# GENERALIZED LAGRANGIAN MEAN CURVATURE FLOWS ON KIM-MCCANN METRICS.

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Abstract. We express the mean curvature flow of Lagrangian submanifolds of the Kim-McCann-Warren metric in the setting generalized mean curvature flow on Kim-McCann metrics. This allows us to tap into previously developed theory to show that the Lagrangian condition is preserved along the flow.

## 1. Introduction

Our goal in this paper is to demonstrate that the setting of generalized Lagrangian mean curvature flows applies in a surprisingly satisfying way to the setting of Kim-McCann metrics [KM10] introduced in the study of regularity theory of optimal transport. Kim-McCann metrics can viewed as a pseudo-Riemmanian analog of almost Calabi-Yau metrics. When adding a conformal factor depending on the measures, it was shown in [KMW10] that graphs of optimal transport maps are minimal surfaces. However as the ambient metric presented in [KMW10] is not Einstein, the mean curvature flow is not at first glance expected to behave well. We see here that the KMW metrics are almost Einstein manifolds and that much of the theory developed for studying Lagrangian mean curvature flow in almost Calabi-Yau metrics be carried over to the Kim-McCann setting.

In particular we prove that a Lagrangian manifold evolving under generalized mean curvature flow in a Kim-McCann metric will stay Lagrangian. The stationary points of generalized mean curvature flow will be stationary for the mean curvature with the metric in [KMW10]

The Kim-McCann metric is still of much interest: See [LV23] for recent geometric exposition of Kim-McCann metric, and also recent results.

Generalized MCF was introdced in [Beh11] and [SW11] (the approaches are slightly different, one focus on a torsion connection, another on a Ricci potential) We will take suggestions from the generalized mean curvature flow in paracomplex manifolds. Some analogies will be used with justifcation, however the proofs of our main result do not require results from this setting, rather they follow from similar arguments, so we will present the arguments directly, without justifying the full analogy. Our angle of approach differs from [CSS11] (which follows [SW11]) where generalized mean cuvature for paracomplex was given, we In [CSS11] the geometry was understood through comparing different connections, as in [SW11]. We follow an approach closer to that of [Beh11] We instead focus an the analogy suggested by the calibrating forms in [KMW10] The goal here is not to provide a complete dictionary of the paracomplex theory, rather simply to show that some results in the optimal transport theory have some nice proof.

## 2. Main statements and outline

The following is the main result. For Kähler-Einstein manifolds, this result was first mapped out in [Smo96], see also [Beh11] and [Woo20].

**Theorem 2.1.** *Suppose that $L$ is a Lagrangian submanifold of a KM manifold, with weighted $\rho, \bar{\rho}$. Then flowing $L$ according the GMCF with the KMW metric preserves the Lagrangian condition.*

This follows by a standard maximum principle argument using the following observation.

**Proposition 2.2.** *Let $F$ be a generalized mean curvature flow for some short time interval $[0, \delta]$. There are geometric quantities $G$ (depending on the particular solution to the flow) such that if we restrict the Kahler form to to a totally real submanifold evolving along the flow we have*

$$\frac{d}{dt}|\omega|^2 = \Delta|\omega|^2 - n\nabla\psi \cdot \nabla|\omega|^2 - |\nabla\omega|^2 + \omega * \omega * G$$

This is proved in several parts.

**Proposition 2.3.** *Suppose that $L$ is a totally real submanifold flowing by GMCF*

$$\left(\frac{dF}{dt}\right)^{\perp} = \vec{N} = H - n(\nabla\psi)^{\perp}$$

*Then (see (4.1)) for definition of $\alpha_{\vec{N}}$)*

(1)
$$\frac{d}{dt}\omega = d\alpha_{\vec{N}}$$

(2)
$$\alpha_N = \alpha_H - n\beta + n\gamma$$

*for forms satisfying*

(3)
$$\begin{aligned} d\alpha_H(e_k, e_l) &= \bar{R}ic(Ke_k, e_l) + \omega * G + g^{ij}\nabla_i\nabla_j\omega_{kl} \\ nd\beta(e_k, e_l) &= \bar{R}ic(Ke_k, e_l) \\ nd\gamma(e_k, e_l) &= \nabla\psi \cdot \nabla\omega_{kl} + G * \omega \end{aligned}$$

We note here that the parabolic flow discussed in [KSW12] provides a vertical flow of Lagrangian submanifolds. One can check that the projection of the flow is onto the normal direction is mean curvature flow in the KMW setting.

## 3. Setup

Recall [KM10]. Given a cost function on $M^n \times \bar{M}^n$ we are considering a metric of the form

$$h := \frac{1}{2}\begin{pmatrix} 0 & -c_{i\bar{s}} \\ -c_{\bar{s}i} & 0 \end{pmatrix}.$$

When the mixed derivative $d\bar{d}c$ is non-singular, this provides a signature $(n, n)$ metric on the product space $M^n \times \bar{M}^n$. The form

$$\omega = -c_{i\bar{s}} dx^i \wedge d\bar{x}^s$$

also provides a symplectic form. If we would like to view the symplectic form as a para-Kähler form, consider the map

$$K : T_{(p,\bar{p})}M \times \bar{M} \to T_{(p,\bar{p})}M \times \bar{M}$$

which is represented as

$$K_{p,\bar{p}} = I_{T_pM} \oplus (-I)_{T_{\bar{p}}\bar{M}}\,.$$

(For a basic introduction to para-Kähler gometry see [CFG96]. The important fact is that complex structure map $J$ satisfying $J^2 = -I$ is replaced by a paracomplex structure map satisfying $K^2 = I$). Then we have

$$\omega(\cdot, \cdot) = h(K\cdot, \cdot).$$

It was noted in [KM10] that graphs $(x, T(x))$ of optimal transport maps are Lagrangian with respect to this symplectic form.

