\Gamma(\omega)\mathbf{1}_K(\omega) dm_{\widehat{G}}(\omega) = \int_{\widehat{G}} \sigma_V^\Gamma(\omega)\mathbf{1}_K(\omega) dm_{\widehat{G}}(\omega).
\end{aligned}
$$

This shows (4.4) and completes the proof of Theorem 4.1. $\qquad\square$

Theorem 4.2 *Suppose that $V \subset L^2(G)$ is both a Γ_1-TI and Γ_2-TI subspace for two closed co-compact subgroups Γ_1 and Γ_2 of G. Then*

$$\sigma_V^{\Gamma_1}(\chi) = \sigma_V^{\Gamma_2}(\chi) \quad \text{for almost all } \chi \in \widehat{G}. \tag{4.5}$$

Consequently, the spectral function σ_V^Γ does not depend on the co-compact subgroup Γ and, after the next proof, we shall omit the superscript Γ by writing σ_V.

Proof We will show that for every element χ_0 in $\widehat{G}$, there is a neighborhood of χ_0 on which (4.5) holds. To retain the almost everywhere feature, it suffices to take a countable subcover of this cover of $\widehat{G}$.

So, consider χ_0 in $\widehat{G}$. Let U be a neighborhood of 0 in $\widehat{G}$ so that

$$U \cap (\Gamma_1^* \cup \Gamma_2^*) = \{0\}. \tag{4.6}$$

Let W be a symmetric neighborhood of 0 so that $W - W \subset U$. We will focus on Γ_1^*. We claim that

$$(\chi_0 + W) \cap (\kappa + \chi_0 + W) = \emptyset \quad \text{for} \quad \kappa \in \Gamma_1^* \setminus \{0\}. \tag{4.7}$$

Otherwise there exist χ_1, χ_2 in W and $\kappa \in \Gamma_1^* \setminus \{0\}$ so that $\chi_0 + \chi_1 = \kappa + \chi_0 + \chi_2$. Then

$$\kappa = \chi_1 - \chi_2 \in W - W \subset U \quad \text{and} \quad \kappa \in \Gamma_1^* \cup \Gamma_2^* \setminus \{0\},$$

which contradicts (4.6).

Now (4.7) implies that for every measurable subset $K \subset \chi_0 + W$ with finite measure,

$$K \cap (\kappa + K) = \emptyset \quad \text{for} \quad \kappa \in \Gamma_1^* \setminus \{0\}. \tag{4.8}$$

From (4.4), we conclude that

$$\int_K \sigma_V^{\Gamma_1}(\chi) d\chi = \| P_V(\check{\mathbf{1}}_K) \|^2 \quad \text{for all finite measure } K \subset \chi_0 + W.$$

This also holds for Γ_2, and the right-side does not depend on any Γ. So

$$\int_K \sigma_V^{\Gamma_1}(\chi) d\chi = \int_K \sigma_V^{\Gamma_2}(\chi) d\chi \quad \text{for all finite measure } K \subset \chi_0 + W.$$

From measure theory, this implies

$$\sigma_V^{\Gamma_1}(\chi) = \sigma_V^{\Gamma_2}(\chi) \quad \text{for almost all } \chi \in \chi_0 + W.$$

This verifies (4.5). □

Example 4.1 Suppose an LCA group G satisfying our standing hypotheses is compact. Then the dual $\widehat{G}$ is discrete and countable. Let $\Gamma \subset G$ be a closed subgroup which is automatically co-compact. Finally, let V be a Γ-TI subspace of $L^2(G)$. Note that Γ might be the trivial subgroup $\{0\}$, so that V could be merely any (closed) subspace of $L^2(G)$ without any additional translation invariance. What is then the spectral function $\sigma_V(\chi)$? With $\Gamma = \{0\}$, we have $\Gamma^* = \widehat{G}$ and $\Omega = \{0\}$. Let $\{e_\kappa\}_{\kappa \in \widehat{G}}$ be the standard basis of $\ell^2(\widehat{G})$. Then the fiberization mapping from Proposition 3.7 is the Fourier transform $L^2(G) \to \ell^2(\widehat{G})$. Thus, the range function from Theorem 3.8 corresponding to V is $J(\{0\}) = \widehat{V}$. Thus,

$$\sigma_V(\kappa) = \| P_{\widehat{V}}(e_\kappa) \|^2 \quad \text{for all } \kappa \in \widehat{G}.$$

We end this section by observing that it is possible to introduce a more general object than the spectral function, the local trace function, introduced for $G = \mathbb{R}^n$ by Dutkay [12]. The nice feature of the local trace function is that it unifies the spectral function and the dimension function into a very general class of functions "measuring the size" of a TI-space. A potential disadvantage is that the local trace function in general is no longer independent of the choice of the underlying co-compact subgroup Γ. Hence, it behaves more like a dimension function.

To introduce the local trace function it is convenient to extend the domain of the range function from the Borel section Ω to the whole group $\widehat{G}$. That is, we require that a *range function* has domain $\widehat{G}$ and satisfies a consistency formula with respect to Γ^*:

$$J(\chi + \kappa_0) = S_{\kappa_0}(J(\chi)) \quad \text{for all } \kappa_0 \in \Gamma^*, \chi \in \widehat{G}, \tag{4.9}$$

where $S_{\kappa_0} : \ell^2(\Gamma^*) \to \ell^2(\Gamma^*)$ is the shift operator defined by

$$S_{\kappa_0}((a_\kappa)_{\kappa \in \Gamma^*}) = (a_{\kappa + \kappa_0})_{\kappa \in \Gamma^*}.$$

Indeed, if J is initially defined on Ω, which is a Borel section of $\widehat{G}/\Gamma^*$, we define

$$J(\omega + \kappa_0) = S_{\kappa_0}(J(\omega)) \qquad \text{for } \omega \in \Omega \text{ and } \kappa_0 \in \Gamma^*.$$

Since $S_{\kappa_0} \circ S_{\kappa_1} = S_{\kappa_0 + \kappa_1}$ is a representation of the group Γ^* we have (4.9).

To interpret Theorem 3.8 with this extended definition of the range function, suppose that $V = S^\Gamma(\mathcal{A})$ for some countable subfamily $\mathcal{A}$ of $L^2(G)$ and J is its corresponding range function. Then for any $\chi = \omega + \kappa_0 \in \widehat{G}$ we have

$$J(\chi) = J(\omega + \kappa_0) = \overline{\text{span}}\{S_{\kappa_0}(\mathcal{T}\phi(\omega)) : \phi \in \mathcal{A}\} = \overline{\text{span}}\{(\hat{\phi}(\chi + \kappa))_{\kappa \in \Gamma^*} : \phi \in \mathcal{A}\}.$$

Thus, (3.9) holds for a.e. $\chi \in \widehat{G}$ under the convention $\mathcal{T}\phi(\chi) = (\hat{\phi}(\chi + \kappa))_{\kappa \in \Gamma^*}$.

Definition 4.2 Let T be a fixed positive (self-adjoint) operator on $\ell^2(\Gamma^*)$ and V a Γ-TI space with corresponding range function J. The *local trace function* associated to the pair (T, V) is defined by

$$\tau_{V,T}^\Gamma(\chi) = \text{trace}(T P_{J(\chi)}) \qquad \text{for } \chi \in \widehat{G}.$$

In particular, [12, Proposition 4.14] holds in our setting. That is, if we take T to be a rank 1 orthogonal projection onto $\text{span}\{e_0\} \subset \ell^2(\Gamma^*)$, then the local trace function coincides with the spectral function

$$\tau_{V,T}^\Gamma(\chi) = \langle P_J(\chi)e_0, e_0 \rangle = \sigma_V(\chi).$$

On the other hand, taking T to be the identity operator I yields

$$\tau_{V,I}^\Gamma(\chi) = \dim_V^\Gamma(\chi).$$

We leave the verification of other properties of the local trace function shown in [12] to the interested reader.

