INTERIOR SCHAUDER ESTIMATES FOR THE FOURTH ORDER HAMILTONIAN STATIONARY EQUATION IN TWO DIMENSIONS

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Abstract. We consider the Hamiltonian stationary equation for all phases in dimension two. We show that solutions that are $C^{1,1}$ will be smooth and we also derive a $C^{2,\alpha}$ estimate for it.

1. Introduction

In this paper, we study the regularity of the Lagrangian Hamiltonian stationary equation, which is a fourth order nonlinear PDE. Consider the function $u : B_1 \to \mathbb{R}$ where $B_1$ is the unit ball in $\mathbb{R}^2$. The gradient graph of $u$, given by $\{(x, Du(x))| x \in B_1\}$ is a Lagrangian submanifold of the complex Euclidean space. The function $\theta$ is called the Lagrangian phase for the gradient graph and is defined by

$$\theta = F(D^2 u) = \text{Im} \log \det(I + iD^2 u)$$

or equivalently,

$$\theta = \sum_i \arctan(\lambda_i)$$

(1.1)

where $\lambda_i$ represents the eigenvalues of the Hessian.

The non-homogeneous special Lagrangian equation is given by the following second order nonlinear equation

$$F(D^2 u) = f(x).$$

(1.2)

The Hamiltonian stationary equation is given by the following fourth order nonlinear PDE

$$\Delta_g \theta = 0$$

(1.3)

where $\Delta_g$ is the Laplace-Beltrami operator, given by:

$$\Delta_g = \sum_{i,j=1}^2 \frac{\partial_i(\sqrt{\det g}g^{ij}\partial_j)}{\sqrt{\det g}}$$
and $g$ is the induced Riemannian metric from the Euclidean metric on $\mathbb{R}^4$, which can be written as

$$g = I + (D^2 u)^2.$$  

Recently, Chen and Warren [CW19] proved that in any dimension, a $C^{1,1}$ solution of the Hamiltonian stationary equation will be smooth with uniform estimates of all orders if the phase $\theta \geq \delta + (n - 2)\pi/2$, or, if the bound on the Hessian is small. In the two dimensional case, using [CW19]'s result, we get uniform estimates for $u$ when $|\theta| \geq \delta > 0$ (by symmetry). In this paper, we consider the Hamiltonian stationary equation for all phases in dimension two without imposing a smallness condition on the Hessian or on the range of $\theta$, and we derive uniform estimates for $u$, in terms of the $C^{1,1}$ bound which we denote by $\Lambda$. We write $||u||_{C^{1,1}(B_1)} = ||Du||_{C^0,1(B_1)} = \Lambda$. Our main results are the following:

**Theorem 1.1.** Suppose that $u \in C^{1,1}(B_1)$ and satisfies (1.3) on $B_1 \subset \mathbb{R}^2$ where $\theta \in W^{1,2}(B_1)$. Then $u$ is a smooth function with interior Hölder estimates of all orders, based on the $C^{1,1}$ bound of $u$.

**Theorem 1.2.** Suppose that $u \in C^{1,1}(B_1)$ and satisfies (1.2) on $B_1 \subset \mathbb{R}^2$. If $f \in C^{\alpha}(B_1)$, then there exists $R = R(2, \Lambda, \alpha) < 1$ such that $u \in C^{2,\alpha}(B_R)$ and satisfies the following estimate

$$|D^2 u|_{C^{\alpha}(B_R)} \leq C_1(||u||_{L^\infty(B_1)}, \Lambda, |f|_{C^{\alpha}(B_1)}).$$

To be clear, for any given function $u$ we denote

$$\theta(x) = F(D^2 u(x))$$

so that for solutions of (1.2) we always have

$$\theta(x) \equiv f(x).$$

Our proof of Theorem 1.1 goes as follows: We start by applying the De Giorgi-Nash theorem to the uniformly elliptic Hamiltonian stationary equation (1.3) on $B_1$ to prove that $\theta \in C^\alpha(B_{1/2})$. Next we consider the non-homogeneous special Lagrangian equation (1.2) where $\theta \in C^\alpha(B_{1/2})$. Using a rotation of Yuan [Yua02] we rotate the gradient graph so that the new phase $\bar{\theta}$ of the rotated gradient graph satisfies $|\bar{\theta}| \geq \delta > 0$. Now we apply [CC03] to the new potential $\bar{u}$ of the rotated graph to obtain a $C^{2,\alpha}$ interior estimate for it. On rotating back the rotated gradient graph to our original gradient graph, we see that our potential $u$ turns out to be $C^{2,\alpha}$ as well. A computation involving change of co-ordinates gives us the corresponding $C^{2,\alpha}$ estimate, shown in (1.4). Once we have a $C^{2,\alpha}$ solution of (1.3), smoothness follows by [CW19, Corollary 5.1].
In two dimensions, solutions to the second order special Lagrangian equation
\[ F(D^2u) = C \]
enjoy full regularity estimates in terms of the potential \( u \) [WY09]. For higher dimensions, such estimates fail [WY13] for \( \theta = C \) with \( |C| < (n-2)\pi/2 \).

