

Math 251

Practice Final Exam - Spring 2013

Your name: SOLUTIONS

Your instructor: _____

Student ID #: _____

You will be asked to present your student ID when you turn in your final.

Instructions:

- To receive credit, you must neatly and clearly show work that justifies your answer.
- Only scientific calculators may be used.
No phones, graphing calculators, electronic dictionaries, headphones, etc.
- Please silence your cell phone: turn off any ringer and turn off vibrate.
If your phone is a distraction to others, you may lose points on your exam.
- Pace yourself. You have approximately 10 minutes per page to complete the exam.
- Check your work once you have worked through all of the problems.

- (1) Two ants named Fred and Velma are walking backwards and forwards along a straight ruler marked in inches. Fred's distance from his starting point is $F(t) = 4t - t^3$ inches after t minutes have passed. Velma's distance from Fred's starting point is $V(t) = t^2 - 12t + 36$ inches after t minutes have passed. Find the minimum distance between the two ants. Make sure to justify that this is a minimum.

$$d(t) = 4t - t^3 - (t^2 - 12t + 36)$$

$$= -t^3 - t^2 + 16t - 36$$

$$d'(t) = -3t^2 - 2t + 16$$

$$d'(t) = 0 \quad \Leftrightarrow \quad -3t^2 - 2t + 16 = 0$$

$$t = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)(16)}}{-6} = \frac{2 \pm 14}{-6} = \underline{\underline{-2\frac{2}{3}, 2}}$$

1st derivative test

critical points

$$\begin{array}{c} d' > 0 \quad d' = 0 \quad d' < 0 \\ \hline | \quad | \quad | \\ t = 0 \quad t = 2 \quad t = 5 \end{array}$$



$\Rightarrow t = 2$ local maximum

$$\text{but } d(2) = -16$$

$$\Rightarrow d(t) < 0 \text{ for all } 0 < t < \infty$$

So the distance is MINIMAL = 16

when $t = 2$.

(2) Find the indicated derivatives.

(a) For $f(x) = \frac{\cos^3(x)}{\sqrt{2x-1}}$, find $f'(x)$.

$$f'(x) = \frac{3 \cos^2(x) \sin x \sqrt{2x-1} - \cos^3(x) \frac{2}{2\sqrt{2x-1}}}{2x-1}$$

(b) For $g(x) = \ln(ax + b)$ where a and b are constants, find $g'(x)$.

$$g'(x) = \frac{1}{ax+b} \cdot a$$

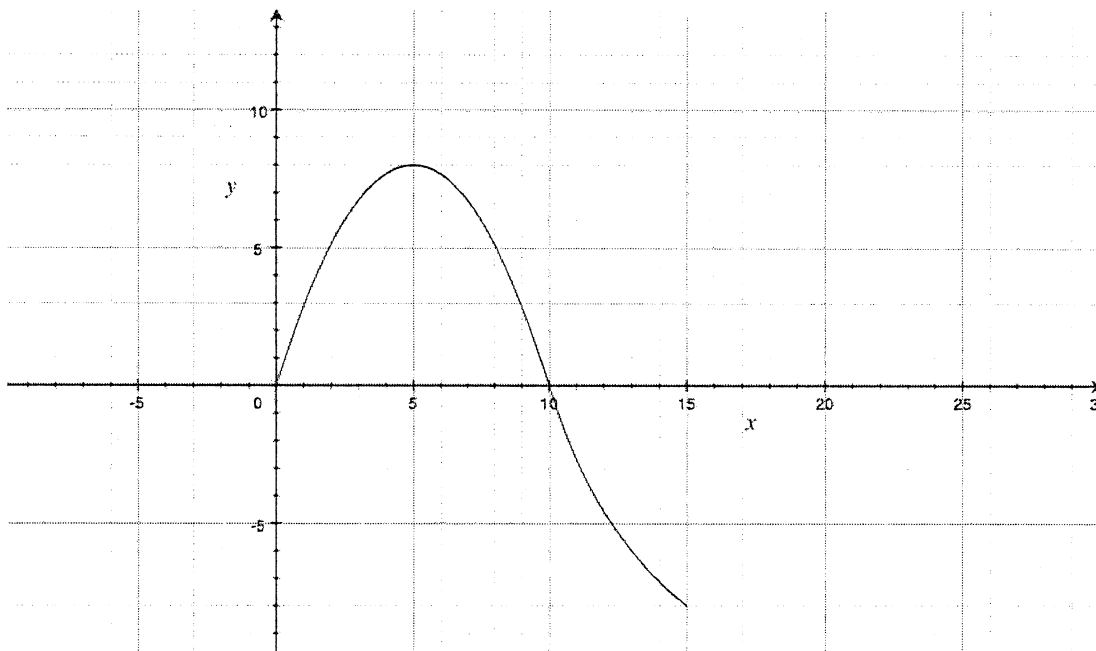
(c) Find $\frac{dy}{dx}$ for $y = (x^2 + 1)^x$.