5 Frames and Riesz Sequences

In this section we obtain a characterization of frame and Riesz sequence property for TI systems in terms of the fiberization operator. This formulation is due to the first author [5, Theorem 2.3]. The extension of this result to the LCA setting for discrete subgroups $\Gamma \subset G$ was done by Cabrelli and Paternostro [9, Theorems 4.1 and 4.3]. In the case when Γ is non-discrete, we need to consider a concept of continuous frames which is a generalization of the usual (discrete) frames proposed by G. Kaiser [21] and independently by Ali, Antoine, and Gazeau [2].

Definition 5.1 Let $\mathcal{A}$ be a countable subset of $L^2(G)$, and $\Gamma \subset G$ be a subgroup. We say that the set

$$E^\Gamma(\mathcal{A}) = \{T_\gamma \phi : \gamma \in \Gamma, \ \phi \in \mathcal{A}\}$$

is a *continuous frame sequence*, or frame for its closed linear span $S^\Gamma(\mathcal{A}) = \overline{\text{span}}E^\Gamma(\mathcal{A})$, if there exist bounds $0 < A \le B < \infty$ such that

$$A||f||^2 \le \sum_{\phi \in \mathcal{A}} \int_\Gamma |\langle f, T_\gamma \phi \rangle|^2 dm_\Gamma(\gamma) \le B||f||^2 \quad \text{for all } f \in S^\Gamma(\mathcal{A}).$$

Likewise, we introduce the concept of continuous Riesz sequences as follows, where $C_{00}(\Gamma)$ is the space of continuous functions on Γ with compact support.

Definition 5.2 We say that $E^\Gamma(\mathcal{A})$ is a *continuous Riesz sequence* if for any collection of functions $a_\phi \in C_{00}(\Gamma)$, which are zero for all but finitely many $\phi \in \mathcal{A}$, we have

$$A \sum_{\phi \in \mathcal{A}} ||a_\phi||^2_{L^2(\Gamma)} \le \left\| \sum_{\phi \in \mathcal{A}} \int_\Gamma a_\phi(\gamma) T_\gamma \phi \, dm_\Gamma(\gamma) \right\|^2 \le B \sum_{\phi \in \mathcal{A}} ||a_\phi||^2_{L^2(\Gamma)}. \quad (5.1)$$

Observe that the integral in (5.1) is vector-valued with values in the space $L^2(G)$. A standard reference for vector-valued integrals is Diestel and Uhl [11]. In the special case when Γ is discrete, the above concepts coincide with the classical definitions of frame and Riesz sequences. However, as we will see later there are no continuous Riesz sequences satisfying Definition 5.2 unless Γ is discrete. The main result of this section is the following generalization of [5, Theorem 2.3] and [9, Theorems 4.1 and 4.3].

Theorem 5.1 *Let $\mathcal{A}$ be a countable subset of $L^2(G)$, let J be the measurable range function associated to $S^\Gamma(\mathcal{A})$, and let $0 < A \le B < \infty$. The following are equivalent:*

  (i) *The set $E^\Gamma(\mathcal{A})$ is a continuous frame (resp., continuous Riesz sequence) for $S^\Gamma(\mathcal{A})$ with bounds A and B.*
 (ii) *For almost every $\omega \in \Omega$, the set $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\} \subset \ell^2(\Gamma^*)$ is a frame (resp., Riesz basis) for $J(\omega)$ with bounds A and B.*

We emphasize that part (i) of Theorem 5.1 requires continuous variants of frame and Riesz sequences unlike part (ii) that deals with the usual (discrete) frames and Riesz bases.

Proof of Theorem 5.1 for frames By Proposition 3.7, we have the following key calculation for $f \in L^2(G)$ and $\phi \in \mathcal{A}$:

$$\int_\Gamma |\langle T_\gamma \phi, f\rangle_{L^2(G)}|^2 dm_\Gamma(\gamma) = \int_\Gamma |\langle \mathcal{T} T_\gamma \phi, \mathcal{T} f\rangle_{L^2(\Omega,\ell^2)}|^2 dm_\Gamma(\gamma)$$

$$= \int_\Gamma \left| \int_\Omega X_\gamma(\omega)\langle \mathcal{T}\phi(\omega), \mathcal{T} f(\omega)\rangle_{\ell^2} dm_{\widehat{G}}(\omega) \right|^2 dm_\Gamma(\gamma) \tag{5.2}$$

$$= \int_\Omega |\langle \mathcal{T}\phi(\omega), \mathcal{T} f(\omega)\rangle_{\ell^2}|^2 dm_{\widehat{G}}(\omega).$$

The last equality follows from Lemma 3.5 with ϕ replaced by $\langle \mathcal{T}\phi(\omega), \mathcal{T} f(\omega)\rangle_{\ell^2}$, which is in $L^1(\Omega, m_{\widehat{G}})$. Summing (5.2) over $\phi \in \mathcal{A}$ we obtain

$$\sum_{\phi\in\mathcal{A}} \int_\Gamma |\langle T_\gamma \phi, f\rangle_{L^2(G)}|^2 dm_\Gamma(\gamma) = \int_\Omega \sum_{\phi\in\mathcal{A}} |\langle \mathcal{T}\phi(\omega), \mathcal{T} f(\omega)\rangle_{\ell^2}|^2 dm_{\widehat{G}}(\omega). \tag{5.3}$$

Note that, a priori, the quantities on either side of (5.3) might be equal to ∞ as in Lemma 3.5. However, either assumption (i) or (ii) implies that both sides of (5.3) are finite.

With (5.3) established, the rest of the proof is a verbatim adaption of [5] and [9, Theorem 4.1]. Indeed, the implication (ii) $\implies$ (i) follows from Proposition 3.7, Theorem 3.8, and the definition of a frame by a direct calculation. The reverse implication requires a straightforward adaptation of the argument from [5]. Consequently, we leave the details to the reader. □

Proof of Theorem 5.1 for Riesz sequences In the case when Γ is discrete this result was shown in [9, Theorem 4.3]. The same result is vacuously true when Γ is not discrete. That is, if either (i) or (ii) holds, then Γ is actually forced to be a discrete subgroup. Indeed, suppose (ii) holds. Then, for any $\phi \in \mathcal{A}$, we have $||\mathcal{T}\phi(\omega)||^2 \geq A$ for a.e. $\omega \in \Omega$. Thus,

$$A m_{\widehat{G}}(\Omega) \leq \int_\Omega ||\mathcal{T}\phi(\omega)||^2 dm_{\widehat{G}}(\omega) < \infty.$$

By Proposition 3.2, Γ is discrete. Likewise, suppose that (i) holds. Since the mapping $\gamma \mapsto T_\gamma \phi$ is continuous from G into $L^2(G)$, for every $\varepsilon > 0$ we can find a neighborhood U of 0 such that

$$||T_\gamma \phi - \phi|| < \varepsilon \quad \text{for } \gamma \in U.$$

We may also select U so that $m_\Gamma(\Gamma \cap U) < \infty$. Then for any $a \in C_{00}(\Gamma)$ with supp $a \subset \Gamma \cap U$ we have

$$\left\| \int_\Gamma a(\gamma) T_\gamma \phi \, dm_\Gamma(\gamma) - \left(\int_\Gamma a(\gamma)\, dm_\Gamma(\gamma) \right)\phi \right\| < \varepsilon \int_\Gamma |a(\gamma)| dm_\Gamma(\gamma).$$

Thus by (5.1) we have

$$\sqrt{A}||a||_{L^2(\Gamma)} \le \left\| \int_\Gamma a(\gamma)T_\gamma\phi\, dm_\Gamma(\gamma) \right\| \le (||\phi|| + \varepsilon)||a||_{L^1(\Gamma)}.$$

Since $C_{00}(\Gamma\cap U)$ is dense in $L^1(\Gamma\cap U, m_\Gamma)$, we have $L^1(\Gamma\cap U, m_\Gamma)\subset L^2(\Gamma\cap U, m_\Gamma)$. By [16, Exercise 5, Chapter 6], $\Gamma \cap U$ cannot have subsets with arbitrarily small measure. Since m_Γ restricted to $\Gamma \cap U$ is a regular measure, it would follow that $m_\Gamma(\{0\}) > 0$. As noted in [20, (15.17.b)], this implies that Γ is discrete. This completes the proof of the theorem by reducing it to [9, Theorem 4.3]. $\square$

Riesz and frame sequences can also be characterized using Gramians and dual Gramians introduced and studied in the setting of SI spaces by Ron and Shen [29,30].