Now given some probability measures $\rho$ and $\bar{\rho}$ that can be locally represented as $\rho(x)dx$ and $\bar{\rho}(\bar{x})d\bar{x}$ (we are using double-dip notation, using $\rho$ and $\bar{\rho}$ to define both the measure and the density function representing the measure). It was show in [KMW10] that the graph of the optimal transport map pairing $\rho$ to $\bar{\rho}$ with cost $c$, is a volume maximizing submanifold of $M \times \bar{M}$ with the following metric:

$$\tilde{h} := \frac{1}{2}\left(\frac{\rho(x)\bar{\rho}(\bar{x})}{\det(-c_{i\bar{s}})}\right)^{1/n}\begin{pmatrix} 0 & -c_{ij} \\ -c_{ij} & 0 \end{pmatrix}. \tag{3.1}$$

These are calibrated by the form $n$-form.

$$\Phi = \frac{1}{2}\left(\rho(x)dx + \bar{\rho}(\bar{x})d\bar{x}\right)$$

In particular, if the graph of the map is Lagrangian and the map is measure preserving, the graph will be a volume maximizing surface. Here we realize $\Phi$ as the real part of a para-holomorphic form.

The conformal factor is somewhat inconvenient, see [KSW12, Section 5, 7.1.2], [War11]. If we would like to view the symplectic form as a Kähler form, the conformal factor either destroys the Kähler compatibility $\omega(X, Y) = h(KX, Y)$, or destroys the closedness $d\omega = 0$ (One can also view this as a weighted manifold, as noted[War14, section 3] and applied in [AK20, pg 2092]. In the para-Kähler setting [CSS11], one considers the different connections that arise. For proving the Lagrangian condition is preserved, we feel it is easier to look directly at the conformal factor and follow the approach in [Beh11].

3.1. **Almost Calabi-Yai metrics.** Recall [Joy07, Section 8.4].

**Definition 3.1.** Let $n \geq 2$. An almost Calabi–Yau $n$-fold, or ACY $n$-fold for short, is a quadruple $(X, J, g, \Omega)$ such that $(X, J, g)$ is a compact $n$ (complex)-dimensional Kähler manifold, and $\Omega$ is a non-vanishing holomorphic $(n, 0)$-form on $X$.

In [Joy07, 8.24] there will be a function $\psi$ such that

$$e^{2n\psi}\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}\left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}. \tag{3.2}$$

We would like to explore the analogy in the paracomplex setting. A para-complex valued form $\eta$ is $(n, 0)$ type if

$$\mathbf{k} \cdot \eta(V_1, .., V_n) = \eta(KV_1, .., V_n) = \eta(V_1, KV_2, .., V_n)$$

etc. Here $\mathbf{k}$ is an algebraic object aking to the more familiar $\mathbf{i}$, but instead satifies $\mathbf{k}^2 = 1$. One can check that for (real valued $u, v$)

$$\Omega = u\,(x)\,dx + v(\bar{x})d\bar{x} + \mathbf{k}\,(u\,(x)\,dx - v(\bar{x})d\bar{x})$$

this is para-holomorphic when $u$ depends only on $x$ and $v$ only on $\bar{x}$, so this satisfies $\bar{\partial}\Omega = 0$. Now we can consider

$$\begin{aligned} \Omega \wedge \bar{\Omega} &= \begin{pmatrix} udx + vd\bar{x} \\ +\mathbf{k}\,(udx - vd\bar{x}) \end{pmatrix} \wedge \begin{pmatrix} udx + vd\bar{x} \\ -\mathbf{k}\,(udx - vd\bar{x}) \end{pmatrix} \\ &= 0 - \mathbf{k}\,(udx + vd\bar{x}) \wedge (udx - vd\bar{x}) + \mathbf{k}\,(udx - vd\bar{x}) \wedge (udx + vd\bar{x}) \\ &= 0 - \mathbf{k}udx\,(-vd\bar{x}) - \mathbf{k}vd\bar{x}udx + \mathbf{k}udxvd\bar{x} - \mathbf{k}vd\bar{x}udx \\ &= 4uv\mathbf{k}dxd\bar{x} \end{aligned}$$

in particular if

$$\Omega = \frac{1}{2}\,(\rho(x)dx + \bar{\rho}(\bar{x})d\bar{x}) + \frac{1}{2}\mathbf{k}\,(\rho(x)dx - \bar{\rho}(\bar{x})d\bar{x})$$

we will have

$$\Omega \wedge \bar{\Omega} = \rho(x)\bar{\rho}(\bar{x})\mathbf{k}.$$

With this set up, consider a function $\psi$ which satisfies

$$e^{2n\psi}\frac{\omega^n}{n!} = (-1)^{n-1}\left(\frac{1}{2}\right)^n \mathbf{k}\Omega \wedge \bar{\Omega}.$$

now as

$$\omega = \frac{1}{2}\sum_{i,\bar{s}}(-c_{i\bar{s}})\,dx^i \wedge d\bar{x}^s$$

we can check that

$$\omega^n = \left(\frac{1}{2}\right)^n n!(-1)^{n-1}\det(-c_{i\bar{s}})\,dxd\bar{x}$$

and to satisfy our version of the equation (3.2), we want

$$e^{2n\psi} = \frac{\rho\bar{\rho}}{\det(-c_{i\bar{s}})}$$

or

$$\psi = \frac{1}{2n}\,(\ln\rho + \ln\bar{\rho} - \ln\det(-c_{i\bar{s}}))$$

The following suggests some strength in the analogy:

**Proposition 3.2.** *The Ricci form satisfies*

$$\bar{R}ic(KX, Y) = 2nd\bar{d}\psi(X, Y)$$

The proof is in the Appendix.

**Proposition 3.3.** *The mean curvature vector for a metric*

$$\tilde{h} = e^{2\psi} h$$

*(which is the metric given in* [KMW10]*) is given by*

$$\tilde{H} = e^{-2\psi} \left( H - n (\nabla\psi)^{\perp} \right) \tag{3.3}$$

We will not prove this here, this is a standard exercise in conformal geometry. Now recall the formula from Berndt [Beh11] in the almost Calabi-Yau setting:

Given a function $\psi$ such that

$$\bar{R}ic(JX, Y) = \lambda\omega + nd\bar{d}\psi$$

define the generalized mean curvature vector to be

$$\hat{H} = H - n (\nabla\psi)^{\perp} . \tag{3.4a}$$

Comparing (3.4a) and (3.3) suggest the obvious analogy.

The generalized mean curvature flow is the normal flow

$$\left( \frac{\partial F}{\partial t} \right)^{\perp} = H - n (\nabla\psi)^{\perp}$$

## 4. Proof of Proposition 2.3

We need some setup and a few more Lemmas.