$$\ln y = x \ln(x^2 + 1)$$

$$\frac{y'}{y} = \ln(x^2 + 1) + x \cdot \frac{2x}{x^2 + 1}$$

$$y' = (x^2 + 1)^x \left(\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right)$$

- (3) Below is the graph of the derivative $C'(t)$ on the interval $[0, 15]$. The equation $C(t)$ describes the temperature of a lake in degrees Celsius after t days, for $0 \leq t \leq 15$. Be careful, the questions below are about $C(t)$, but the graph is for $C'(t)$.



- (a) For what intervals is C increasing?

$$C'(t) > 0$$

$$[0, 10]$$

- (b) For what intervals is C concave up?

$$C''(t) > 0 \Rightarrow C'(t) \text{ increasing}$$

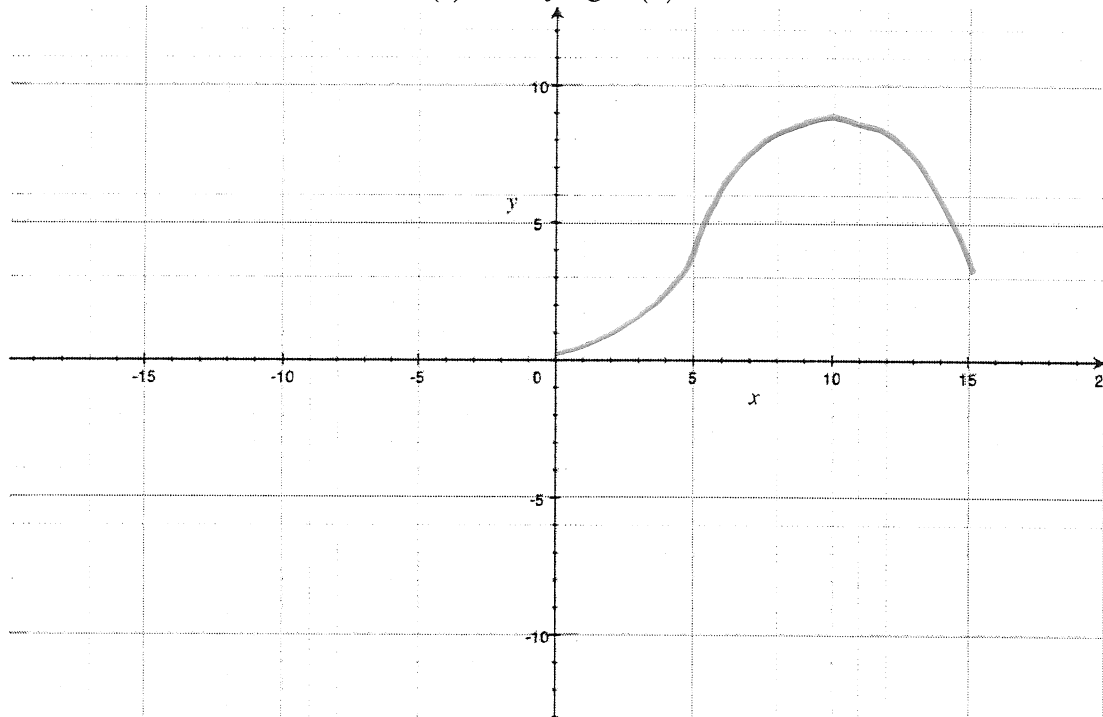
$$[0, 5]$$

- (c) What do your answers from part (a) and (b) tell us about how the temperature of the lake changes over time? Describe the behavior of C from 0 to 15 hours in a sentence or two.