Definition 5.3 Let $\mathcal{A}$ be a countable subset of $L^2(G)$ such that

$$\sum_{\phi\in\mathcal{A}} |\hat{\phi}(\chi)|^2 < \infty \quad \text{for almost all } \chi \in \widehat{G}. \tag{5.4}$$

For almost all $\omega \in \Omega$, define Gramian and dual Gramian as (possibly) infinite matrices

$$\mathcal{G}_\omega = \left(\sum_{\kappa\in\Gamma^*} \hat{\phi}_1(\omega + \kappa)\overline{\hat{\phi}_2(\omega + \kappa)} \right)_{\phi_1,\phi_2\in\mathcal{A}} \tag{5.5}$$

and

$$\widetilde{\mathcal{G}}_\omega = \left(\sum_{\phi\in\mathcal{A}} \hat{\phi}(\omega + \kappa_1)\overline{\hat{\phi}(\omega + \kappa_2)} \right)_{\kappa_1,\kappa_2\in\Gamma^*}. \tag{5.6}$$

By standard results in frame theory [10, Chapter 3], the following are equivalent:

(i) $\mathcal{G}_\omega$ defines a bounded operator on $\ell^2(\mathcal{A})$,
(ii) $\widetilde{\mathcal{G}}_\omega$ defines bounded operator on $\ell^2(\Gamma^*)$,
(iii) the synthesis operator $K_\omega : \ell^2(\mathcal{A}) \to \ell^2(\Gamma^*)$ corresponding to $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}$ is bounded,
(iv) the analysis operator $K_\omega^* : \ell^2(\Gamma^*) \to \ell^2(\mathcal{A})$ corresponding to $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}$ is bounded.

Recall that for $\phi \in \mathcal{A}$, $K_\omega(1_\phi) = \mathcal{T}\phi(\omega)$, where 1_ϕ is the sequence that is 1 at ϕ and 0 elsewhere. For $v \in \ell^2(\Gamma^*)$, $K_\omega^*(v) = (\langle v, \mathcal{T}\phi\rangle(\omega))_{\phi\in\mathcal{A}}$. Moreover, $\mathcal{G}_\omega = K_\omega^*K_\omega$ and $\widetilde{\mathcal{G}}_\omega = K_\omega K_\omega^*$.

Then, we have a generalization of [5, Theorem 2.5] to the setting of LCA groups which was shown for discrete Γ in [9, Proposition 4.9]. Compare also [23, Theorem 4.1].

Theorem 5.2 *Let $\mathcal{A}$ be a countable subset of $L^2(G)$ satisfying (5.4), and consider $0 < A \le B$. Then the following are equivalent:*

(i) *The set $E^\Gamma(\mathcal{A})$ is a continuous frame for $S^\Gamma(\mathcal{A})$ with constants A and B.*
(ii) *For almost all $\omega \in \Omega$,*

$$A\|v\|^2 \le \langle \widetilde{\mathcal{G}}_\omega v, v \rangle \le B\|v\|^2 \quad \text{for all } v \in \text{span}\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}.$$

(iii) *For almost all $\omega \in \Omega$, the spectrum $\sigma(\widetilde{\mathcal{G}}_\omega) \subset \{0\} \cup [A, B]$.*

Theorem 5.2 has a variant for Riesz sequences which we will not use in this paper. In this variant, condition (ii) is replaced by

$$A\|v\|^2 \le \langle \mathcal{G}_\omega v, v \rangle \le B\|v\|^2 \quad \text{for all } v = (v_\phi)_{\phi \in \mathcal{A}} \in \ell^2(\mathcal{A}),$$

and condition (iii) by $\sigma(\mathcal{G}_\omega) \subset [A, B]$.

Finally, we will need the following decomposition theorem which was shown in [5, Theorem 3.3]. The extension of this result when Γ is a discrete co-compact subgroup of an LCA group was formulated in [9, Theorem 4.11]. In the case when $\Gamma \subset G$ is merely co-compact, we have the following result.

Theorem 5.3 *Let V be a Γ-TI subspace of $L^2(G)$. Then there exist functions $\phi_n \in V$, $n \in \mathbb{N}$ such that:*

(i) *$E^\Gamma(\phi_n)$ is a continuous Parseval frame for $S^\Gamma(\phi_n)$,*
(ii) *V can be decomposed as an orthogonal sum*

$$V = \bigoplus_{n \in \mathbb{N}} S^\Gamma(\phi_n).$$

In particular, $E^\Gamma(\{\phi_n : n \in \mathbb{N}\})$ is a continuous Parseval frame for V.

Proof By Theorem 3.8, the space $W = \mathcal{T}(V)$ is of the form

$$W = \{\Phi \in L^2(\Omega, \ell^2(\Gamma^*)) : \Phi(\omega) \in J(\omega) \text{ for a.e. } \omega \in \Omega\}$$

for some measurable range function J. Theorem 2.6 yields the existence of functions $\Phi_n \in L^\infty(\Omega, \ell^2(\Gamma^*))$ such that $\|\Phi_n(\omega)\| \in \{0, 1\}$ and

$$J(\omega) = \bigoplus_{n \in \mathbb{N}} J_n(\omega), \quad \text{where } J_n(\omega) = \text{span } \Phi_n(\omega).$$

In the case when Γ is discrete, and hence Ω has finite measure, we can take $\phi_n = \mathcal{T}^{-1}(\Phi_n)$. However, if Ω has infinite measure, then we need to partition Ω as a countable union of sets of finite measure $\Omega = \dot{\bigcup}_{m \in \mathbb{N}} \Omega_m$. In this case we define functions $\phi_{n,m} = \mathcal{T}^{-1}(\Phi_n \mathbf{1}_{\Omega_m})$. By Theorem 5.1, each $E^\Gamma(\phi_{n,m})$ is a continuous Parseval frame for $S^\Gamma(\phi_{n,m})$. Since

$$J(\omega) = \bigoplus_{n,m \in \mathbb{N}} J_{n,m}(\omega) \quad \text{where } J_{n,m}(\omega) = \text{span } \mathcal{T}\phi_{n,m}(\omega),$$

by (2.10) we have

$$V = \bigoplus_{n,m \in \mathbb{N}} S^\Gamma(\phi_{n,m}).$$

A simple reindexing yields the conclusion of Theorem 5.3.   □

Theorem 5.3 enables us to employ the following equivalent way of defining the spectral function; see [7, Lemma 2.3] or [8, Prop. 2.1].

Theorem 5.4 *Let $\mathcal{A}$ be a countable subset of $L^2(G)$, and let $V = S^\Gamma(\mathcal{A})$. If $E^\Gamma(\mathcal{A})$ forms a Parseval frame for the space V, then*

$$\sigma_V(\chi) = \sum_{\phi \in \mathcal{A}} |\hat{\phi}(\chi)|^2 \quad \text{for almost all } \chi \in \widehat{G}. \tag{5.7}$$

In particular, (5.7) does not depend on the choice of $\mathcal{A}$ as long as $E^\Gamma(\mathcal{A})$ is a Parseval frame for V.