A total real submanifold $L$ in para-Kähler setting is a submanifold whose tangent space does not contain any para-complex planes, that is for every $V \in T_pL$ we have $KV \notin T_pL$.

Given a totally real submanifold $L$, we define

$$\tilde{K} : T_pL \to \left( T_pL \right)^{\perp}$$
$$\tilde{K}(X) = (K(X))^{\perp}$$

Taking $\{e_i\}$ to be a basis for the tangent space to $L^n$, one can check that defining

$$\omega_i^l = \omega_{ij} g^{jl}$$

for

$$\omega_{ij} = \omega(e_i, e_j)$$

and

$$g_{ij} = h(e_i, e_j)$$

the indcued metric, then

$$\tilde{K}(e_i) = K(e_i) - \omega_i^s e_s.$$

Notice that

$$\begin{aligned} K\tilde{K} &= K^2(e_i) - \omega_i^l K e_l \\ &= e_i - \omega_i^l K e_l. \end{aligned}$$

We assume a Kähler condition

$$\omega = h(K\cdot, \cdot)$$

Note that in a para-Kähler manifold, we have

$$h(KX, Y) = -h(K^2 X, KY) = -h(X, KY).$$

We will also use the (negative definite) metric on $\left\{\tilde{K}e_i\right\}$ which forms a basis for the normal space

$$\eta_{ij} = h(\tilde{K}e_i, \tilde{K}e_j).$$

Define components of the second fundamental form via

$$a(X, Y, Z) = h(\bar{\nabla}_X Y, \tilde{K}Z)$$

and then define an associated 1-form to any normal vector via

$$\alpha_{\vec{N}} = \omega(\vec{N}, \cdot). \tag{4.1}$$

For the mean curvature vector, this gives

$$\begin{aligned} \alpha_H &= \omega(H, \cdot) = h(KH, \cdot) = -h(H, K\cdot) \\ &= -h(g^{ij}\bar{\nabla}_{e_i} e_j, \tilde{K}\cdot) \end{aligned}$$

so

$$\alpha_H = -g^{ij} a_{ijk} dx^k. \tag{4.2}$$

### 4.1. **More Claims.**

**Claim 4.1.**

$$\nabla_l a_{jki} - \nabla_k a_{jli} = R(e_l, e_k, e_j, \tilde{K}(e_i)) + \omega * A * A$$

**Claim 4.2.**

$$\nabla_l a_{ijk} - \nabla_k a_{ijl} = \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) + \omega * G + \nabla_j \nabla_i \omega_{kl}$$

**Claim 4.3.**

$$g^{ij}\bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) = \bar{R}ic(Ke_k, e_l) + G * \omega$$

4.2. **Proof of Proposition 2.3 assuming claims.** The first part is standard: Since $\omega$ is parallel, we have

$$\begin{aligned}\frac{d}{dt}\omega\left(\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\right) &= \omega\left(\frac{\partial}{\partial t}\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\right)+\omega\left(\frac{\partial F}{\partial x^i},\frac{\partial}{\partial t}\frac{\partial F}{\partial x^j}\right)\\ &= \omega\left(\frac{\partial}{\partial x^i}\frac{\partial F}{\partial t},\frac{\partial F}{\partial x^j}\right)+\omega\left(\frac{\partial F}{\partial x^i},\frac{\partial}{\partial x^j}\frac{\partial F}{\partial t}\right)\\ &= \omega\left(\frac{\partial}{\partial x^i}\vec{N},\frac{\partial F}{\partial x^j}\right)+\omega\left(\frac{\partial F}{\partial x^i},\frac{\partial}{\partial x^j}\vec{N}\right)\\ &= \omega\left(\frac{\partial}{\partial x^i}\vec{N},\frac{\partial F}{\partial x^j}\right)+\omega\left(\frac{\partial F}{\partial x^i},\frac{\partial}{\partial x^j}\vec{N}\right)\\ &= \frac{\partial}{\partial x^i}\omega\left(\vec{N},\frac{\partial F}{\partial x^j}\right)-\omega\left(\frac{\partial}{\partial x^i}\vec{N},\frac{\partial}{\partial x^i}\frac{\partial F}{\partial x^j}\right)\\ &\quad+\frac{\partial}{\partial x^j}\omega\left(\frac{\partial F}{\partial x^i},\vec{N}\right)-\omega\left(\frac{\partial}{\partial x^j}\frac{\partial F}{\partial x^i},\vec{N}\right)\end{aligned}$$

noting in the last line that

$$\frac{\partial}{\partial x^i}\omega\left(\vec{N},\frac{\partial F}{\partial x^j}\right)=\omega\left(\frac{\partial}{\partial x^i}\vec{N},\frac{\partial F}{\partial x^j}\right)+\omega\left(\vec{N},\frac{\partial}{\partial x^i}\frac{\partial F}{\partial x^j}\right)$$

whenever the form is parallel. Now notice that

$$\begin{aligned}\omega\left(\vec{N},\frac{\partial F}{\partial x^j}\right) &= h(K\vec{N},\frac{\partial F}{\partial x^j})\\ &= -h(\vec{N},K\frac{\partial F}{\partial x^j})=-h(\vec{N},\tilde{K}\frac{\partial F}{\partial x^j})=N(\frac{\partial F}{\partial x^j})\end{aligned}$$

so t

$$\begin{aligned}\frac{d}{dt}\omega\left(\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\right) &= \frac{\partial}{\partial x^i}\alpha_N(\frac{\partial F}{\partial x^j})-\omega\left(\frac{\partial}{\partial x^i}\vec{N},\frac{\partial}{\partial x^i}\frac{\partial F}{\partial x^j}\right)\\ &\quad-\frac{\partial}{\partial x^j}\alpha_N\left(\frac{\partial F}{\partial x^i}\right)-\omega\left(\frac{\partial}{\partial x^j}\frac{\partial F}{\partial x^i},\vec{N}\right)\end{aligned}$$

Now $\omega$ is alternating so the mixed terms are canceling. Recall that

$$d\alpha(X,Y)=X\alpha(Y)-Y\alpha(X)-\alpha([X,Y])$$

so we have

$$d\alpha_N\left(\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\right)=\frac{d}{dt}\omega\left(\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\right).$$

Next, we write the generalized mean curvature vector as follows

$$\tilde{H}=H-n(\nabla\psi)^{\perp}=H-n\nabla\psi+n(\nabla\psi)^T$$

and associate the forms

$$\begin{aligned}\beta(\cdot) &= \omega(\nabla\psi,\cdot)\\ \gamma &= \omega((\nabla\psi)^T,\cdot)\end{aligned}$$

Then the decomposition in (2) follows.