Temp. increases for the first 10 days.
 The rate at which temp. increases has
 a maximum when $t=5$.
 When $t=10$, temp. has a maximum &
 decreases after that.

(problem continues on next page)

(d) Give a reasonable sketch for $C(t)$ satisfying $C(0) = 0$.



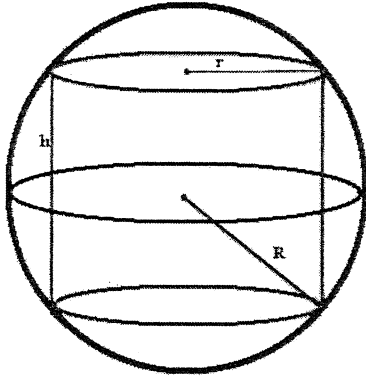
(4) Evaluate each limit, justifying your answers.

(a) $\lim_{x \rightarrow 3^+} \ln(3-x)$ \leftarrow problem doesn't make sense
 since $x > 3 \Rightarrow 3-x < 0$ & $\ln(3-x)$
 is undefined
 so DNE

(b) $\lim_{x \rightarrow 3^-} \ln(3-x) = \lim_{t \rightarrow 0^+} \ln t = \boxed{-\infty}$

(c) $\lim_{x \rightarrow 3} \ln(3-x) \leftarrow$ DNE

- (5) A cylinder is inscribed in a sphere of radius $R = 6$ meters. Find the largest possible volume of such a cylinder, including units. The volume of a cylinder is $V = \pi r^2 h$.



Problem solved in class

Answer $\underline{\underline{96\pi\sqrt{3}}} \approx \underline{\underline{522.37 \text{ m}^3}}$

(6) Consider the function $f(x) = x^5 - 5x - 1$.

(a) Find the intervals on which f is increasing or decreasing.

$$f'(x) = 5x^4 - 5$$

$$f'(x) > 0 \Leftrightarrow 5x^4 - 5 > 0 \quad x^4 > 1$$

$$\Rightarrow x > 1 \quad \text{or} \quad x < -1$$

$(-\infty, -1) \cup (1, \infty)$ - increasing
 $(-1, 1)$ - decreasing

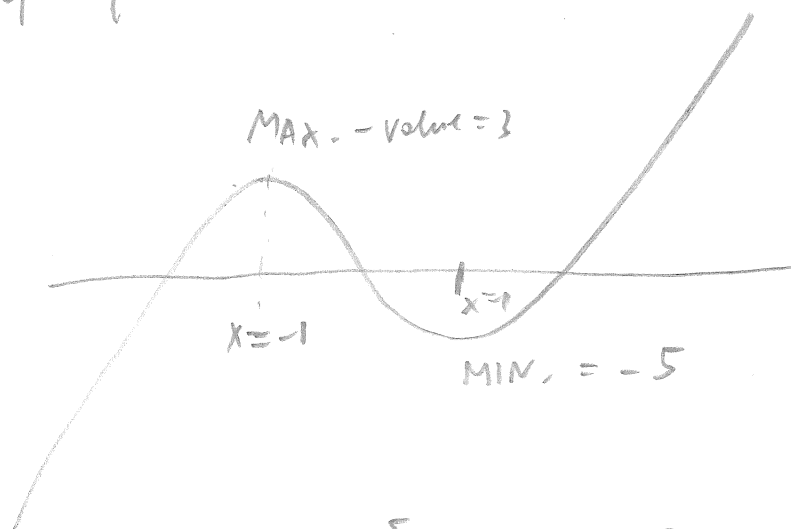
(b) Find the local maximum and minimum values of f .

