MINIMAL LAGRANGIAN SUBMANIFOLDS OF WEIGHTED KIM-MCCANN METRICS

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Abstract. We explore the regularity theory of optimal transport maps for costs satisfying a Ma-Trudinger-Wang condition, using the graphs of the transport maps as maximal Lagrangian surfaces with respect to a weighted metric on the product space. For computational simplicity, we restrict to two dimensions.

1. Introduction

In this note, we explore maximal n-dimensional submanifolds of codimension n, in a class of pseudo-Riemmanian metrics with signature (n, n). Kim and McCann [KM10] defined these metric on the product space $M \times \bar{M}$, when giving an alternate formulation of the Ma-Trudinger-Wang regularity theory. In this setting we show that the regularity properties can be obtained by looking at the graph of the optimal transport map as a maximal surface, with appropriate weighting. If the metric is of Kim-McCann type, a calibrated Lagrangian submanifold is either the graph of a solution to an optimal transportation problem, or, if the manifold has topology, could be the graph of a Lie solution to the optimal transportation problem. We slightly reformulate the setting of Kim-McCann [KM10] and Kim-McCann-Warren [KMW10]. Recall first the setting of Kim-McCann where metrics on the product space $M \times \bar{M}$ are locally given by

$$h = \begin{pmatrix} 0 & h_{i\bar{j}}(x, \bar{x}) \\ h_{i\bar{j}}(x, \bar{x}) & 0 \end{pmatrix} \tag{1.1}$$

where

$$h_{i\bar{j}}(x, \bar{x}) = -\frac{1}{2}\partial_{\bar{j}}\partial_i c(x, \bar{x}) \tag{1.2}$$

for some cost function $c : M \times \bar{M} \to \mathbb{R}$. We assume here and in the sequel that c is twice differentiable and satisfies the (A2) condition on $N = M \times \bar{M} \backslash \mathcal{C}$ where $\mathcal{C}$ is a measure zero set which we call the 'cut locus'. In particular, $h_{i\bar{j}}(x, \bar{x})$ is non-degenerate almost everywhere and h defines an (n, n) signature metric away from the cut locus.

We are concerned with an underlying optimal transportation problem: Find the optimal transportation map between the measures defined by bounded mass densities ρ and $\bar{\rho}$ for the cost function c. In particular, minimize

$$\int_M c(x, T(x))\rho(x)dx$$

over the space of maps T satisfying

$$T_{\#}d\rho = d\bar{\rho}.$$

In [KMW10] it was illustrated that when taking the conformal metric

$$\tilde{h} = \left[\frac{(\rho(x)\bar{\rho}(\bar{x}))}{\det\left(-\partial_{\bar{j}}\partial_i c(x,\bar{x})\right)}\right]^{1/n} \begin{pmatrix} 0 & h_{i\bar{j}}(x,\bar{x}) \\ h_{i\bar{j}}(x,\bar{x}) & 0 \end{pmatrix} \tag{1.3}$$

graphs of optimal transportion plans T are calibrated maximal submanifolds with respect to this metric [KMW10, Theorem 1.1]. Unfortunately the conformal approach can lead to some technical computational issues, especially when the dimension satisfies $n = 2$. So here we suggest that a weighted approach should work better: Consider instead the weighted manifold

$$\left(M \times \bar{M}, h, \left(\frac{\rho(x)\bar{\rho}(\bar{x})}{\det\left(-\partial_{\bar{j}}\partial_i c(x,\bar{x})\right)}\right)^{1/2}\right). \tag{1.4}$$

One can check that the volume of an n-submanifold with this weight is the same as the volume of the n-submanifold in the conformal setting defined by (1.3). Thus the minimal surfaces are the same in either setting. However, instead of the minimal surface equation occuring in the setting (1.3), in the latter setting (1.4), the manifolds will be locally defined by the weighted minimal surface equation

$$H + (\nabla f)^N = 0 \tag{1.5}$$

where

$$e^{-f} = \left(\frac{\rho(x)\bar{\rho}(\bar{x})}{\det\left(-\partial_{\bar{j}}\partial_i c(x,\bar{x})\right)}\right)^{1/2}.$$

Note that when the dimension and codimension of a space-like manifold are both n, in a metric of signature (n, n) one can try to locally maximize the submanifold. This leads to the minimal surface equation (1.5) satisfied by the maximizers. In fact, a calibration argument shows that [KMW10] the submanifolds are maximizing with respect to an appropriate class of space-like submanifolds. Recall that the graph of an optimal transportation map will be space-like with respect to h,[KM10, Section 5.]