Now using (4.2) we have

$$d\alpha_H(e_k, e_l) = e_k\left(-g^{ij}a_{ijl}\right) - e_l\left(-g^{ij}a_{ijk}\right)$$

evaluating in normal coordinates, this becomes

$$\begin{aligned} d\alpha_H(e_k, e_l) &= g^{ij}a_{ijk,l} - g^{ij}a_{ijl,k} \\ &= \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) + \omega * A * A + \nabla_j\nabla_i\omega_{lk} + R * \omega \end{aligned}$$

by Claim 4.2. Then by Claim 4.3

$$d\alpha_H(e_k, e_l) = \bar{R}ic(Ke_k, e_l) + G * \omega + \nabla_j\nabla_i\omega_{lk}$$

proving the first part of (3).

The second part is Proposition 3.2 together with Claim 6.2.

The third part is on the immersed manifold: compute $\gamma$

$$\begin{aligned} \gamma(e_m) &= \omega\left((\nabla\psi)^T, e_m\right) \\ &= h\left(K(\nabla\psi)^T, e_m\right) \\ &= -h\left((\nabla\psi)^T, Ke_m\right) \\ &= -h\left((\nabla\psi)^T, \tilde{K}e_m + (K - \tilde{K})e_m\right) \\ &= -h\left((\nabla\psi)^T, (K - \tilde{K})e_m\right) \\ &= h((\nabla\psi)^T, \omega_m^l e_l) = h(\nabla\psi, \omega_m^p e_p) \\ &= \omega_m^p\psi_p \end{aligned}$$

So

$$\begin{aligned} d\gamma(e_k, e_l) &= e_k\left(\omega_l^p\psi_p\right) - e_l\left(\omega_k^p\psi_p\right) \\ &= \omega * D^2\psi + \left(\omega_{l,k}^p - \omega_{k,l}^p\right)\psi_p \\ &= \omega * D^2\psi + (\omega_{lm,k}g^{mp} - \omega_{km,l}g^{mp})\psi_p \end{aligned}$$

Now use closedness (in normal coordinates can ignore derivative from upper indices)

$$\omega_{lm,k} + \omega_{mk,l} + \omega_{kl,m} = 0$$

so we have

$$(\omega_{lm,k}g^{mp} - \omega_{km,l}g^{mp})\psi_p = -\omega_{kl,m}g^{mp}\psi_p.$$

## 5. Putting it together

IT can be shown that

$$\frac{d}{dt}g_{ij} = -2h(\vec{N}, A_{ij})$$

(cf. anywhere evolution equations are proved)

So

$$\begin{aligned} \frac{d}{dt}|\omega|^2 &= -2h(\vec{N}, A_{ij}) * \omega * \omega + 2\omega^{kl}\frac{d}{dt}\omega_{kl} \\ &= \omega * \omega + 2\omega^{kl}(d\alpha_H - nd\beta + nd\gamma)(e_k, e_l). \end{aligned}$$

Then from Proposition 2.3 we get

$$\frac{d}{dt}|\omega|^2 = G * \omega * \omega + \Delta|\omega|^2 - |\nabla\omega|^2 - n\nabla|\omega|^2 \cdot \nabla\psi.$$

Now we apply the standard maximum principle argument: Let

$$f_\varepsilon = |\omega|^2 - \varepsilon e^{2Ct}$$

and compute

$$\begin{aligned}\frac{d}{dt}f_\varepsilon &\le C|\omega|^2 + \Delta|\omega|^2 - n\nabla|\omega|^2 \cdot \nabla\psi - 2C\varepsilon e^{2ct} \\ &= C|\omega|^2 - 2C\varepsilon e^{2ct} + \Delta f_\varepsilon - n\nabla\psi \cdot \nabla f_\varepsilon \\ &= Cf_\varepsilon - C\varepsilon e^{2ct} + \Delta f_\varepsilon - n\nabla\psi \cdot \nabla f_\varepsilon < Cf_\varepsilon + \Delta f_\varepsilon - n\nabla\psi \cdot \nabla f_\varepsilon.\end{aligned}$$

So the function $f_\varepsilon$ which is negative at $t = 0$ if $L$ is Lagrangian, satisfies a parabolic maximum principle:The function $f_\varepsilon$ must be negative for all $t$ in the domain. Taking $\varepsilon \to 0$ proves that $|\omega|^2$ must always be 0 along the flow.

## 6. Appendix

We will use use $E_i$ is a basis for $T_pM$ and $E_{\bar{s}}$ and a basis for $T_{\bar{p}}\bar{M}$. Together these form a basis for the product $T_{(p,\bar{p})}M \times \bar{M}$. Note that if we fix a coordinate system for $M$ near a point $p$ and given a choice of chart near $\bar{p}$ we may choose a local change of coordinates for $\bar{M}$ such that

$$-c_{is} = \delta_{is}.$$

Note that we may do this wehile choosing a normal set of coordinates for the induces metric on $M$ at the point $p$.

**Lemma 6.1.** *(This is* [KM10, Lemma 4.1] *In a KM metric we have*

(1)

$$\Gamma^k_{bc} = c^{k\bar{s}} c_{\bar{s}b,c}$$

*and all mixed Christofel symbols vanish.*

(2) *For curvature, if the indices $\alpha, \beta, \gamma, \delta$ are $E_i$ or $E_{\bar{s}}$*

$$R_{\alpha\beta\gamma\delta} = 0$$

*unless two of $\alpha\beta\gamma\delta$ are the same type and two are of the opposite type.*

(3)

$$R_{\bar{a}bc\bar{d}} = \frac{1}{2}c^{ms}c_{\bar{d}ma}c_{\bar{s}b,c} - \frac{1}{2}c_{\bar{d}b,c\bar{a}}$$