1st derivative test $\Rightarrow x = -1$ local max.

$x = 1$ local min

(c) How many real number solutions does $x^5 - 5x - 1 = 0$ have? Explain why your answer is correct.

Rough Graph of f



$$f(-1) = (-1)^5 - 5(-1) - 1 = 3$$

$$f(1) = -5$$

So there are exactly 3 solutions of $f(x) = 0$.

- (7) You watch a friend throw a pumpkin off the roof of the Wells Fargo Center, which is 40 floors tall. You observe how many floors the pumpkin has fallen at intervals of 1 second:

| | | | | | |
|---------------|---|---|---|----|----|
| time (sec) | 0 | 1 | 2 | 3 | 4 |
| floors fallen | 0 | 2 | 7 | 15 | 27 |

- (a) Use a linear approximation to estimate when the pumpkin hits the ground.

$f(t)$ = number of floors fallen at time t

$$f'(4) \approx \frac{f(3) - f(4)}{3 - 4} = \frac{15 - 27}{-1} = 12 \text{ floors/sec.}$$

$$f(t) \approx f(4) + f'(4)(t - 4) \quad \leftarrow \text{LINEAR APPROX.}$$

$$= 27 + 12(t - 4)$$

Solve $\underbrace{40}_{\text{\# of floors of the building}} = 27 + 12(t - 4)$

$$13 = 12(t - 4)$$

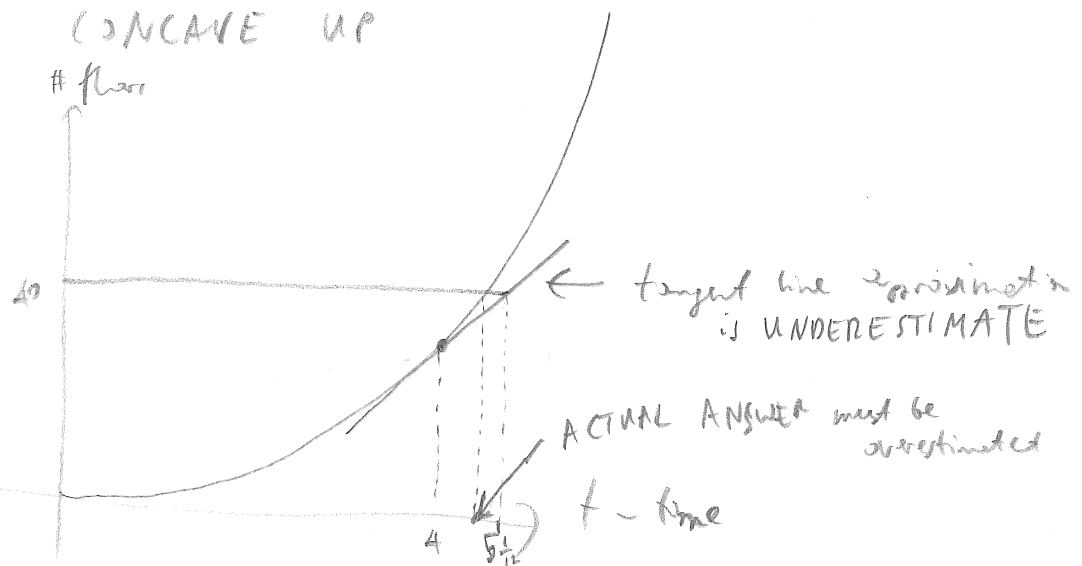
of floors of the building

$$\frac{13}{12} = t - 4 \Rightarrow t = 5\frac{1}{12} \text{ sec}$$

- (b) Is your estimate an underestimate or an overestimate? Why?

the answer in (a) is **OVERESTIMATE** since

$f(t)$ is **CONCAVE UP**



Also $f'(4)$ is also underestimated since $f'(t)$ is increasing.