In this paper, we would like to demonstrate that the regularity theory can be derived geometrically, obtaining yet another approach for the Ma Trudinger Wang regularity theory. The original regularity paper [MTW05] is a maximum principle argument applied to the Monge-Ampère equation (2.1), in which the authors identify a 4th order condition on the cost function which leads to regularity for general smooth densities. The approach by Loeper [Loe09] is based on analysis of the cost function requiring less regularity. The setting of Kim-McCann gives a solid pseudo-Riemannian geometric formulation of the approach offered in Loeper, in particular, showing that c-segments are geodesics with respect to this metric on the product manifold.

To significantly simplify the computation we restrict to the case where both manifolds are compact and have dimension $n = 2$. We also restrict to the case where the metric is of the Kim-McCann form (1.2), however, it may be possible to loosen this to more general pseudo-Riemmanian submanifolds satisfying a suitable curvature condition. Indeed, our approach is motivated by the paper of Li and Salavessa [LS11] where general regularity result follow from curvature conditions.

2. Prelimaries

We will consider a product of compact, oriented manifolds of the form $M \times \bar{M}$, with metrics of the form (1.1) and consider some function

$$T : M \to \bar{M}.$$

Recall $h_{i\bar{j}}(x, \bar{x}) = -\frac{1}{2}\partial_{\bar{j}}\partial_i c(x, \bar{x})$ for some cost function c that is twice differentiable with non-degenerate mixed Hessian on $N = M \times \bar{M} \backslash \mathcal{C}$. We denote the graph of the map T by

$$\Gamma = \{(x, T(x) : x \in M\} \subset M \times \bar{M}.$$

The following is the Ma-Trudinger-Wang curvature condition as uncovered by Kim and McCann in the product manifold setting [KM10, Definition 2.3].

Definition 2.1. *The metric h is strictly regular, if whenever $(x, \bar{x}) \in N$ and*

$$0 \neq \partial_i \in T_x M$$
$$0 \neq \partial_{\bar{j}} \in T_{\bar{x}} M$$
$$h_{(x,\bar{x})}(\partial_i, \partial_{\bar{j}}) = 0,$$

then the curvature of the metric h satisfies

$$R\left(\partial_i, \partial_{\bar{j}}, \partial_{\bar{j}}, \partial_i\right) > 0.$$

The following is our main theorem.

Theorem 2.2. *Suppose that $(x, T(x))$ is a maximal Lagrangian submanifold in a weighted Kim-McCann metric, where $T(x)$ is locally an optimal transport map, M and $\bar{M}$ are compact, oriented of dimension 2, and the metric h is strictly regular. Suppose also that the graph Γ has positive distance from the cut locus $\mathcal{C}$. Suppose the mass densities ρ and $\bar{\rho}$ being paired are bounded above and away from zero. Then, given a metric g_1 on M and metric g_2 on $\bar{M}$ there is a constant such that*

$$\|DT\| \leq C.$$

Here the norm is defined as

$$\|DT\| = \sup_{x \in M, v \in T_x M, \|v\|_{g_1} = 1} \|DT(v)\|_{g_2}.$$

This establishes the C^1 estimates for the optimal transport maps. Recall [MTW05, (2.17)] that solutions of this optimal transportation problem can be determined by a potential function u satisfying

$$\det(u_{ij} + c_{ij}) = \left|\det(-c_{i\bar{j}})\right| \frac{\rho(x)}{\bar{\rho}(T_u(x))} \tag{2.1}$$

where T_u is cost transform of the function u defining a map from M to $\bar{M}$ implicitly [MTW05, (2.12)] via

$$Du(x) = D_x c(x, T_u(x)). \tag{2.2}$$

Form the perspective of Monge-Ampère type equations, this C^1 estimates obtained above is on the map T_u which has the same regularity as Du. Thus the above theorem provides $C^{1,1}$ estimates on the potential, which puts bounds on the ellipticity of the Monge-Ampère equation (2.1). Once this ellipticity is in place, one can

apply Schauder theory to obtain higher order estimates (provided the densities are smooth enough).

Note that this is true for Lie solutions of optimal transport equations, see [Del08], which will exist when M has nontrivial first Betti number, [War11].