*Proof.* Recall

$$h = \frac{1}{2}(-c_{i\bar{s}}dx^i \otimes d\bar{x}^s)$$

$$\begin{aligned} R_{abcd} &= \left(\bar{\nabla}_a\bar{\nabla}_b c - \bar{\nabla}_b\bar{\nabla}_a c, d\right) \\ &= \left(\bar{\nabla}_a \Gamma^k_{bc} e_k - \bar{\nabla}_b \Gamma^k_{ac} e_k, d\right) \\ &= \left(\partial_a \Gamma^k_{bc} e_k + \Gamma^k_{bc}\Gamma^m_{ak} e_m - \partial_b \Gamma^k_{ac} e_k - \Gamma^k_{ac}\Gamma^m_{bk} e_m, d\right) \\ &= \left(\partial_a \Gamma^k_{bc} - \partial_b \Gamma^k_{ac}\right) g_{kd} + \left(\Gamma^k_{bc}\Gamma^m_{ak} - \Gamma^k_{ac}\Gamma^m_{bk}\right) g_{md} \end{aligned}$$

In this case

$$\Gamma^k_{bc} = \frac{1}{2} h^{km} \left(h_{mb,c} + h_{mc,b} - h_{bc,m}\right)$$

Now

$$h^{k\bar{s}} = 2\left(-c^{k\bar{s}}\right)$$

so

$$\begin{aligned} \Gamma^k_{bc} &= \left(-c^{k\bar{s}}\right)\left(h_{\bar{s}b,c} + h_{\bar{s}c,b} - h_{bc,\bar{s}}\right) \\ &= -\frac{1}{2}\left(-c^{k\bar{s}}\right)\left(c_{\bar{s}b,c} + c_{\bar{s}c,b} - c_{bc,\bar{s}}\right) \end{aligned}$$

so if all $b, c, k$the same type

$$\begin{aligned} \Gamma^k_{bc} &= -\frac{1}{2}\left(-c^{k\bar{s}}\right)\left(2c_{\bar{s}b,c}\right) \\ &= c^{k\bar{s}} c_{\bar{s}b,c} \end{aligned}$$

$$\begin{aligned} R_{\bar{a}bc\bar{d}} &= \left(\partial_{\bar{a}} \Gamma^k_{bc} - \partial_b \Gamma^k_{\bar{a}c}\right) g_{k\bar{d}} + \left(\Gamma^k_{bc}\Gamma^m_{\bar{a}k} - \Gamma^k_{\bar{a}c}\Gamma^m_{bk}\right) g_{m\bar{d}} \\ &= \partial_{\bar{a}} \Gamma^k_{bc} g_{k\bar{d}} = \partial_{\bar{a}}\left(c^{k\bar{s}} c_{\bar{s}b,c}\right) \frac{1}{2}\left(-c_{k\bar{d}}\right) \\ &= \left(-c^{k\bar{p}} c^{m\bar{s}} c_{\bar{p}ma} c_{\bar{s}b,c} + c^{k\bar{s}} c_{\bar{s}b,c\bar{a}}\right) \frac{1}{2}\left(-c_{k\bar{d}}\right) \\ &= \frac{1}{2} c^{ms} c_{\bar{d}ma} c_{\bar{s}b,c} - \frac{1}{2} c_{\bar{d}b,c\bar{a}} \\ &= R_{b\bar{a}\bar{d}c} \end{aligned}$$

□

6.1. **Proof of Proposition 3.2.** First observe that

**Claim 6.2.**

$$d\beta = 2d\bar{d}\psi.$$

*Proof.* For $\beta$ we look at the ambient manifold

$$\begin{aligned} \beta(E_k) &= \omega\left(\nabla\psi, E_k\right) = h\left(K\nabla\psi, E_k\right) \\ &= -h\left(\nabla\psi, KE_k\right) = -h\left(\nabla\psi, E_k\right) \\ &= -\psi_k \end{aligned}$$

meanwhile

$$\begin{aligned}\beta(E_{\bar{p}}) &= -h\left(\nabla\psi, KE_{\bar{p}}\right)\\ &= h\left(\nabla\psi, E_{\bar{p}}\right)\\ &= \psi_{\bar{p}}\end{aligned}$$

so

$$\beta = -\psi_k dx^k + \psi_{\bar{p}} dx^{\bar{p}}$$

and then

$$\begin{aligned}d\beta &= -\psi_{k\bar{p}} dx^{\bar{p}} dx^k + \psi_{\bar{p}k} dx^k dx^{\bar{p}}\\ &= 2\psi_{\bar{p}k} dx^k dx^{\bar{p}}\end{aligned}$$

This proves the claim. □

Next we prove Proposition 3.2

*Proof.* Obviously,

$$\begin{aligned}d\beta(E_i, E_j) &= 0\\ d\beta(E_{\bar{s}}, E_{\bar{p}}) &= 0\\ d\beta(E_k, E_{\bar{p}}) &= 2\psi_{\bar{p}k}\end{aligned}$$

We compute an expression for this directly: recall

$$\psi = \frac{1}{2n}\left(\ln\rho + \ln\bar{\rho} - \ln\det\left(-c_{i\bar{s}}\right)\right).$$

First note that because $\rho$ depends only and $x$ and $\bar{\rho}$ only on $\bar{x}$, the first two terms to do not survive differentiation in both variables. Thus

$$\begin{aligned}\psi_{\bar{p}k} &= -\frac{1}{2n}\left[\ln\det\left(-c_{i\bar{s}}\right)\right]_{\bar{p},k}\\ &= -\frac{1}{2n}\left[\left(-c^{i\bar{s}}\right)\left(-c_{i\bar{s}\bar{p}}\right)\right]_k\\ &= -\frac{1}{2n}\left[\left(-c^{i\bar{s}}\right)\left(-c_{i\bar{s}\bar{p}k}\right) - \left(-c^{i\bar{s}}\right)_k\left(-c_{i\bar{s}\bar{p}}\right)\right]\\ &= -\frac{1}{2n}\left[c^{i\bar{s}}c_{i\bar{s}\bar{p}k} - \left(-c^{i\bar{a}}\right)(-c^{b\bar{s}})(-c_{\bar{a}bk})\left(-c_{i\bar{s}\bar{p}}\right)\right]\\ &= -\frac{1}{2n}\left[c^{i\bar{s}}c_{i\bar{s}\bar{p}k} - c^{i\bar{a}}c^{b\bar{s}}c_{\bar{a}bk}c_{i\bar{s}\bar{p}}\right]\end{aligned}$$

so

$$d\beta = -\frac{1}{n}\left[c^{i\bar{s}}c_{i\bar{s}\bar{p}k} - c^{i\bar{a}}c^{b\bar{s}}c_{\bar{a}bk}c_{i\bar{s}\bar{p}}\right]dx^k dx^{\bar{p}}$$

$$\begin{aligned}d\beta(E_k, E_{\bar{p}}) &= -\frac{1}{n}\left[c^{i\bar{s}}c_{i\bar{s}\bar{p}k} - c^{i\bar{a}}c^{b\bar{s}}c_{\bar{a}bk}c_{i\bar{s}\bar{p}}\right]\\ &= \sum_i \frac{1}{n}\left[c_{i\bar{i}\bar{p}k} + c_{\bar{i}sk}c_{i\bar{s}\bar{p}}\right]\end{aligned} \tag{6.1}$$