Proof The proof closely follows that of [7, Lemma 2.3]. Let $J(\omega)$ and $P_J(\omega)$ be as in Definition 4.1. By Theorem 5.1, with $A = B = 1$, $E^\Gamma(\mathcal{A})$ is a Parseval frame for V if and only if $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}$ is a Parseval frame for $J(\omega)$, for almost all ω. So for almost all $\omega \in \Omega$ and all $v \in J(\omega)$, we have

$$\|v\|^2 = \sum_{\phi \in \mathcal{A}} |\langle v, \mathcal{T}\phi(\omega)\rangle|^2.$$

Hence

$$\|P_J(\omega)v\|^2 = \sum_{\phi \in \mathcal{A}} |\langle v, \mathcal{T}\phi(\omega)\rangle|^2 \quad \text{for all } v \in \ell^2(\Gamma^*).$$

In particular, for (almost all) $\omega \in \Omega$ and $\kappa \in \Gamma^*$, we have

$$\sigma_V(\omega + \kappa) = \|P_J(\omega)e_\kappa\|^2 = \sum_{\phi \in \mathcal{A}} |\langle e_\kappa, \mathcal{T}\phi(\omega)\rangle|^2 = \sum_{\phi \in \mathcal{A}} |\hat{\phi}(\omega + \kappa)|^2,$$

which completes the proof.   □

As a corollary, Theorem 5.2 describes the spectral function as diagonal entries of the dual Gramian $\langle \tilde{\mathcal{G}}_\omega e_\kappa, e_\kappa \rangle$ of a Γ-TI system $\mathcal{A}$ as in Theorem 5.4. Finally, Theorem 5.5 gives yet another equivalent way of defining the spectral function, see [7, Proposition 2.2].

Theorem 5.5 *Let $\mathfrak{S}$ be the set of all Γ-TI subspaces of $L^2(G)$. Then the spectral function σ_V of V in $\mathfrak{S}$ is determined as the unique mapping $\sigma : \mathfrak{S} \to L^\infty(\widehat{G})$ that satisfies*

$$\sigma_{S^\Gamma(\phi)}(\chi) = \begin{cases} |\hat{\phi}(\chi)|^2 \left(\sum_{\kappa \in \Gamma^*} |\hat{\phi}(\chi + \kappa)|^2 \right)^{-1} & \text{for } \chi \in \operatorname{supp} \hat{\phi}, \\ 0 & \text{otherwise,} \end{cases} \tag{5.8}$$

which is additive with respect to orthogonal sums, i.e.,

$$V = \bigoplus_{i \in \mathbb{N}} V_i \quad \text{for some} \quad V_i \in \mathfrak{S} \quad \text{implies} \quad \sigma_V = \sum_{i \in \mathbb{N}} \sigma_{V_i}. \tag{5.9}$$

Proof The property (5.9) is an immediate consequence of (2.10), Theorem 3.8, and Definition 4.1. Hence, it remains to show (5.8). Given any $\phi \in L^2(G)$ we define $\Phi \in L^\infty(\Omega, \ell^2(\Gamma^*))$ by

$$\Phi(\omega) = \begin{cases} \mathcal{T}\phi(\omega)/\|\mathcal{T}\phi(\omega)\| & \omega \in \operatorname{supp} \mathcal{T}\phi, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|\Phi(\omega)\| \in \{0, 1\} \quad \text{for almost all } \omega \in \Omega. \tag{5.10}$$

Define functions $\phi_m = \mathcal{T}^{-1}(\Phi \mathbf{1}_{\Omega_m})$, where Ω_m's are the same as in the proof of Theorem 5.3. By (5.10) and Theorem 5.1, each $E^\Gamma(\phi_m)$ is a continuous Parseval frame for $S^\Gamma(\phi_m)$. Thus, by (5.9) and Theorem 5.4 for any $\omega \in \Omega$ and $\kappa \in \Gamma^*$ we have

$$\sigma_{S^\Gamma(\phi)}(\omega + \kappa) = \sum_{m \in \mathbb{N}} \sigma_{S^\Gamma(\phi_m)}(\omega + \kappa) = \sum_{m \in \mathbb{N}} |\hat{\phi}_m(\omega + \kappa)|^2 = \sum_{m \in \mathbb{N}} |\langle \mathcal{T}\phi_m(\omega), e_\kappa \rangle|^2$$
$$= |\langle \Phi(\omega), e_\kappa \rangle|^2 = |\hat{\phi}(\omega + \kappa)|^2 / \|\mathcal{T}\phi(\omega)\|^2.$$

This completes the proof of Theorem 5.5.   □

6 Modulations, Dilations, and Epimorphisms

The last equality holds when $\omega \in \operatorname{supp} \mathcal{T}\phi$, and otherwise equals 0.

In this section we shall show how the spectral functions behave under modulations and epimorphisms on G.

For a character χ in $\widehat{G}$, we define the *modulation operator* M_χ on $L^2(G)$ by

$$M_\chi(f)(x) = \chi(x)f(x) \quad \text{for all } x \in G.$$

We have the following generalization of [7, equation (2.7)].

Theorem 6.1 *Let $V \subset L^2(G)$ be a Γ-TI subspace. Let χ_0 be in $\widehat{G}$. Then $M_{\chi_0}(V)$ is a Γ-TI subspace and*

$$\sigma_{M_{\chi_0}(V)}(\chi) = \sigma_V(\chi - \chi_0) \quad \text{for almost all } \chi \in \widehat{G}. \tag{6.1}$$

Before we generalize dilations to the present setting, we introduce some notation and terminology. As is standard, we write **Aut**(G) for the group of topological automorphisms α of G onto itself. Also, we write **Epi**(G) for the semigroup of continuous group homomorphisms α of G onto G. We also write **Epick**(G) for the set of $\alpha \in$ **Epi**(G) having compact kernel ker α. Thus **Aut**$(G) \subset$ **Epick**$(G) \subset$ **Epi**(G). Let Mor(G, H) be the set of all continuous group homomorphisms between LCA groups G and H.

Example 6.1

(a) For $G = \mathbb{R}^n \times \mathbb{Z}^m$, we have

$$\mathbf{Aut}(G) = \mathbf{Epick}(G) = \mathbf{Epi}(G) = \mathrm{Mor}(\mathbb{Z}^m, \mathbb{R}^n) \rtimes (\mathbf{Aut}(\mathbb{R}^n) \times \mathbf{Aut}(\mathbb{Z}^m)).$$

This follows immediately from [34, Theorem 25.8].

(b) Let $G = \prod_{n=1}^{\infty} \mathbb{T} \times \sum_{n=1}^{\infty} \mathbb{Z}$, where the sum is the discrete "weak direct sum", i.e., all $(k_1, k_2, \ldots)$ in $\mathbb{Z}^{\aleph_0}$ such that all but finitely many k_i equal 0. Define $\alpha : G \to G$ by

$$\alpha(z_1, z_2, \ldots; k_1, k_2, \ldots) = (z_2, z_3, \ldots; k_2, k_3, \ldots).$$

Then $\alpha \in$ **Epi**(G) and ker $\alpha = \{(z_1, 1, 1, \ldots; k_1, 0, 0, \ldots) : z_1 \in \mathbb{T}, k_1 \in \mathbb{Z}\} \cong \mathbb{T} \times \mathbb{Z}$. This shows that ker α need not be compact or discrete, even though G satisfies our standing hypotheses.

(c) Each α in **Epi**$(\mathbb{T})$ has the form $\alpha_n(z) = z^n$ (for all z in $\mathbb{T}$) for some $n \in \mathbb{Z} \setminus \{0\}$. Note that **Epi**$(\mathbb{T})$ is a semigroup under composition since $\alpha_n \circ \alpha_{n'} = \alpha_{nn'}$ for $n, n' \in \mathbb{Z} \setminus \{0\}$.