(8) For a certain computer chip, the cost of manufacturing a single chip depends on the market's demand for the chip. If the cost is C \$ per chip, and the demand is D chips, we can write $C = f(D)$. Assume $f'(D) = -5D^2$.

(a) As the demand for chips increases, is the cost per chip increasing or decreasing? Why?

$$f'(D) = -5D^2 < 0$$

\Rightarrow cost is decreasing

(b) Based on your answer to part (a), as the demand for chips increases, is the cost per chip increasing/decreasing more rapidly or more slowly? Why?

$$f''(D) = -10D < 0$$

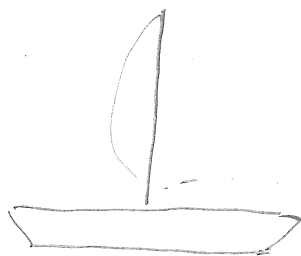
cost is decreasing more rapidly as D increases.

In other words, slope of f is decreasing

$\Rightarrow f$ is concave down

& decreasing.

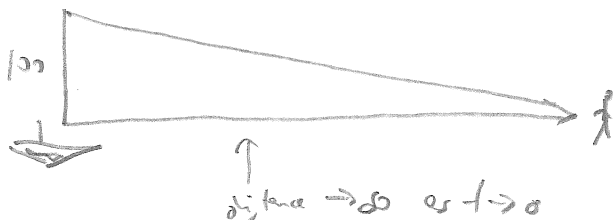
- (9) A woman is standing on the shore of a beach. There is a ship 100 ft out in the water from the woman, and the ship is sailing away from the woman. The mast of the ship is 100 ft tall. Let $\theta(t)$ denote the angle from the woman up to the top of the mast.
- (a) What is $\theta(0)$?



$$\tan \theta(0) = \frac{100}{100} = 1$$

$$\Rightarrow \theta(0) = \frac{\pi}{4} \text{ radians}$$

- (b) What is $\lim_{t \rightarrow \infty} \theta(t)$?



$$\tan \theta(t) = \frac{100}{\text{distance}} \rightarrow 0 \text{ as } t \rightarrow \infty$$

$$\Rightarrow \lim_{t \rightarrow \infty} \theta(t) = 0$$

- (c) Is $\theta(t)$ increasing or decreasing?

$\theta(t)$ is decreasing since distance is increasing

$$\& \tan \theta(t) = \frac{100}{\text{distance}}$$

distance = $d(t)$

- (d) Is $\theta(t)$ concave up or concave down?

Take derivative \Rightarrow

$$\theta(t) = \arctan\left(\frac{100}{d}\right)$$

$$\theta'(t) = \frac{1}{1 + \left(\frac{100}{d}\right)^2} \cdot \frac{100}{-d^2} \cdot d'(t) = \frac{-100}{d^2 + 100} \cdot d'(t)$$

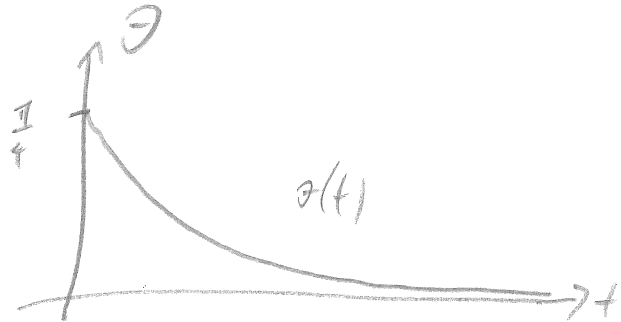
$\Rightarrow \theta(t)$ CONCAVE UP

(problem continues on the next page)

is increasing (assuming $d'(t)$ is constant,
that is ship moves away at
constant speed)

(e) Use this information to sketch a graph of $\theta(t)$ for $t \geq 0$.

$\theta(t)$ - decreasing & concave up



(10) Compute the following limits exactly. You must show your work.