Here and in the following assume that at a point we have

$$-c_{i\bar{s}} = \delta_{is} \tag{6.2}$$

Next we compute $\bar{R}ic(KE_k, E_{\bar{p}})$. Using (6.2)

$$E_i + E_{\bar{\imath}}$$
$$E_i - E_{\bar{\imath}}$$

form an orthonormal basis, and we can compute the Ricci

$$\begin{aligned}
&Ric(KE_k, E_{\bar{p}}) \\
&= \sum_i \bar{R}(E_i + E_{\bar{\imath}}, KE_k, E_{\bar{p}}, E_i + E_{\bar{\imath}}) - \sum_i \bar{R}(E_i - E_{\bar{\imath}}, KE_k, E_{\bar{p}}, E_i - E_{\bar{\imath}}) \\
&= \sum_i \bar{R}(E_i + E_{\bar{\imath}}, E_k, E_{\bar{p}}, E_i + E_{\bar{\imath}}) - \sum_i \bar{R}(E_i - E_{\bar{\imath}}, E_k, E_{\bar{p}}, E_i - E_{\bar{\imath}})
\end{aligned}$$

using $KE_k = E_k$ then using KM relations (Lemma 6.1)

$$\begin{aligned}
&= \sum_i \bar{R}(E_{\bar{\imath}}, E_k, E_{\bar{p}}, E_i) - \sum_i \bar{R}(-E_{\bar{\imath}}, E_k, E_{\bar{p}}, E_i) \\
&= 2\sum_i \bar{R}(E_{\bar{\imath}}, E_k, E_{\bar{p}}, E_i) \\
&= -2\sum_i \bar{R}(E_k, E_{\bar{\imath}}, E_{\bar{p}}, E_i)
\end{aligned}$$

o plug this in

$$\begin{aligned}
Ric(KE_k, E_{\bar{p}}) &= -2\sum_i \bar{R}(E_k, E_{\bar{\imath}}, E_{\bar{p}}, E_i) \\
&= -2\sum_i R_{k\bar{\imath}\bar{p}i} = -2\sum_i \left(\frac{1}{2}c^{ms}c_{\bar{p}m\bar{\imath}}c_{\bar{s}k,i} - \frac{1}{2}c_{\bar{p}k,i\bar{\imath}}\right) \\
&= \sum_i \left(c_{\bar{p}k,i\bar{\imath}} - c^{ms}c_{\bar{p}m\bar{\imath}}c_{\bar{s}k,i}\right) \\
&= \sum_i \left(c_{\bar{p}k,i\bar{\imath}} + c_{\bar{p}s\bar{\imath}}c_{\bar{s}ki}\right)
\end{aligned}$$

Recall (6.1) we confirm that

$$nd\beta(E_k, E_{\bar{p}}) = Ric(KE_k, E_{\bar{p}})$$

□

6.2. **Proof of Claims.**

**Claim 6.3.**

$$a_{ijk} = a_{jki} + \nabla_j \omega_{ki}$$

*Proof.* Symmetric in first two, so

$$\begin{aligned} a_{ijk} &= h(\bar{\nabla}_{e_j} e_i, \tilde{K} e_k) = h(\bar{\nabla}_{e_j} e_i, K e_k) - h(\bar{\nabla}_{e_j} e_i, \omega_k^l e_l) \\ &= e_j h(e_i, K e_k) - h(e_i, K \bar{\nabla}_{e_j} e_k) - \omega_k^l h(\bar{\nabla}_{e_j} e_i, e_l) \\ &= \partial_j \omega_{ki} + h(K e_i, \bar{\nabla}_{e_j} e_k) - \omega_k^l h(\bar{\nabla}_{e_j} e_i, e_l) \\ &= \partial_j \omega_{ki} + h(K e_i - \omega_i^l e_l, \bar{\nabla}_{e_j} e_k) + h(\omega_i^l e_l, \bar{\nabla}_{e_j} e_k) - \omega_k^l h(\bar{\nabla}_{e_j} e_i, e_l) \\ &= \partial_j \omega_{ki} + h(\tilde{K} e_i, \bar{\nabla}_{e_j} e_k) + \omega_i^l h(e_l, \bar{\nabla}_{e_j} e_k) - \omega_k^l h(\bar{\nabla}_{e_j} e_i, e_l) \\ &= a_{jki} + \partial_j \omega_{ki} + \omega_i^l h(e_l, \bar{\nabla}_{e_j} e_k) - \omega_k^l h(\bar{\nabla}_{e_j} e_i, e_l) \end{aligned}$$

Now we may assume at point at the Christoffel symbols vanish on the induced manifold. Then the ambient connection on pairs of tangent vectors lies in the normal direction, and we also have $\partial_j \omega_{ki} = \nabla_j \omega_{ki}$. Conclusion follows □

Proof of Claim 4.1:

$$\nabla_l a_{jki} - \nabla_k a_{jli} = \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) + \omega * A * A$$