Theorem 6.2 *Given G, **Epick**(G) is a semigroup (under composition). Moreover, there is a semigroup homomorphism $\Delta :$ **Epick**$(G) \to (0, \infty)$ such that*

$$\int_G (f \circ \alpha)(x) \, dm_G(x) = \Delta(\alpha) \int_G f(x) \, dm_G(x) \tag{6.2}$$

for all integrable functions f on G with respect to the Haar measure m_G.

Proof To check that **Epick**(G) is a semigroup, we need that ker$(\beta \circ \alpha) = \alpha^{-1}(\beta^{-1}(0))$ is compact for $\alpha, \beta \in$ **Epick**(G). Thus it suffices to show that if $\alpha \in$ **Epick**(G), then $\alpha^{-1}(K)$ is compact for compact $K \subset G$. Let $H = \ker \alpha = \alpha^{-1}(0)$. By [20, Theorem (5.27)], G and G/H are topologically isomorphic, and the topological isomorphism is given by $\Phi(x) = \alpha^{-1}(x)$. Clearly $\Phi(K)$ is compact. Also $\Phi(K) = \{\alpha^{-1}(k) : k \in K\} = \{x + H : x \in X\}$ for some subset $X \subset G$. By [20, (5.24.a)], $X + H$ is compact in G. Therefore

$$X + H = \bigcup_{x \in X} (x + H) = \bigcup_{k \in K} \alpha^{-1}(k) = \alpha^{-1}(K)$$

is compact.

To obtain Δ as described, it suffices by [20, Theorem (15.5)] to show that $J_\alpha(f) = \int_G (f \circ \alpha)(x) \, dm_G(x)$ defines a positive translation-invariant linear functional on the space $C_{00}(G)$ of continuous functions on G with compact support. The verifications are routine, once it is checked that $J_\alpha(f)$ is finite for all $f \in C_{00}(G)$. This is because $f \circ \alpha$ also has compact support, which follows from the conclusion in the second sentence of this proof. □

We call Δ the *modular function* on $\mathbf{Epick}(G)$. It extends the modular function defined on $\mathbf{Aut}(G)$; see, for example [20, (15.26) and (26.21)].

Theorem 6.3 *If G is compactly generated, then $\mathbf{Epick}(G) = \mathbf{Epi}(G)$.*

Proof Consider a compactly generated LCA group G. By the structure theorem [20, Theorem (9.8)], $G = A \times F$, where $A = \mathbb{R}^n \times \mathbb{Z}^m$ and F is compact. Let $K = \{0\} \times F$, which is compact in G. Consider α in $\mathbf{Epi}(G)$. Since α takes compact subgroups into compact subgroups, we have $\alpha(K) \subset K$. We define

$$\beta : G/K \to G/K \text{ where } \beta(g + K) = \alpha(g) + K$$

for all $g + K$ in G/K; β is well defined since $\alpha(K) \subset K$. It easy to verify that β is a continuous group homomorphism; see [20, (5.15), (5.17)]. Since α maps G onto G, it follows that β is in $\mathbf{Epi}(G/K)$. Since G/K is topologically isomorphic with $A = \mathbb{R}^n \times \mathbb{Z}^m$, Example 6.1(a) shows that $\mathbf{Epi}(G/K) = \mathbf{Aut}(G/K)$, so $\ker \beta = \{K\}$. This implies that $\ker \alpha \subset K$, since $\alpha(g) = 0 \implies \beta(g + K) = K \implies g + K \in \ker \beta \implies g + K = K \implies g \in K$. Since K is compact, so is $\ker \alpha$. □

A simplified proof of [20, Theorem (9.12)], which is about compactly generated LCA groups, is given in the 'Appendix'.

Proposition 6.4 *Consider a closed co-compact subgroup Γ of G, and consider α in $\mathbf{Epi}(G)$. Then $\alpha^{-1}\Gamma$ is co-compact.*

Proof Since G is σ-compact, the open mapping theorem for topological groups [20, Theorem (5.29)] shows that α is an open mapping. Then [20, Theorem (5.34)] implies $G/\alpha^{-1}\Gamma$ and G/Γ are topologically isomorphic, so if G/Γ is compact, so is $G/\alpha^{-1}\Gamma$. □

Proposition 6.5 *Let $\alpha \in \mathbf{Epi}(G)$, and let $\hat{\alpha}$ be the adjoint homomorphism defined by $\hat{\alpha}(\chi) = \chi \circ \alpha$ for $\chi \in \widehat{G}$. Then $\hat{\alpha}$ is a topological isomorphism of $\widehat{G}$ onto the annihilator K^* of $K = \ker \alpha$. Also, we have*

$$\hat{\alpha}\Gamma^* = (\alpha^{-1}\Gamma)^*. \tag{6.3}$$

Proof Since α maps G onto G, and G is σ-compact, the open mapping theorem [20, Theorem (5.29)] gives that α is open and continuous. By [20, Theorem (24.40)], $\hat{\alpha}$ is an open and continuous isomorphism of $\widehat{G}$ onto K^*.

To show (6.3), consider $\chi \in \Gamma^*$ and $x \in \alpha^{-1}\Gamma$. Then $\alpha(x) \in \Gamma$, so $\hat{\alpha}(\chi)(x) = \chi(\alpha(x)) = 1$. Since this holds for all $x \in \alpha^{-1}\Gamma$, we conclude that $\hat{\alpha}\Gamma^* \subset (\alpha^{-1}\Gamma)^*$.

Now suppose that χ is in $(\alpha^{-1}\Gamma)^*$. Since $K \subset \alpha^{-1}\Gamma$, we have $(\alpha^{-1}\Gamma)^* \subset K^*$, so $\chi \in K^*$. Therefore there is a (unique) $\psi \in \widehat{G}$ so that $\hat{\alpha}(\psi) = \chi$. It suffices to show that ψ is in Γ^*. Let x be in Γ. Since α maps G onto G, we have $x = \alpha(y)$ for some $y \in G$. Then $\alpha(y) \in \Gamma$, so $y \in \alpha^{-1}\Gamma$. Since χ is in $(\alpha^{-1}\Gamma)^*$, we conclude $\psi(x) = \psi(\alpha(y)) = \hat{\alpha}(\psi)(y) = \chi(y) = 1$.   □

Definition 6.1 For $\alpha \in \mathbf{Epick}(G)$, we define the *$\alpha$-dilation operator* on $L^2(G)$ by

$$D_\alpha f(x) = f(\alpha(x)) \quad \text{for all } x \in G.$$

We need the following lemma.