2. Proof of theorems:

We first prove Theorem 1.2, followed by the proof of Theorem 1.1. We prove Theorem 1.2 using the following lemma. Recalling (1.5, 1.6) we state the following lemma:

**Lemma 2.1.** Suppose that \( u \in C^{1,1}(B_1) \) satisfies (1.2) on \( B_1 \subset \mathbb{R}^2 \). Suppose
\[ 0 \leq \theta(0) < (\pi/2 - \arctan \Lambda)/4. \]
If \( \theta \in C^\alpha(B_1) \), then there exists \( 0 < \alpha < \bar{\alpha} \) and \( C_0 \) such that
\[ |D^2u(x) - D^2u(0)| \leq C_0(||u||_{L^\infty(B_1)}, \Lambda, |\theta|_{C^\alpha(B_1)}) \ast |x|^{\alpha}. \]

**Proof.** Consider the gradient graph \( \{(x, Du(x))| x \in B_1 \} \) where \( u \) has the following Hessian bound
\[ -\Lambda I_n \leq D^2u \leq \Lambda I_n \]
a.e. where it exists.

Define \( \delta \) as
\[ \delta = (\pi/2 - \arctan \Lambda)/2 > 0. \]
Since by (2.1) we have \( 0 \leq \theta(0) < \delta/2 \), there exists \( R'(\delta, |\theta|_{C^\alpha}) > 0 \) such that
\[ |\theta(x) - \theta(0)| < \delta/2 \]
for all \( x \in B_{R'} \subseteq B_1 \). This implies for every \( x \) in \( B_{R'} \) for which \( D^2u \) exists, we have
\[ \delta > \theta > \theta(0) - \delta/2. \]
So now we rotate the gradient graph \( \{(x, Du(x))| x \in B_{R'} \} \) downward by an angle of \( \delta \).

Let the new rotated co-ordinate system be denoted by \((\bar{x}, \bar{y})\) where
\[ \bar{x} = \cos(\delta)x + \sin(\delta)Du(x) \]
(2.3)
\[ \bar{y} = -\sin(\delta)x + \cos(\delta)Du(x). \]

On differentiating \( \bar{x} \) (2.3) with respect to \( x \) we see that
\[ \frac{d\bar{x}}{dx} = \cos(\delta)I_n + \sin(\delta)D^2u(x) \leq \cos(\delta)I_n + \Lambda \sin(\delta)I_n \]
Thus
\[
\cos(\delta)I_n - \Lambda \sin(\delta)I_n \leq \frac{d\bar{x}}{dx} \leq \cos(\delta)I_n + \Lambda \sin(\delta)I_n.
\]
To obtain Lipschitz constants so that
\[
\left(2.5\right) \frac{1}{L_2} I_n \leq \frac{d\bar{x}}{dx} \leq L_1 I_n
\]
let
\[
L_1 = \cos(\delta) + \Lambda \sin(\delta)
\]
\[
L_2 = \max\left\{\left|\frac{1}{\cos(\sigma)I_n + D^2u(x)\sin(\sigma)}\right| \mid x \in B_{R'}\right\}.
\]
To find the value of \(L_2\), we see that in \(B_{R'}\) we have the following:
let \(\min\{\theta_1, \theta_2\} \geq -A\) where \(A = \arctan \Lambda\).
\[
\cos(\delta)I_n + \sin(\delta)D^2u(x) \geq \cos(\delta) - \sin(\delta)\tan(A)
\]
\[
= \cos(\delta)(1 - \tan(\delta)\tan(A))
\]
\[
= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\delta + A)}
\]
\[
= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - A + A)}
\]
\[
= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - \delta)}.
\]
This shows that
\[
\frac{1}{L_2} = \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - \delta)}.
\]
Clearly \(1/L_2\) is positive.

Now, by [CW19, Prop 4.1] we see that there exists a function \(\bar{u}\) such that
\[
\bar{y} = D_x \bar{u}(\bar{x})
\]
where
\[
(2.6) \quad \bar{u}(x) = u(x) + \sin \delta \cos \delta \frac{|Du(x)|^2 - |x|^2}{2} - \sin^2(\delta)Du(x) \cdot x
\]
defines \(\bar{u}\) implicitly in terms of \(\bar{x}\) (since \(\bar{x}\) is invertible). Here \(\bar{x}\) refers to the rotation map (2.3).