*Proof.* In normal coordinates, just need to compute

$$\partial_l a_{jki} - \partial_k a_{jli}$$

now

$$\begin{aligned} \partial_l a_{jki} = \partial_l a_{kji} &= \partial_l h(\bar{\nabla}_{e_k} e_j, \tilde{K}(e_i)) \\ &= h(\bar{\nabla}_l \bar{\nabla}_{e_k} e_j, \tilde{K}(e_i)) + h(\bar{\nabla}_{e_k} e_j, \bar{\nabla}_l \tilde{K}(e_i)) \end{aligned}$$

so (using $a_{jli} = a_{lji}$

$$\begin{aligned} \partial_l a_{jki} - \partial_k a_{jli} &= \partial_l a_{kji} - \partial_k a_{lji} \\ &= h(\bar{\nabla}_{e_l} \bar{\nabla}_{e_k} e_j, \tilde{K}(e_i)) + h(\bar{\nabla}_{e_k} e_j, \bar{\nabla}_l \tilde{K}(e_i)) \\ &\quad - h(\bar{\nabla}_{e_k} \bar{\nabla}_{e_l} e_j, \tilde{K}(e_i)) - h(\bar{\nabla}_{e_l} e_j, \bar{\nabla}_k \tilde{K}(e_i)) \\ &= h(\bar{\nabla}_{e_l} \bar{\nabla}_{e_k} e_j - \bar{\nabla}_{e_k} \bar{\nabla}_{e_l} e_j, \tilde{K}(e_i)) \\ &\quad + h(\bar{\nabla}_{e_k} e_j, \bar{\nabla}_l \tilde{K}(e_i)) - h(\bar{\nabla}_{e_l} e_j, \bar{\nabla}_k \tilde{K}(e_i)) \end{aligned}$$

Now the first line is just curvature: Note that

$$[e_l, e_k] = \bar{\nabla}_{e_l} e_k - \bar{\nabla}_{e_k} e_l$$

by the torsion free. We also know that by the choice of normal coordinates, we have $=\left(\bar{\nabla}_{e_l} e_k\right)^{\perp} = \bar{\nabla}_{e_l} e_k$ and because the normal parts are symmetric, we must have $[e_l, e_k] = 0$.

So we get

$$\begin{aligned} \partial_l a_{jki} - \partial_k a_{jli} &= \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) \\ &\quad + h(\bar{\nabla}_{e_k} e_j, \bar{\nabla}_l \tilde{K}(e_i)) - h(\bar{\nabla}_{e_l} e_j, \bar{\nabla}_k \tilde{K}(e_i)) \end{aligned}$$

Now using formula

$$\bar{\nabla}_{e_l} e_j = \Gamma_{jl}^n e_n - \eta^{mn} a_{mlj} \tilde{K}(e_n)$$

(easily derived)

$$\begin{aligned} h(\bar{\nabla}_{e_k} e_j, \bar{\nabla}_l \tilde{K}(e_i)) &= h(\bar{\nabla}_{e_k} e_j, K\bar{\nabla}_l e_i - \bar{\nabla}_l \omega_i^s e_s)) \\ &= h(\bar{\nabla}_{e_k} e_j, K\left[\Gamma_{il}^n e_n - \eta^{mn} a_{mli} \tilde{K}(e_n)\right]) \\ &- h(\bar{\nabla}_{e_k} e_j, \bar{\nabla}_l \omega_i^s e_s)) \end{aligned}$$

now at origin conditions

$$\begin{aligned} h(\bar{\nabla}_{e_k} e_j, \bar{\nabla}_l \tilde{K}(e_i)) &= h(\bar{\nabla}_{e_k} e_j, -\eta^{mn} a_{mli} K\tilde{K}(e_n)) \\ &- h(\bar{\nabla}_{e_k} e_j, \partial_l \omega_i^s e_s + \omega_i^s \bar{\nabla}_l e_s)) \end{aligned}$$

Now for the first term, because $\bar{\nabla}_{e_k} e_j$ is normal, we only need to look at the normal part of $K\tilde{K}(e_n)$ which has a $\omega$ component, so this is $A * \omega$. But pairing with the second fundamental form gives $A * \omega * \omega$. For the second term, we have again normal $\bar{\nabla}_{e_k} e_j$ and the other term is $\omega * A * A$.Combining all the above

$$\begin{aligned} \partial_l a_{jki} - \partial_k a_{jli} &= \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) \\ &+ A * \omega * \omega + A * \omega * \omega. \end{aligned}$$

□

Proof of Claim 4.2:

$$\nabla_l a_{ijk} - \nabla_k a_{ijl} = \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) + \omega * G + \nabla_j \nabla_i \omega_{kl}$$

*Proof.* From Claim 6.3

$$\begin{aligned} \nabla_l a_{ijk} &= \nabla_l a_{jki} + \nabla_l \nabla_j \omega_{ki} \\ \nabla_k a_{ijl} &= \nabla_k a_{jli} + \nabla_k \nabla_j \omega_{li} \end{aligned}$$

Thus

$$\begin{aligned} \nabla_l a_{ijk} - \nabla_k a_{ijl} &= \nabla_l a_{jki} - \nabla_k a_{jli} \\ &+ \nabla_l \nabla_j \omega_{ki} - \nabla_k \nabla_j \omega_{li} \\ &= \nabla_l a_{jki} - \nabla_k a_{jli} \\ &+ \nabla_k \nabla_j \omega_{il} - \nabla_l \nabla_j \omega_{ik} \\ &= \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) + \omega * A * A \\ &+ \nabla_k \nabla_j \omega_{il} - \nabla_l \nabla_j \omega_{ik} \end{aligned}$$

Now using Ricci identity

$$\nabla_k \nabla_j \omega_{il} = \nabla_j \nabla_k \omega_{il} + R * \omega$$

so

$$\begin{aligned} \nabla_k \nabla_j \omega_{il} - \nabla_l \nabla_j \omega_{ik} &= \nabla_j \nabla_k \omega_{il} - \nabla_l \nabla_j \omega_{ik} + R * \omega \\ &= \nabla_j \nabla_k \omega_{il} - \nabla_j \nabla_l \omega_{ik} + R * \omega \end{aligned}$$

again using the RIcci identity so we have

$$\nabla_k \nabla_j \omega_{il} - \nabla_l \nabla_j \omega_{ik} = \nabla_j \left(\nabla_k \omega_{il} + \nabla_l \omega_{ki}\right) + R * \omega$$

Now by closedness of the Kähler form

$$\nabla_k \omega_{il} + \nabla_l \omega_{ki} + \nabla_i \omega_{lk} = 0$$

so we get

$$\begin{aligned}\nabla_k \nabla_j \omega_{il} - \nabla_l \nabla_j \omega_{ik} &= -\nabla_j \nabla_i \omega_{lk} + R * \omega \\ &= \nabla_j \nabla_i \omega_{kl} + R * \omega\end{aligned}$$