Lemma 6.6 *Let $\alpha \in \mathbf{Epick}(G)$ and $K = \ker \alpha$. Then for any $f \in L^2(G)$ we have*

$$\widehat{(D_\alpha f)}(\chi) = \begin{cases} \Delta(\alpha)\hat{f}(\hat{\alpha}^{-1}(\chi)) & \text{for } \chi \in \hat{\alpha}(\widehat{G}) = K^*, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Proof It suffices to prove this for f in $L^1(G) \cap L^2(G)$. We have

$$\widehat{(D_\alpha f)}(\chi) = \int_G \overline{\chi(x)} f(\alpha(x)) \, dm(x). \tag{6.5}$$

If $\chi \in \hat{\alpha}(\widehat{G})$, and we define $\psi = \hat{\alpha}^{-1}(\chi)$, then $\chi = \psi \circ \alpha$. Hence, using equation (6.2), we have

$$\widehat{(D_\alpha f)}(\chi) = \int_G \overline{\psi(\alpha(x))} f(\alpha(x)) \, dm(x) = \Delta(\alpha) \int_G \overline{\psi(x)} f(x) \, dm(x)$$
$$= \Delta(\alpha) \hat{f}(\hat{\alpha}^{-1}(\chi)), \tag{6.6}$$

which confirms equation (6.4) in this case. Otherwise, $\chi \notin K^*$, and we use equation (6.5) and [19, Theorem (28.54)] to obtain

$$\widehat{(D_\alpha f)}(\chi) = \int_G \overline{\chi(x)} f(\alpha(x)) \, dm(x)$$
$$= \int_{G/K} \int_K \overline{\chi(x+y)} f(\alpha(x+y)) \, dm_K(y) \, dm_{G/K}(x+K), \tag{6.7}$$

with the Haar measures suitably normalized, say with $m_K(K) = 1$. Observe that for each $x \in G$, equivalently each $x + K \in G/K$, we have $\alpha(x + y) = \alpha(x)$ for all $y \in K$, since α is constant on $x + K$. So $f(\alpha(x + y)) = f(\alpha(x))$ for all $y \in K$. Therefore the inside integral above is equal to

$$\overline{\chi(x)} \cdot f(\alpha(x)) \int_K \overline{\chi(y)} \, dm_K(y),$$

and this equals 0 because $\overline{\chi}$ is not the constant character 1 on K (since $\chi \notin K^*$); see [20, Lemma (23.19)]. Since all the inside integrals in Eq. (6.7) equal 0, $\widehat{(D_\alpha f)}(\chi) = 0$. This completes the proof of (6.4). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We have the following formula for the spectral function of $D_\alpha(V)$, which is a generalization of [7, equation (2.6)].

Theorem 6.7 *Let $V \subset L^2(G)$ be a Γ-TI subspace. Let $\alpha \in \mathbf{Epick}(G)$. Then $D_\alpha(V)$ is an $\alpha^{-1}\Gamma$-TI subspace and*

$$\sigma_{D_\alpha(V)}(\chi) = \begin{cases} \sigma_V((\hat{\alpha})^{-1}\chi) & \text{for } \chi \in \hat{\alpha}(\widehat{G}) = K^*, \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

Proof Consider any $f \in D_\alpha(V)$ and $\tilde{\gamma} \in \alpha^{-1}\Gamma$, i.e., $f = D_\alpha g$ for some $g \in V$ and $\alpha(\tilde{\gamma}) = \gamma$ for some $\gamma \in \Gamma$. Then,

$$T_{\tilde{\gamma}} f(x) = g(\alpha(x - \tilde{\gamma})) = g(\alpha(x) - \gamma) = D_\alpha(T_\gamma g)(x) \quad \text{for} \quad x \in G.$$

Since $T_\gamma g \in V$, this shows that $D_\alpha(V)$ is $\alpha^{-1}\Gamma$-TI.

Let $E^\Gamma(\mathcal{A})$ be a Parseval frame of V for some countable $\mathcal{A} \subset L^2(G)$. By Theorem 5.1, this means that $\{\mathcal{T}\phi(\omega)\}_{\phi \in \mathcal{A}}$ is a Parseval frame of $J(\omega)$ for a.e. $\omega \in \Omega$. The existence of such $\mathcal{A}$ is guaranteed by Theorem 5.3. Then, by Theorem 5.4, the spectral function of V is given by

$$\sigma_V(\chi) = \sum_{\phi \in \mathcal{A}} |\hat{\phi}(\chi)|^2 \quad \text{for almost all } \chi \in \widehat{G}.$$

Let Ω_0 be a Borel section of $\widehat{G}/K^*$, which is topologically isomorphic with $\widehat{K}$. Since $K = \ker \alpha$ is compact, Ω_0 is a discrete subset of $\widehat{G}$. Since $\hat{\alpha}(\Omega) \oplus \hat{\alpha}(\Gamma^*) = \hat{\alpha}(\widehat{G})$, and since $\hat{\alpha}\Gamma^* = (\alpha^{-1}\Gamma)^*$ by Proposition 6.5, we have

$$\hat{\alpha}(\Omega) \oplus (\alpha^{-1}\Gamma)^* = \hat{\alpha}(\widehat{G}) = K^*, \tag{6.9}$$

so that $\hat{\alpha}(\Omega)$ is a Borel section of $\hat{\alpha}(\widehat{G})/(\alpha^{-1}\Gamma)^*$. Since $\widehat{G} = K^* \oplus \Omega_0 = \hat{\alpha}(\Omega) \oplus \Omega_0 \oplus \hat{\alpha}(\Gamma^*)$, the set $\tilde{\Omega} = \hat{\alpha}(\Omega) \oplus \Omega_0$ is a Borel section of $\widehat{G}/(\alpha^{-1}\Gamma)^*$ such that $\hat{\alpha}(\Omega) \subset \tilde{\Omega}$.

Let $\tilde{\mathcal{T}} : L^2(G) \to L^2(\tilde{\Omega}, \ell^2(\hat{\alpha}(\Gamma^*)))$ be the fiberization operator corresponding to the subgroup $\alpha^{-1}\Gamma \subset G$. That is,

$$\tilde{\mathcal{T}} f(\tilde{\omega}) = \{\hat{f}(\tilde{\omega} + \tilde{\kappa})\}_{\tilde{\kappa} \in \hat{\alpha}(\Gamma^*)} \quad \text{for } \tilde{\omega} \in \tilde{\Omega}. \tag{6.10}$$

Taking $f = D_\alpha \phi$ in (6.10), by Lemma 6.6 we have

$$\tilde{\mathcal{T}} f(\tilde{\omega}) = \begin{cases} \Delta(\alpha)\{\hat{\phi}(\hat{\alpha}^{-1}(\tilde{\omega} + \tilde{\kappa}))\}_{\tilde{\kappa} \in \hat{\alpha}(\Gamma^*)} & \tilde{\omega} \in \hat{\alpha}(\Omega), \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

Thus, if $\tilde{\omega} = \hat{\alpha}(\omega)$ for some $\omega \in \Omega$, then $\tilde{\mathcal{T}}f(\tilde{\omega}) = \Delta(\alpha)\mathcal{T}\phi(\omega)$, where we identify $\ell^2(\hat{\alpha}(\Gamma^*))$ with $\ell^2(\Gamma^*)$ in a natural way using the action of $\hat{\alpha}$. Consequently, $\Delta(\alpha)^{-1}\tilde{\mathcal{T}}(D_\alpha\phi)(\tilde{\omega})$ is a Parseval frame for the range function $\tilde{J}(\tilde{\omega})$ corresponding to the space $D_\alpha(V)$. By Theorem 5.1, $E^{\alpha^{-1}\Gamma}(\tilde{\mathcal{A}})$ is a Parseval frame of $D_\alpha(V)$, where $\tilde{\mathcal{A}} = \Delta(\alpha)^{-1}D_\alpha(\mathcal{A})$. By Theorem 5.4 and Lemma 6.6 we obtain (6.8). $\qquad\square$

We end this section by summarizing several equivalent ways of introducing the spectral function associated to TI subspaces of $L^2(G)$ with respect to arbitrary co-compact closed subgroups Γ of a second countable LCA group G. The spectral function of the TI-space V can be defined using:

(i) diagonal of orthogonal projections onto the range function,
(ii) decomposition of V into orthogonal sums of principal TI spaces,
(iii) generators of a Parseval frame of V,
(iv) diagonal of dual Gramians.

We can also collect together all the main properties of the spectral function in a similar way as [7, Proposition 2.6].