3. More Setup

We start by fixing g_1, g_2 to be arbitrary Riemmanian metrics on $M, \bar{M}$. These will be fixed and will serve as a gauge against which to obtain estimates. On the other hand, the metric g on M will be the metric induced on Γ, that is

$$g\left(\partial_i, \partial_j\right) = h_{j\bar{k}} \frac{\partial T^{\bar{k}}}{\partial x_i} + h_{i\bar{k}} \frac{\partial T^{\bar{k}}}{\partial x_j}.$$

As M is oriented, at each point there exists a local oriented orthonormal frame (w.r.t g_1), e_1, e_2. Let $U \subset M$ be a neighborhood on which this frame exists. These define a local frame for the cotangent space

$$\omega_i(\cdot) = g_1(e_i, \cdot) \tag{3.1}$$

We then define the forms

$$\nu_i = h(e_i, \cdot), \ i = 1, 2$$

on $U \times \bar{M}$.

Now ω_i extends to a form on $U \times \bar{M}$:

$$\hat{\omega}_i = \omega_i \circ P_{T_x M}$$

where $P_{T_x M}$ is the projection onto the tangent space using the natural decomposition. Add the two forms and create a form as follows:

$$\Omega = (\hat{\omega}_1 + \nu_1) \wedge (\hat{\omega}_2 + \nu_2).$$

Lemma 3.1. *The form Ω is globally well-defined.*

Proof. We need to show that any chart we choose, we get the same form. At any point, choose any arbitrary oriented orthonormal frame $\tilde{e}_1, \tilde{e}_2$. This is related to any other oriented orthonormal frame via a an orthogonal transformation O. In this case, the form is defined as

$$\begin{aligned}
&(\hat{\omega}_1 + \nu_1)^{\tilde{}} \wedge (\hat{\omega}_2 + \nu_2)^{\tilde{}} \\
&= O^{j_1 1}(\hat{\omega}_{j_1} + \nu_{j_1}) \wedge O^{j_2 2}(\hat{\omega}_{j_2} + \nu_{j_2}) \\
&= (\det O)\,(\hat{\omega}_1 + \nu_1) \wedge (\hat{\omega}_2 + \nu_2).
\end{aligned}$$

Because the basis is assumed to be oriented, we will always have $\det O = 1$. □

Our goal now is to control above and away from zero the ratio

$$\mu = \frac{\Omega}{dVol_g}.$$

Because Ω and $dVol_g$ are top forms, the ratio is a well-defined quantity on M.

3.1. **Setting up coordinate systems near a point.** Because M is compact, a maximum for μ must happen at some point x_0. To control μ we are going to compute the Laplacian operator with respect to g at this point. The coordinates setup is crucial and so we go through the setup of a coordinate system around x_0 in detail here. We begin with a set of normal coordinates generated by an arbitrary tangent basis e_1', e_2' for g_1 at x_0 (call this System 1'). Now the (induced) metric g_{x_0} is a symmetric matrix when written in terms of basis $\{e_1', e_2'\}$, so is orthogonally diagonalizable, with respect to some basis $\{e_1, e_2\}$, which is still a basis for a normal tangent system at the point x_0 with respect to g_1. We may thus choose normal coordinates (System 1) with respect to g_1 that are diagonal with respect to g, that is

$$\begin{aligned} \|e_i\|_g^2 &= \lambda_i^2,\ i = 1, 2. \\ \langle e_1, e_2 \rangle_g &= 0 \\ \|e_i\|_{g_1} &= 1,\ i = 1, 2. \end{aligned}$$

As the submanifold is space-like g is positive definite. Rotating e_1, e_2 if necessary, we choose $\lambda_1 \geq \lambda_2 > 0$. (We will use e_i in two senses, identifying e_i first as an abstract tangent vector on the manifold M, which is measured by g or g_1, but also as a submanifold tangent vector to the graph Γ, which is measured by h.) Take an arbitrary, but controlled chart (say normal coordinates at $\bar{x}_0$ w.r.t. g_2) for $\bar{M}$. In these coordinates (together with System 1 in the first variable), the metric h has the form

$$\frac{1}{2}\begin{pmatrix} 0 & h_{i\bar{j}}(x, \bar{x}) \\ h_{i\bar{j}}(x, \bar{x}) & 0 \end{pmatrix}.$$

We can take a local change of coordinates φ of $\bar{M}$ at $\bar{x}_0$, so that $D\varphi_{\bar{x}_0} = h_{i\bar{j}}(x_0, \bar{x}_0)^{-1}$, as h_{ij} is nondegenerate. In this case, the metric at the point $(x_0, \bar{x}_0)$ becomes

$$h = \frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3.2}$$

Notice here that so far our choice of coordinates only depends on rotational properties of g, that is, the dependence is on the direction of diagonalization. Thus the derivatives of objects (such as Ω) can be controlled in a uniform way in terms of the background metric g_1, g_2. We continue to call this system (extended to a neighborhood in the product $M \times \bar{M}$) Product System 1. We note that particular, the vertical basis vectors $\{\bar{e}_1, \bar{e}_2\}$ satisfy

$$\|\bar{e}_2\|_{g_2} \leq C_1 \tag{3.3}$$

for some universal constant, in particular does not depend on $\|DT\|$.