Note that
\[
\bar{\theta}(\bar{x}) - \bar{\theta}(\bar{y}) = \theta(x) - \theta(y)
\]
which implies that \( \bar{\theta} \) is also a \( C^{\bar{\alpha}} \) function

\[
\frac{|\bar{\theta}(\bar{x}_1) - \bar{\theta}(\bar{x}_2)|}{|\bar{x}_1 - \bar{x}_2|^{\bar{\alpha}}} = \frac{|\theta(x_1) - \theta(x_2)|}{|x_1 - x_2|^{\bar{\alpha}}} \ast \frac{|x_1 - x_2|^{\bar{\alpha}}}{|\bar{x}_1 - \bar{x}_2|^{\bar{\alpha}}}
\]

thus,

\[
|\bar{\theta}|_{C^{\bar{\alpha}}(B_{r_0})} \leq L_{\bar{\alpha}} \bar{\theta}|_{C^{\bar{\alpha}}(B_{r'})}.
\]

Let \( \Omega = \bar{x}(B_{r'}) \). Note that \( B_{r_0} \subset \Omega \) where \( r_0 = R'/2L_2 \). So our new gradient graph is \( \{(\bar{x}, D_2 \bar{u}(\bar{x}))|\bar{x} \in \Omega\} \). The function \( \bar{u} \) satisfies the equation

\[
F(D^2 \bar{u}) = \bar{\theta}(\bar{x})
\]

in \( B_{r_0} \) where \( \bar{\theta} \in C^{\bar{\alpha}}(B_{r_0}) \). Observe that on \( B_{r_0} \) we have

\[
\bar{\theta} = \theta - 2\delta < \delta - 2\delta = -\delta < 0
\]
as \( \theta < \delta \) on \( B_{r'} \).

**Claim 2.2.** : If \( |\bar{\theta}| > \delta \), then \( F(D^2 \bar{u}) = \bar{\theta} \) is a solution to a uniformly elliptic concave equation.

**Proof.** The proof follows from [CPW17, lemma 2.2] and also from [CW19, pg 24]. \( \square \)

Now using [CC03, Corollary 1.3] we get interior Schauder estimates for \( \bar{u} \):

\[
|D^2 \bar{u}(\bar{x}) - D^2 \bar{u}(0)| \leq C(||\bar{u}||_{L^\infty(B_{r_0/2})} + |\bar{\theta}|_{C^{\bar{\alpha}}(B_{r_0/2})})
\]

for all \( \bar{x} \) in \( B_{r_0/2} \) where \( C = C(\Lambda, \alpha) \). This is our \( C^{2,\alpha} \) estimate for \( \bar{u} \).

Next, in order to show the same Schauder type inequality as (2.7) for \( u \) in place of \( \bar{u} \), we establish relations between the following pairs:

(i) oscillations of the Hessian of \( D^2 u \) and \( D^2 \bar{u} \)
(ii) oscillations of \( \theta \) and \( \bar{\theta} \)
(iii) the supremum norms of \( u \) and \( \bar{u} \).

We rotate back to our original gradient graph by rotating up by an angle of \( \delta \) and consider again the domain \( B_{r'}(0) \). This gives us the following relations:

\[
x = \cos(\delta)x - \sin(\delta)D_2 \bar{u}(\bar{x})
\]

(2.8)

\[
y = \sin(\delta)x + \cos(\delta)D_2 \bar{u}(\bar{x}).
\]

This gives us:

\[
\frac{dx}{d\bar{x}} = \cos(\delta)I_n - \sin(\delta)D_2^2 \bar{u}(\bar{x})
\]

\[
D_2y = \sin(\delta)I_n + \cos(\delta)D_2^2 \bar{u}(\bar{x}).
\]
So we have
\[ D_x^2 u(x) = D_x y \frac{d \bar{x}}{dx} = \left[ \sin(\delta)I_n + \cos(\delta) D_x^2 \bar{u}(\bar{x}) \right] \left[ \cos(\delta)I_n - \sin(\delta) D_x^2 \bar{u}(\bar{x}) \right]^{-1}. \]
The above expression is well defined everywhere because \( D_x^2 \bar{u}(\bar{x}) < \cot(\delta)I_n \) for all \( \bar{x} \in B_{r_0} \).

Note that we have
\[ \cos(\delta)I_n - D_x^2 \bar{u}(\bar{x}) \sin(\delta) \geq \frac{1}{L_1}, \]
by (2.5).