Proposition 6.8 *Let $\mathfrak{S}$ be the set of all Γ-TI subspaces of $L^2(G)$. For $V, W \in \mathfrak{S}$, we have:*

(a) $0 \leq \sigma_V(\chi) \leq 1$ *for almost all $\chi \in \widehat{G}$.*
(b) $V = \bigoplus_{i\in\mathbb{N}} V_i$ *for some $V_i \in \mathfrak{S}$* $\implies \sigma_V = \sum_{i\in\mathbb{N}} \sigma_{V_i}$.
(c) $V \subset W \implies \sigma_V \leq \sigma_W$.
(d) *If $V \subset W$, then $V = W \iff \sigma_V = \sigma_W$.*
(e) $\sigma_V = \mathbf{1}_E$ *for some measurable subset $E \subset \widehat{G}$* $\iff V = \{f \in L^2(G) : \text{supp } \hat{f} \subset E\}$.
(f) $\sigma_{M_{\chi_0}(V)}(\chi) = \sigma_V(\chi - \chi_0)$ *for almost all $\chi \in \widehat{G}$, where M_{χ_0} is a modulation by $\chi_0 \in \widehat{G}$.*
(g) $\sigma_{D_\alpha(V)}(\chi) = \sigma_V((\hat{\alpha})^{-1}\chi)$ *for almost all $\chi \in \hat{\alpha}(\widehat{G})$ and 0 otherwise, where D_α is a dilation by $\alpha \in \mathbf{Epick}(G)$.*
(h) $\dim_V^\Gamma(\chi) = \sum_{\kappa\in\Gamma^*} \sigma_V(\chi + \kappa)$ *for almost all $\chi \in \widehat{G}$.*

7 The Spectral Function as a Pointwise Limit

Rzeszotnik and the first author [7] have shown that the spectral function can also be defined pointwise using the Lebesgue Differentiation Theorem. To generalize this result to the LCA setting we need the following concept; compare [13, Definition (2.1)] or [19, Definition (44.10)].

Definition 7.1 In an LCA group G with Haar measure $m = m_G$, a decreasing sequence $\{U_j\}$ of finite measure sets is called a D'-sequence if every neighborhood of 0 contains some U_j, and if there is a constant $C > 0$ satisfying

$$0 < m(U_j - U_j) \leq Cm(U_j) \quad \text{for all } j \in \mathbb{N}. \tag{7.1}$$

Remark 7.1 If G has an invariant metric d satisfying the doubling property

$$m(B(0, 2r)) \le Cm(B(0, r)) \quad \text{for some constant } C > 0, \tag{7.2}$$

where $B(0, r)$ is the ball about 0 of radius r, then $\{B(0, 2^{-j})\}$ is a D'-sequence. Indeed,

$$m(B(0, 2^{-j}) - B(0, 2^{-j})) \le m(B(0, 2^{-j+1})) \le Cm(B(0, 2^{-j})) \quad \text{for all } j \in \mathbb{N}.$$

Proposition 7.1 *If G is 0-dimensional or if G contains an open subgroup topologically isomorphic to $\mathbb{R}^a \times \mathbb{T}^b$ for nonnegative integers a and b, then G has a D'-sequence.*

Proof First note that, in reading [19, Section 44] or [13], every D''-sequence is a D'-sequence. Now, if G is 0-dimensional, then it is clear from [20, Theorem (7.7)] that G has a D'-sequence using open subgroups, as noted in [13, (2.9)]. If G contains an open subgroup topologically isomorphic to $\mathbb{R}^a \times \mathbb{T}^b$, then apply [19, Theorem (44.30)]; this isn't surprising either, since $\mathbb{R}^a \times \mathbb{T}^b$ has a D'-sequence. □

The next theorem is the Lebesgue Differentiation Theorem [19, Theorem (44.18)].

Theorem 7.2 *If $\{U_j\}$ is a D'-sequence in G, then for every $f \in L^p(G)$, $1 \le p < \infty$, we have*

$$\lim_{j \to \infty} \frac{1}{m(U_j)} \int_{x+U_j} f \, dm = f(x) \quad \text{for almost all } x \in G. \tag{7.3}$$

Combining this theorem with Theorem 4.1 gives the following; compare [8, Theorem 2.6].

Theorem 7.3 *Suppose that $\widehat{G}$ has a D'-sequence $\{U_j\}$ and that $V \subset L^2(G)$ is a Γ-TI space for a closed co-compact subgroup Γ of G. Then*

$$\sigma_V(\chi) = \lim_{j \to \infty} \frac{\|P_V(\check{\mathbf{1}}_{U_j+\chi})\|^2}{m_{\widehat{G}}(U_j)} \quad \text{for almost all } \chi \in \widehat{G}. \tag{7.4}$$

Proof Choose j_0 so that $U_j \cap (\kappa + U_j) = \emptyset$ for all $\kappa \in \Gamma^* \setminus \{0\}$ and for $j \ge j_0$. By Theorem 4.1, for $\chi \in \widehat{G}$, we have

$$\|P_V(\check{\mathbf{1}}_{U_j+\chi})\|^2 = \int_{U_j+\chi} \sigma_V \, dm_{\widehat{G}}.$$

Now by Theorem 7.2, for almost all $\chi \in \widehat{G}$, we have

$$\sigma_V(\chi) = \lim_{j \to \infty} \frac{1}{m_{\widehat{G}}(U_j)} \int_{\chi+U_j} \sigma_V \, dm_{\widehat{G}} = \lim_{j \to \infty} \frac{\|P_V(\check{\mathbf{1}}_{U_j+\chi})\|^2}{m_{\widehat{G}}(U_j)}.$$

□

Example 7.2 It has been noted that the "tubby torus" $\mathbb{T}^{\aleph_0}$, which is compact, does not have a D'-sequence; see, for example, the Notes to [19, Section 44] or [13, p. 194]. In Proposition 7.4 we will show that fact. Hence not every compactly generated LCA group has a D'-sequence. However, by Proposition 7.1 all groups in the category **CGAL** of compactly generated abelian Lie groups do. This category, which consists of all groups of the form $\mathbb{R}^a \times \mathbb{T}^b \times \mathbb{Z}^c \times F$ where F is finite, is studied by M. Stroppel [34]. The character groups $\widehat{G}$ of groups G in **CGAL** also have D'-sequences, because the category **CGAL** is closed under taking character groups. In fact, it is the smallest category containing $\mathbb{R}$ and closed with respect to taking closed subgroups, quotients by closed subgroups, and finite products [34, Corollary (21.20)].

Proposition 7.4 $\mathbb{T}^{\aleph_0}$ *has no D'-sequences.*

Proof For finite n, we write λ_n and m_n for Lebesgue measure on $\mathbb{R}^n$ and Haar measure on $\mathbb{T}^n$, respectively. We write m for the Haar measure on the tubby torus $\mathbb{T}^{\aleph_0}$. Let $\pi_n : \mathbb{R}^n \to \mathbb{T}^n$ be the quotient map given by $\pi_n(\mathbf{x}) = \mathbf{x} + \mathbb{Z}^n$ for $\mathbf{x} \in \mathbb{R}^n$. Let $I_n = [-\frac{1}{2}, \frac{1}{2})^n$. Then π_n restricted to I_n is a measure-preserving map.

Consider any n. Since any basis of neighborhoods of $\mathbf{0}$ has members that are subsets of $\pi_n([-\frac{1}{4}, \frac{1}{4})^n) \times \mathbb{T}^{\aleph_0}$, it suffices to show

$$m(U - U) \geq 2^{n-1} m(U) \quad \text{for all open} \quad U \subset \pi_n\left(\left[-\frac{1}{4}, \frac{1}{4}\right)^n\right) \times \mathbb{T}^{\aleph_0}. \quad (7.5)$$

We will show that for sufficiently large k, there is an open subset W of $\mathbb{T}^{n+k}$ satisfying

$$W \times \mathbb{T}^{\aleph_0} \subset U \quad \text{and} \quad m_{n+k}(W) > \frac{1}{2} m(U) \quad (7.6)$$

and

$$m_{n+k}(W - W) \geq 2^n m_{n+k}(W). \quad (7.7)$$

For then

$$m(U - U) \geq m_{n+k}(W - W) \geq 2^n m_{n+k}(W) > 2^n \frac{1}{2} m(U) = 2^{n-1} m(U),$$

which verifies (7.5).