We now define another set of coordinates (System 2) that we will use later in our geometric computations. Namely take the coordinates given by exponentiating the tangent space with respect to g, where

$$\begin{aligned} \partial_1 &= \frac{1}{\lambda_1} e_1 \\ \partial_2 &= \frac{1}{\lambda_2} e_2 \end{aligned}$$

noting that

$$\|\partial_1\|_g = \|\partial_2\|_g = 1.$$

In (Product System 1) coordinates on $M \times \bar{M}$ we have

$$\partial_1 = (\frac{1}{\lambda_1}, 0, \lambda_1, *)$$
$$\partial_2 = (0, \frac{1}{\lambda_2}, *, \lambda_2).$$

Now if we are assuming that the submanifold is Lagrangian, then the symplectic form given by ([KM10, 5.3])

$$h_{ij} dx^i \wedge d\bar{x}^j$$

is simply $dx \wedge d\bar{x}$ at the point $(x_0, \bar{x}_0)$. The submanifold is Lagrangian ([KM10, Theorem A.1]): combining this with the fact that the tangent vectors are orthogonal with respect to (3.2) the tangent vectors must take the form:

$$\partial_1 = (\frac{1}{\lambda_1}, 0, \lambda_1, 0)$$
$$\partial_2 = (0, \frac{1}{\lambda_2}, 0, \lambda_2).$$

We also have in the ambient space the representation (Product System 1)

$$e_1 = (1, 0, \lambda_1^2, 0) \tag{3.4}$$
$$e_2 = (0, 1, 0, \lambda_2^2)$$

and at the point $(x_0, \bar{x}_0)$ we have

$$\Omega = (dx_1 + dy_1) \wedge (dx_2 + dy_2)$$

still with respect to Product System 1. Thus

$$\mu(x_0) = \frac{\Omega(\partial_1, \partial_2)}{dVol_g(\partial_1, \partial_2)} = \Omega(\partial_1, \partial_2) = (g_1(e_1, \cdot) + h(e_1, \cdot)) \wedge (g_1(e_2, \cdot) + h(e_2, \cdot))(\partial_1, \partial_2) \tag{3.5}$$
$$= (g_1(e_1, \partial_1) + h(e_1, \partial_1))(g_1(e_2, \partial_2) + h(e_2, \partial_2))$$
$$- (g_1(e_1, \partial_2) + h(e_2, \partial_2))(g_1(e_2, \partial_1) + h(e_2, \partial_1)) \tag{3.6}$$
$$= \left(\frac{1}{\lambda_1} + \lambda_1\right)\left(\frac{1}{\lambda_2} + \lambda_2\right) = \frac{1}{\lambda_1 \lambda_2} + \lambda_1 \lambda_2 + \lambda_1 + \lambda_2.$$

We will also use the following normal vectors at the point $(x_0, \bar{x}_0)$

$$n_1 = (\frac{1}{\lambda_1}, 0, -\lambda_1, 0)$$
$$n_2 = (0, \frac{1}{\lambda_2}, 0, -\lambda_2).$$

4. Computations

The major technical goal is to show the following.

Proposition 4.1. *Assume the same conditions as in Theorem 2.2. There exists a C depending on h, ρ_1, ρ_2, and the uniform regularity of c etc such that*

$$\mu \leq C_0.$$

Proposition 4.2. *The above proposition implies Theorem 2.2*

Proof of Proposition 4.2 . Take Product System 1 coordinates as above at a point x_0. For any unit vector v with respect to g_1 we have

$$v = v_1 e_1 + v_2 e_2, \quad v_1^2 + v_2^2 = 1.$$

We have (3.4) with respect to Product System 1

$$DT = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \tag{4.1}$$

that is

$$DT(v) = v_1 \lambda_1^2 \bar{e}_1 + v_2 \lambda_2^2 \bar{e}_2. \tag{4.2}$$

We have by (3.5), and Proposition 4.1

$$\frac{1}{\lambda_1 \lambda_2} + \lambda_1 \lambda_2 + \lambda_1 + \lambda_2 \leq C_0 \tag{4.3}$$

which implies that the largest eigenvalue $\lambda_1 \leq C_0$. Combining (4.2) and (4.3) with (3.3) we have

$$\|DT(v)\| \leq C_0^2 C_1.$$

□

Now in order to prove Proposition 4.1, we will use System 2 coordinates on M, so we may do the following geometric computation.