Next,
\[ D_x^2 u(x) - D_x^2 u(0) = \left[ \sin(\delta)I_n + \cos(\delta) D_x^2 \bar{u}(\bar{x}) \right] \left[ \cos(\delta)I_n - \sin(\delta) D_x^2 \bar{u}(\bar{x}) \right]^{-1} \]
\begin{equation}
(2.9)
\end{equation}
\[ - \left[ \sin(\delta)I_n + \cos(\delta) D_x^2 \bar{u}(0) \right] \left[ \cos(\delta)I_n - \sin(\delta) D_x^2 \bar{u}(0) \right]^{-1}. \]

For simplification of notation we write
\[ D_x^2 \bar{u}(\bar{x}) = A \]
\[ D_x^2 \bar{u}(0) = B \]
\[ \cos(\delta) = c, \sin(\delta) = s. \]

Noting that \([sI_n + cA]\) and \([cI_n - sA]^{-1}\) commute with each other we can write (2.9) as the following equation
\[ D_x^2 u(x) - D_x^2 u(0) = \]
\[ [cI_n - sB]^{-1}[cI_n - sB][sI_n + cA][cI_n - sA]^{-1} - \]
\[ [cI_n - sB]^{-1}[sI_n + cB][cI_n - sA][cI_n - sA]^{-1}. \]

Again we see that
\[ [cI_n - sB][sI_n + cA] - [sI_n + cB][cI_n - sA] = A - B. \]
This means
\[ D_x^2 u(x) - D_x^2 u(0) = [cI_n - sB]^{-1}[A - B][cI_n - sA]^{-1}. \]
We have already shown that
\[ |cI_n - sA| \geq \frac{1}{L_1} \]
which implies
\[ |cI_n - sA|^{-1} \leq L_1. \]
Thus we get
\[
|D_x^2 u(x) - D_x^2 u(0)| \leq L_2^2 |D_x^2 \bar{u}(\bar{x}) - D_x^2 \bar{u}(0)|.
\]
(2.10)
\[
\leq C L_1^2 \left( ||\bar{u}||_{L^\infty(B_{r_0/2})} + |\theta|_{C^\alpha(B_{r_0/2})} |\bar{x}|^\alpha \right)
\]
where \( L_1 \) is the Lipschitz constant of the co-ordinate change map. This implies
\[
\frac{1}{L_1^{\alpha+2}} |D_x^2 u(x)|_{C^\alpha(B_R)} \leq |D_x^2 u(\bar{x})|_{C^\alpha(B_{r_0/2})}.
\]
(2.11)
Recall from (2.6) that
\[
\bar{u}(x) = u(x) + v(x).
\]
This shows
\[
||\bar{u}(\bar{x})||_{L^\infty(B_{r_0/2})} = ||\bar{u}(x)||_{L^\infty(B^{-1}_{r_0/2})} \leq ||\bar{u}(x)||_{L^\infty(B_{r'})}
\]
(2.12)
\[
\leq ||u(x)||_{L^\infty(B_{r'})} + ||v||_{L^\infty(B_{r'})}.
\]
Note that
\[
||v||_{L^\infty(B_R)} \leq R ||Du||_{L^\infty(B_{r'})} + \frac{1}{2} R^2 + ||Du||_{L^\infty(B_{r'})}^2
\]
(2.13)
and combining (2.11), (2.12), (2.13) with (2.10) we get
\[
|D_x^2 u(x) - D_x^2 u(0)| \leq C L_1^{\alpha+2} \left\{ \frac{1}{2} R^2 + ||Du||_{L^\infty(B_{r'})}^2 + L_0^2 |\theta|_{C^\alpha(B_{r'})} \right\} |x|^{\alpha}.
\]
This proves the Lemma.

**Proof of Theorem 1.2.** First note that the lemma provides a bound for the Hölder norm of the Hessian on any interior ball, so by a rescaling of the form
\[
u_\rho(x) = \frac{u(\rho x)}{\rho^2}
\]
for values of \( \rho > 0 \) and translation of any point to the origin. Consider the gradient graph \((x, Du(x)) | x \in B_1\) where \( u \) satisfies
\[F(D^2 u) = \theta\]
on \( B_1 \) and \( \theta \in C^\alpha(B_1)\). Then there exists a ball of radius \( r \) inside \( B_1 \) on which \( \text{osc} \theta < \delta/4 \) where \( \delta \) is as defined in (2.2).
Now this means that either we have \( \theta(x) < \delta/2 \) in which case, by the above lemma we see that \( u \in C^{2,\alpha}(B_r) \) satisfying the given estimates; or we have \( \theta(x) > \delta/4 \) in which case \( u \in C^{2,\alpha}(B_r) \) with uniform estimates, by claim (2.2) and [CC03, Corollary 1.3].
Proof of Theorem 1.1. Since $u \in C^{1,1}(B_1)$ and $\theta \in W^{1,2}(B_1)$ satisfies the uniformly elliptic equation

$$\Delta g \theta = 0,$$

by the De Giorgi-Nash Theorem we have that $\theta \in C^\alpha(B_{1/2}).$ This means that $u$ satisfies

$$F(D^2 u) = \theta.$$ 

By Theorem 1.2 we see that $u \in C^{2,\alpha}(B_r)$ where $r < 1/2$. Smoothness follows by [CW19, Corollary 5.1].

References