There exists a compact set $K \subset U$ so that $m(K) > \frac{1}{2} m(U)$. The set K can be covered by finitely many translates $(\mathbf{x}_j + W_j) \times \mathbb{T}^{\aleph_0}$ of neighborhoods of $\mathbf{0}$, each of which is a subset of U, where $\mathbf{x}_j \in \mathbb{T}^{n_j}$ and W_j is open in $\mathbb{T}^{n_j}$. The union of these translates can be written as $W \times \mathbb{T}^{\aleph_0}$ where $W \subset \mathbb{T}^{n+k}$ for sufficiently large k. Since $m_{n+k}(W) = m(W \times \mathbb{T}^{\aleph_0}) \geq m(K)$, condition (7.6) holds.

To verify (7.7), we let ψ be the function from $A = [-\frac{1}{2}, \frac{1}{2})^n \times [-1, 1)^k$ onto I_{n+k}, so that the entry of each $\psi(\mathbf{x})$ is equal to the corresponding entry of $\mathbf{x}$ modulo 1. Note that $\pi_{n+k}(\psi(\mathbf{x})) = \pi_{n+k}(\mathbf{x})$ for all $\mathbf{x}$ in A. Also, $\psi^{-1}(\mathbf{x})$ has 2^k elements for each $\mathbf{x}$ in I_{n+k}, and $\lambda_{n+k}(\psi^{-1}(C)) = 2^k \lambda_{n+k}(C)$ for $C \subset I_{n+k}$. Hence

$$\lambda_{n+k}(E) \leq 2^k \lambda_{n+k}(\psi(E)) \quad \text{for measurable } E \subset A, \quad (7.8)$$

since $E \subset \psi^{-1}(\psi(E))$. Since $W \subset \mathbb{T}^{n+k}$ and $W \times \mathbb{T}^{\aleph_0} \subset U \subset \pi_n([\frac{1}{4}, \frac{1}{4})^n) \times \mathbb{T}^{\aleph_0}$, we have $W = \pi_{n+k}(V)$ for an open subset V of $[-\frac{1}{4}, \frac{1}{4})^n \times [-\frac{1}{2}, \frac{1}{2})^k$, so that $V - V \subset A$. Since $\pi_{n+k}(\psi(V - V)) = \pi_{n+k}(V) - \pi_{n+k}(V)$, we obtain

$$
\begin{aligned}
m_{n+k}(W - W) &= m_{n+k}(\pi_{n+k}(V) - \pi_{n+k}(V)) \\
&= m_{n+k}(\pi_{n+k}(\psi(V - V)) = \lambda_{n+k}(\psi(V - V)).
\end{aligned}
$$

Therefore, by (7.8) and the Brunn-Minkowski inequality ([27, Theorem 1.1], [35, Theorem 3.16]), applied to $V - V \subset \mathbb{R}^{n+k}$ we have

$$
\begin{aligned}
m_{n+k}(W - W) &\geq 2^{-k}\lambda_{n+k}(V - V) \geq 2^{-k}2^{n+k}\lambda_{n+k}(V) \\
&= 2^n m_{n+k}(\pi_{n+k}(V)) = 2^n m_{n+k}(W).
\end{aligned}
$$

This shows (7.7) and completes the proof of the proposition.   □

It is an interesting problem whether Theorem 7.3 can be generalized to an arbitrary second countable LCA group G. More generally, one can ask whether the Lebesgue Differentiation Theorem 7.2 holds for such groups. This question is ultimately related to the boundedness of the Hardy-Littlewood maximal operator. The renowned result of Stein and Strömberg [33] guarantees dimensionless bounds of strong type (p, p), $1 < p < \infty$, of the Hardy-Littlewood maximal operator with respect to centered balls. However, a remarkable recent result of Aldez [1] shows that weak type $(1, 1)$ bounds of the Hardy-Littlewood maximal operator with respect to centered cubes grow to ∞ as dimension increases. Hence, it is far from trivial what kind of boundedness results hold for the Hardy-Littlewood maximal operator on infinite dimensional LCA groups such as the "tubby torus" $\mathbb{T}^{\aleph_0}$.

7.1 Open Question

Suppose that G is a second countable LCA group. Let d be an invariant metric on G and let $B(x, r) = \{y \in G : d(x, y) < r\}$ be the ball of radius $r > 0$ centred at $x \in G$. Is it true that for any $f \in L^p(G)$, $1 \leq p \leq \infty$, we have

$$
\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} f \, dm = f(x) \quad \text{for almost all } x \in G?
$$

8 Appendix

Here we give an alternative and simpler proof of [20, Theorem (9.12)]; all number references in this appendix are to [20].

Theorem (9.12) *Let τ be a topological isomorphism of $\mathbb{R}^a \times \mathbb{Z}^b \times F$ into $\mathbb{R}^c \times \mathbb{Z}^d \times E$, where a, b, c, d are nonnegative integers and F and E are compact groups (not necessarily abelian). Then $a \leq c$ and $a + b \leq c + d$.*

First a lemma. An element in a topological group is *compact* if it belongs to a compact subgroup of the group.

Lemma *Let τ be a topological isomorphism of H into $G \times K$, where H and G are locally compact and K is a compact group. If H has no compact elements other than the identity, then H is topologically isomorphic to a closed subgroup of G.*

Proof Let π be the projection of $G \times K$ onto G. Since $\tau(H)$ is closed in $G \times K$ by (5.11), the image $\pi(\tau(H))$ is closed in G by (5.18). Also, π is one-to-one on $\tau(H)$. [If (x, k_1) and (x, k_2) in $\tau(H)$ have the same image x in G, then $(e, k_1 k_2^{-1})$ is in $\tau(H)$. Since K is compact, $(e, k_1 k_2^{-1})$ is a compact element in $\tau(H)$. Since only the identity of $\tau(H)$ is a compact element, $k_1 = k_2$.]

Since π is a closed mapping by (5.18), it is also a closed mapping of the closed subgroup $\tau(H)$ onto $\pi(\tau(H))$. Since π is one-to-one on this closed subgroup, it is also an open mapping, so that π is a topological isomorphism of $\tau(H)$ onto $\pi(\tau(H))$. Thus $\pi \circ \tau$ is a topological isomorphism of H into G. $\qquad\square$

Proof of (9.12) First, we show $a + b \leq c + d$. Restricting τ to $\mathbb{R}^a \times \mathbb{Z}^b \times \{e\}$ gives a topological isomorphism of $\mathbb{R}^a \times \mathbb{Z}^b$ into $\mathbb{R}^c \times \mathbb{Z}^d \times E \subset \mathbb{R}^{c+d} \times E$. Applying the Lemma with $H = \mathbb{R}^a \times \mathbb{Z}^b$, $G = \mathbb{R}^{c+d}$ and $K = E$, we see that H is topologically isomorphic to a closed subgroup of $\mathbb{R}^{c+d}$. Hence $a + b \leq c + d$ by (9.11).

To prove $a \leq c$, note that restricting τ to $\mathbb{R}^a \times \{0\} \times \{e\}$ gives a topological isomorphism of $\mathbb{R}^a$ into $\mathbb{R}^c \times \mathbb{Z}^d \times E$. Since $\mathbb{R}^a$ is connected, τ maps $\mathbb{R}^a$ into $\mathbb{R}^c \times \{0\} \times E$. Applying the Lemma with $H = \mathbb{R}^a$, $G = \mathbb{R}^c$ and $K = E$, we see that $\mathbb{R}^a$ is topologically isomorphic to a closed subgroup of $\mathbb{R}^c$. Hence $a \leq c$ by (9.11). $\qquad\square$

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