Lemma 4.3. *Suppose that $\Gamma = (x, T(x))$ is a weighted minimal submanifold. At the point $(x_0, \bar{x}_0) \in \Gamma$, working in normal coordinates for M with respect to g, we have*

$$\begin{aligned}
&\sum_k \bar{\nabla}_{\partial_k} \bar{\nabla}_{\partial_k} \Omega(\partial_1, \partial_2) - \sum_k \nabla_k \nabla_k \Omega(\partial_1, \partial_2) \\
&= - \|B\|^2 \Omega(\partial_1, \partial_2) - 2 \sum_k \Omega(B(\partial_k, \partial_1), B(\partial_k, \partial_2)) \\
&- 2 \sum_k \left\{ \bar{\nabla}_k \Omega(B_{k1}, \partial_2) + \bar{\nabla}_k \Omega(\partial_1, B_{k2}) \right\} - \sum_k \bar{\nabla}_{B(k,k)} \Omega(\partial_1, \partial_2) \\
&+ \sum_s \left\{ \Omega(Rm_h(\partial_2, \partial_1, \partial_2, n_s) n_s, \partial_2) + \Omega(\partial_1, Rm_h(\partial_1, \partial_2, \partial_1, n_s) n_s) \right\} \\
&- \Omega(\left(\bar{\nabla}^2 f(n_p, \partial_1) - \left(\bar{\nabla} f\right)^T \cdot \bar{\nabla}_1 n_p \right) n^{ps} n_s, \partial_2) \\
&- \Omega(\partial_1, \left(\bar{\nabla}^2 f(n_p, \partial_2) - \left(\bar{\nabla} f\right)^T \cdot \bar{\nabla}_2 n_p \right) n^{ps} n_s)
\end{aligned}$$

Here, $\|B\|^2$ is the positive norm of the second fundamental form B, of the submanifold, B_{ki} is the normal vector field defined by $B(\partial_k, \partial_i)$, Rm_h the curvature tensor of h and n^{pq} is the (negative definite) inverse metric on the normal vectors.

4.0.1. *Proof of Lemma.* Our goal is to come up with an expression for the following quantity, taken in normal coordinates, at a point.

$$\bar{\nabla}_{\partial_k} \bar{\nabla}_{\partial_k} \Omega(\partial_1, \partial_2) - \nabla_{\partial_k} \nabla_{\partial_k} \Omega(\partial_1, \partial_2).$$

Here ∂_k is a (System 2) normal coordinate derivative along M with respect to g, ∇ denotes differentiation w.r.t. g, while $\bar{\nabla}_{\partial_k}$ denote covariant differentiation with respect to h.

To begin

$$\bar{\nabla}_{\partial_k}\Omega(\partial_1,\partial_2) = \partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2).$$

Differentiating again

$$\begin{aligned}\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\Omega(\partial_1,\partial_2) &= \partial_k\left\{\partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\right\}\\ &\quad - \left\{\begin{array}{c}\bar{\nabla}_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_2)\\ \partial_k\Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\\ \partial_k\Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_2)\end{array}\right\}.\end{aligned}$$

Now repeat, but with respect to the submanifold connection

$$\begin{aligned}\nabla_{\partial_k}\nabla_{\partial_k}\Omega(\partial_1,\partial_2) &= \partial_k\left\{\partial_k\Omega(\partial_1,\partial_2) - \Omega(\nabla_k\partial_1,\partial_2) - \Omega(\partial_1,\nabla_{\partial_k}\partial_2)\right\}\\ &\quad - \left\{\begin{array}{c}\nabla_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\nabla_{\nabla_{\partial_k}\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\nabla_{\nabla_{\partial_k}\partial_k}\partial_2)\\ \partial_k\Omega(\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\nabla_{\partial_k}\partial_2)\\ \partial_k\Omega(\partial_1,\nabla_{\partial_k}\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\nabla_{\partial_k}\partial_2) - \Omega(\partial_1,\nabla_{\partial_k}\nabla_{\partial_k}\partial_2)\end{array}\right\}.\end{aligned}$$