So

$$\nabla_l a_{ijk} - \nabla_k a_{ijl} = \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) + \omega * G + \nabla_j \nabla_i \omega_{kl}$$

□

Proof of Claim 4.3:

$$g^{ij} \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) = \bar{R}ic(Ke_k, e_l) + G * \omega$$

*Proof.* Consider

$$g^{ij} \bar{R}(X, Y, e_j, \tilde{K}(e_i))$$

use Bianchi identity

$$= -g^{ij} \bar{R}(e_j, X, Y, \tilde{K}(e_i)) - g^{ij} \bar{R}(Y, e_j, X, \tilde{K}(e_i))$$

Now for the first term, hit with $K$ both of the first, for the second hit the last two (checking that the parallel operator $K$ behaves on pairs of vectors in the curvature tensor as it would in the metric)

$$\begin{aligned}&= g^{ij} \bar{R}(Ke_j, KX, Y, \tilde{K}(e_i)) + g^{ii} \bar{R}(Y, e_j, KX, K\tilde{K}(e_i)) \\ &= g^{ij} \bar{R}(Ke_j, KX, Y, \tilde{K}(e_i) - g^{ii} \bar{R}(e_j, Y, KX, K\tilde{K}(e_i)) \\ &= g^{ii} \bar{R}(\tilde{K}e_j, KX, Y, \tilde{K}(e_i)) - g^{ii} \bar{R}(e_j, Y, KX, e_i)) \\ &+ g^{ii} \bar{R}(\left(K - \tilde{K}\right) e_i, KX, Y, \tilde{K}(e_i)) - g^{ii} \bar{R}(e_i, Y, KX, \left(K\tilde{K} - I\right), e_i) \\ &= \left(g^{ij} + \eta^{ij}\right) \bar{R}(\tilde{K}e_j, KX, Y, \tilde{K}(e_i)) - \eta^{ij} \bar{R}(\tilde{K}e_j, KX, Y, \tilde{K}(e_i)) \\ &- g^{ij} \bar{R}(e_j, Y, KX, e_i) \\ &+ g^{ii} \bar{R}(\left(K - \tilde{K}\right) e_i, KX, Y, \tilde{K}(e_i)) - -g^{ii} \bar{R}(e_i, Y, KX, \left(K\tilde{K} - I\right), e_i)\end{aligned}$$

Now by using the symmetric of the trace in $i, j$ and Bianchi, we may swap the $E_{\bar{p}}, KE_k$ and get

$$\begin{aligned}&= \left(g^{ij} + \eta^{ij}\right) \bar{R}(\tilde{K}e_j, KX, Y, \tilde{K}(e_i)) - Ric(KX, Y) \\ &+ g^{ii} \bar{R}(\left(K - \tilde{K}\right) e_i, KX, Y, \tilde{K}(e_i)) - g^{ii} \bar{R}(e_i, Y, KX, \left(K\tilde{K} - I\right), e_i) \\ &= -Ric(KX, Y) + \omega * R\end{aligned}$$

Applying this

$$\begin{aligned}g^{ij} \bar{R}(e_l, e_k, e_j, \tilde{K}(e_i)) &= -\bar{R}ic(Ke_l, e_k) + \omega * R \\ &= \bar{R}ic(Ke_k, e_l) + \omega * R\end{aligned}$$

using the fact that this Ricci form is alternating. □

## References


[AK20] Farhan Abedin and Jun Kitagawa, *Exponential convergence of parabolic optimal transport on bounded domains*, Anal. PDE **13** (2020), no. 7, 2183–2204. MR 4175823

[Beh11] Tapio Behrndt, *Generalized Lagrangian mean curvature flow in Kähler manifolds that are almost Einstein*, Complex and differential geometry, Springer Proc. Math., vol. 8, Springer, Heidelberg, 2011, pp. 65–79. MR 2964468

[CFG96] V. Cruceanu, P. Fortuny, and P. M. Gadea, *A survey on paracomplex geometry*, Rocky Mountain J. Math. **26** (1996), no. 1, 83–115. MR 1386154

[CSS11] Mykhaylo Chursin, Lars Schäfer, and Knut Smoczyk, *Mean curvature flow of space-like Lagrangian submanifolds in almost para-Kähler manifolds*, Calc. Var. Partial Differential Equations **41** (2011), no. 1-2, 111–125. MR 2782799

[Joy07] Dominic D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics, vol. 12, Oxford University Press, Oxford, 2007. MR 2292510

[KM10] Young-Heon Kim and Robert J. McCann, *Continuity, curvature, and the general covariance of optimal transportation*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 4, 1009–1040. MR 2654086

[KMW10] Young-Heon Kim, Robert J. McCann, and Micah Warren, *Pseudo-Riemannian geometry calibrates optimal transportation*, Math. Res. Lett. **17** (2010), no. 6, 1183–1197. MR 2729641

[KSW12] Young-Heon Kim, Jeffrey Streets, and Micah Warren, *Parabolic optimal transport equations on manifolds*, Int. Math. Res. Not. IMRN (2012), no. 19, 4325–4350. MR 2981711

[LV23] Flavien Léger and François-Xavier Vialard, *A geometric Laplace method*, Pure Appl. Anal. **5** (2023), no. 4, 1041–1080. MR 4680531

[Smo96] Knut Smoczyk, *A canonical way to deform a lagrangian submanifold*, 1996.

[SW11] Knut Smoczyk and Mu-Tao Wang, *Generalized Lagrangian mean curvature flows in symplectic manifolds*, Asian J. Math. **15** (2011), no. 1, 129–140. MR 2786468

[War11] Micah Warren, *A McLean Theorem for the moduli space of Lie solutions to mass transport equations*, Differential Geom. Appl. **29** (2011), no. 6, 816–825. MR 2846278

[War14] ______, *On solutions to Cournot-Nash equilibria equations on the sphere*, Pacific J. Math. **272** (2014), no. 2, 423–437. MR 3284893

[Woo20] Albert Wood, *Singularities of lagrangian mean curvature flow*, Doctoral thesis, UCL (University College London) (2020).



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