(a) $\lim_{\theta \rightarrow \pi/2} \tan \theta - \sec \theta$

$$= \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin \theta - 1}{\cos \theta}$$

L'Hôpital rule
 $\frac{0}{0} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\cos \theta}{-\sin \theta} = \frac{0}{-1} = \underline{\underline{0}}$

(b) $\lim_{x \rightarrow 0^-} \frac{\sin(x^2)}{x^3}$ $\xrightarrow{\text{L'Hôpital } \frac{0}{0}}$ $\lim_{x \rightarrow 0^-} \frac{\cos(x^2) \cdot 2x}{3x^2}$

$$= \lim_{x \rightarrow 0^-} \frac{2 \cos(x^2)}{3x} = \frac{2}{0} = \underline{\underline{-\infty}}$$

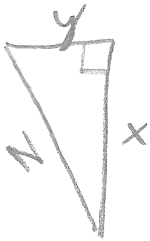
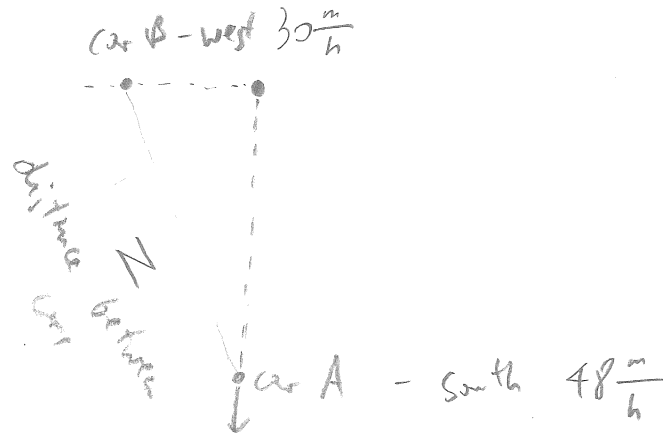
b/c $3x < 0$ for $x < 0$

$$(c) \lim_{t \rightarrow 1} \frac{3^t - 3}{t - 1} \stackrel{\text{L'Hôpital } \frac{0}{0}}{=} \lim_{t \rightarrow 1} \frac{3^t \cdot \ln 3}{1} = \underline{\underline{3 \cdot \ln 3}}$$

$$(d) \lim_{x \rightarrow \infty} x^2 e^{-x} = \lim_{x \rightarrow \infty} \frac{x^2}{e^x} \stackrel{\text{L'Hôpital } \frac{\infty}{\infty}}{=} \lim_{x \rightarrow \infty} \frac{2x}{e^x}$$

$$\stackrel{\text{" } \frac{\infty}{\infty} \text{ L'Hôpital}}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

- (11) Two cars start out from the same point. Car A starts traveling south at 48 mi/hr. One hour later, Car B starts traveling west at 30 mi/hr. At what rate is the distance between the cars increasing 3 hours after Car A sets out?



$$z^2 = x^2 + y^2$$

$$2z z' = 2x x' + 2y y'$$

time $t = 3$ hours

$$\Rightarrow x = 48 \frac{m}{h} \cdot 3h = 144 m$$

$$y = 30 \frac{m}{h} \cdot \underbrace{2h}_{\text{one hour later } 3-1=2} = 60 m$$

$$z = \sqrt{144^2 + 60^2} = \sqrt{144 \cdot 144 + 12 \cdot 5 \cdot 12 \cdot 5} = \\ = \sqrt{144} \sqrt{144 + 25} = 12 \cdot \sqrt{169} = 12 \cdot 13 = 156 m$$

$$\begin{array}{l} z \cdot z' = x \cdot x' + y \cdot y' \\ 156 \cdot ? = 144 \cdot 48 + 60 \cdot 30 \end{array}$$

$$z' = \frac{12 \cdot 144 \cdot 48 + 1800}{12 \cdot 13} = \frac{576 + 150}{13}$$

$$= \frac{726}{13} \frac{m}{h} \approx 55.8 \frac{m}{h}$$