Now, consider the difference:

$$\begin{aligned}&\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\Omega(\partial_1,\partial_2) - \nabla_{\partial_k}\nabla_{\partial_k}\Omega(\partial_1,\partial_2)\\ &= \partial_k\left\{\begin{array}{c}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\\ -\partial_k\Omega(\partial_1,\partial_2) + \Omega(\nabla_{\partial_k}\partial_1,\partial_2) + \Omega(\partial_1,\nabla_{\partial_k}\partial_2)\end{array}\right\}\\ &- \left\{\begin{array}{c}\bar{\nabla}_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_2)\\ \partial_k\Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\\ \partial_k\Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_2)\end{array}\right\}\\ &+ \left\{\begin{array}{c}\nabla_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\nabla_{\nabla_{\partial_k}\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\nabla_{\nabla_{\partial_k}\partial_k}\partial_2)\\ \partial_k\Omega(\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\nabla_{\partial_k}\partial_2)\\ \partial_k\Omega(\partial_1,\nabla_{\partial_k}\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\nabla_{\partial_k}\partial_2) - \Omega(\partial_1,\nabla_{\partial_k}\nabla_{\partial_k}\partial_2)\end{array}\right\}.\end{aligned}$$

At the origin in normal coordinates, we can eliminate terms of the form $\nabla_{\partial_k}\partial_j$ and we get

$$\begin{aligned}&\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\Omega(\partial_1,\partial_2) - \nabla_{\partial_k}\nabla_{\partial_k}\Omega(\partial_1,\partial_2)\\ &= \partial_k\left\{\Omega(\nabla_{\partial_k}\partial_1 - \bar{\nabla}_{\partial_k}\partial_1,\partial_2) + \Omega(\partial_1,\nabla_{\partial_k}\partial_2 - \bar{\nabla}_{\partial_k}\partial_2)\right\}\\ &- \left\{\begin{array}{c}\bar{\nabla}_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_2)\\ \partial_k\Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\\ \partial_k\Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_2)\end{array}\right\}\\ &+ \left\{\begin{array}{c}\partial_k\Omega(\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\nabla_k\nabla_k\partial_1,\partial_2)\\ \partial_k\Omega(\partial_1,\nabla_{\partial_k}\partial_2) - \Omega(\partial_1,\nabla_k\nabla_k\partial_2)\end{array}\right\}.\end{aligned}$$

$$\begin{aligned}&= \partial_k\left\{\begin{array}{c}\Omega(\nabla_{\partial_k}\partial_1 - \bar{\nabla}_{\partial_k}\partial_1,\partial_2) + \Omega(\partial_1,\nabla_{\partial_k}\partial_2 - \bar{\nabla}_{\partial_k}\partial_2)\\ +\Omega(\nabla_{\partial_k}\partial_1,\partial_2) + \Omega(\partial_1,\nabla_{\partial_k}\partial_2)\\ -\Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\end{array}\right\}\\ &- \left\{\begin{array}{c}\bar{\nabla}_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_2)\\ -\Omega(\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\\ -\Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_2)\end{array}\right\}\\ &+ \left\{\begin{array}{c}-\Omega(\nabla_{\partial_k}\nabla_{\partial_k}\partial_1,\partial_2)\\ -\Omega(\partial_1,\nabla_{\partial_k}\nabla_{\partial_k}\partial_2)\end{array}\right\}\end{aligned}$$

$$
\begin{aligned}
&= 2\partial_k \left\{ \ \Omega(\nabla_{\partial_k}\partial_1 - \bar\nabla_{\partial_k}\partial_1, \partial_2) + \Omega(\partial_1, \nabla_{\partial_k}\partial_2 - \bar\nabla_{\partial_k}\partial_2) \ \right\} \\
&+ 2\Omega(\bar\nabla_{\partial_k}\partial_1, \bar\nabla_{\partial_k}\partial_2) - \bar\nabla_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) \\
&+ \left\{ \begin{array}{c} \Omega(\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_1 - \nabla_{\partial_k}\nabla_{\partial_k}\partial_1 + \bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_1, \partial_2) \\ +\Omega(\partial_1, \bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_2 - \nabla_{\partial_k}\nabla_{\partial_k}\partial_2 + \bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_2) \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= -2\partial_k \left\{ \Omega(B(\partial_k,\partial_1),\partial_2) + \Omega(\partial_1, B(\partial_k,\partial_2)) \right\} \\
&+ 2\Omega(B(\partial_k,\partial_1), B(\partial_k,\partial_2)) - B(\partial_k,\partial_k)\Omega(\partial_1,\partial_2) \\
&+\Omega(\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_1 - \nabla_k\nabla_k\partial_1 + \bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_1, \partial_2) \\
&+\Omega(\partial_1, \bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_2 - \nabla_k\nabla_k\partial_2 + \bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_2) \ .
\end{aligned}
$$

Now letting $B_{k1} := B(\partial_k, \partial_1)$ etc, denote the normal vector field along M, covariantly differentiating $\Omega\left(B_{k1}, \partial_2\right)$ etc gives

$$
\begin{aligned}
&\partial_k \left\{ \Omega(B(\partial_k,\partial_1),\partial_2) + \Omega(\partial_1, B(\partial_k,\partial_2)) \right\} \\
&= \bar\nabla_k\Omega(B_{k1},\partial_2) + \Omega(\bar\nabla_k B_{k1},\partial_2) + \Omega(B_{k1}, \bar\nabla_k\partial_2) \\
&+ \bar\nabla_k\Omega(\partial_1, B_{k2}) + \Omega(\partial_1, \bar\nabla_k B_{k2}) + \Omega(\bar\nabla_k\partial_1, B_{k2}).
\end{aligned}
$$

Recalling that we are at the origin of a normal coordinate system

$$
\begin{aligned}
&= \bar\nabla_k\Omega(B_{k1},\partial_2) + \Omega(\bar\nabla_k B_{k1},\partial_2) + \Omega(B_{k1}, B_{k2}) \\
&+ \bar\nabla_k\Omega(\partial_1, B_{k2}) + \Omega(\partial_1, \bar\nabla_k B_{k2}) + \Omega(B_{k1}, B_{k2}).
\end{aligned}
$$

Now plug this back into the larger expression

$$
\begin{aligned}
&\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\Omega(\partial_1,\partial_2) - \nabla_k\nabla_k\Omega(\partial_1,\partial_2) \\
&= -2\left\{ \begin{array}{c} \bar\nabla_k\Omega(B_{k1},\partial_2) + \Omega(\bar\nabla_k B_{k1},\partial_2) + \Omega(B_{k1}, B_{k2}) \\ +\bar\nabla_k\Omega(\partial_1, B_{k2}) + \Omega(\partial_1, \bar\nabla_k B_{k2}) + \Omega(B_{k1}, B_{k2}) \end{array} \right\} \\
&+ 2\Omega(B(\partial_k,\partial_1), B(\partial_k,\partial_2)) - B(\partial_k,\partial_k)\Omega(\partial_1,\partial_2) \\
&+\Omega(\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_1 - \nabla_{\partial_k}\nabla_{\partial_k}\partial_1 + \bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_1, \partial_2) \\
&+\Omega(\partial_1, \bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_2 - \nabla_{\partial_k}\nabla_{\partial_k}\partial_2 + \bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_2)
\end{aligned}
$$

$$
\begin{aligned}
&= -2\left\{ \begin{array}{c} \bar\nabla_k\Omega(B_{k1},\partial_2) + \Omega(\bar\nabla_k B_{k1},\partial_2) \\ \bar\nabla_k\Omega(\partial_1, B_{k2}) + \Omega(\partial_1, \bar\nabla_k B_{k2}) \end{array} \right\} \\
&- B_{kk}\Omega(\partial_1,\partial_2) - 2\Omega(B_{k1}, B_{k2}) \\
&+ \left\{ \begin{array}{c} \Omega(\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_1 - \nabla_{\partial_k}\nabla_{\partial_k}\partial_1 + \bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_1, \partial_2) \\ +\Omega(\partial_1, \bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_2 - \nabla_{\partial_k}\nabla_{\partial_k}\partial_2 + \bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_2) \end{array} \right\}.
\end{aligned}
$$

Now we inspect the last two terms

$$
\begin{aligned}
&\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\partial_1 - \nabla_{\partial_k}\nabla_{\partial_k}\partial_1 \\
&= \bar\nabla_{\partial_k}\left\{\nabla_{\partial_k}\partial_1 + B_{k1}\right\} - \nabla_{\partial_k}\nabla_{\partial_k}\partial_1 \\
&= \bar\nabla_{\partial_k}\left\{\Gamma^l_{k1}\partial_l + B_{k1}\right\} - \nabla_{\partial_k}\left(\Gamma^l_{k1}\partial_l\right) \\
&= \left\{\partial_k\Gamma^l_{k1}\partial_l + \Gamma^l_{k1}\left(\nabla_k\partial_l + B(k,l)\right) + \bar\nabla_{\partial_k}B_{k1}\right\} - \left(\partial_k\Gamma^l_{k1}\partial_l + \Gamma^l_{k1}\nabla_k\partial_l\right) \\
&= \bar\nabla_{\partial_k}B_{k1}.
\end{aligned}
$$

again using the fact that that Christoffel symbols vanish at the point. Thus

$$\begin{aligned}\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\Omega(\partial_1,\partial_2)-\nabla_k\nabla_k\Omega(\partial_1,\partial_2) &= -2\left\{\begin{array}{c}\bar\nabla_k\Omega(B_{k1},\partial_2)+\Omega(\bar\nabla_k B_{k1},\partial_2)\\ \bar\nabla_k\Omega(\partial_1,B_{k2})+\Omega(\partial_1,\bar\nabla_k B_{k2})\end{array}\right\}\\ &\quad - B_{kk}\Omega(\partial_1,\partial_2)-2\Omega(B_{k1},B_{k2})\\ &\quad +\left\{\begin{array}{c}\Omega(\bar\nabla_{\partial_k}B_{k1},\partial_2)\\ +\Omega(\partial_1,\bar\nabla_{\partial_k}B_{k2})\end{array}\right\}+\left\{\begin{array}{c}\Omega(\bar\nabla_{\partial_k}\bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_1,\partial_2)\\ +\Omega(\partial_1,\bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_2)\end{array}\right\}\end{aligned}$$

which further simplifies

$$\begin{aligned}&= -2\left\{\bar\nabla_k\Omega(B_{k1},\partial_2)+\bar\nabla_k\Omega(\partial_1,B_{k2})\right\}-2\Omega(B_{k1},B_{k2})\\ &- \left\{\Omega(\bar\nabla_{\partial_k}B_{k1},\partial_2)+\Omega(\partial_1,\bar\nabla_{\partial_k}B_{k2})\right\}\\ &+\Omega(\bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_1,\partial_2)+\Omega(\partial_1,\bar\nabla_{\bar\nabla_{\partial_k}\partial_k}\partial_2)-B_{kk}\Omega(\partial_1,\partial_2)\end{aligned}$$

$$\begin{aligned}&= -2\left\{\bar\nabla_k\Omega(B_{k1},\partial_2)+\bar\nabla_k\Omega(\partial_1,B_{k2})\right\}-2\Omega(B(\partial_k,\partial_1),B(\partial_k,\partial_2))\\ &- \left\{\Omega(\bar\nabla_{\partial_k}B_{k1},\partial_2)+\Omega(\partial_1,\bar\nabla_{\partial_k}B_{k2})\right\}\\ &-\bar\nabla_{B_{kk}}\Omega(\partial_1,\partial_2).\end{aligned}\tag{4.4}$$

Now using alternating nature of Ω and tangential and normal decomposition, we have

$$\begin{aligned}\Omega(\bar\nabla_k B_{k1},\partial_2) &= \Omega(\left(\bar\nabla_k B_{k1}\right)^T+\left(\bar\nabla_k B_{k1}\right)^N,\partial_2)\\ &= \Omega(\left(\left(\bar\nabla_k B_{k1}\right)\cdot\partial_1\right)\partial_1,\partial_2)+\Omega(\left(\bar\nabla_k B_{k1}\right)^N,\partial_2)\\ &= -B_{k1}\cdot B_{k1}\Omega(\partial_1,\partial_2)+\Omega(\left(\bar\nabla_k B_{k1}\right)^N,\partial_2)\end{aligned}$$

where we have used

$$\begin{aligned}0 &= \bar\nabla_k\left(B_{k1}\cdot\partial_1\right)\\ &= \bar\nabla_k B_{k1}\cdot\partial_1+\left(B_{k1}\cdot\bar\nabla_k\partial_1\right)\\ &= \bar\nabla_k B_{k1}\cdot\partial_1+B_{k1}\cdot B_{k1}.\end{aligned}$$

again using $\nabla_k\partial_1=0$ at the point. Thus (4.4) becomes

$$\begin{aligned}\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\Omega(\partial_1,\partial_2)-\nabla_k\nabla_k\Omega(\partial_1,\partial_2) &= -2\left\{\bar\nabla_k\Omega(B_{k1},\partial_2)+\bar\nabla_k\Omega(\partial_1,B_{k2})\right\}-2\Omega(B_{k1},B_{k2})\\ &\quad +B_{k1}\cdot B_{k1}\Omega(\partial_1,\partial_2)-\Omega(\left(\bar\nabla_k B_{k1}\right)^N,\partial_2)\\ &\quad +B_{k2}\cdot B_{k2}\Omega(\partial_1,\partial_2)-\Omega(\partial_1,\left(\bar\nabla_k B_{k1}\right)^N)\\ &\quad -\bar\nabla_{B_{kk}}\Omega(\partial_1,\partial_2).\end{aligned}$$

At this point we begin summing over k

$$\begin{aligned}\sum_k\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\Omega(\partial_1,\partial_2)-\nabla_k\nabla_k\Omega(\partial_1,\partial_2) &= -2\left\{\bar\nabla_k\Omega(B_{k1},\partial_2)+\bar\nabla_k\Omega(\partial_1,B_{k2})\right\}-2\Omega(B_{k1},B_{k2})\\ &\quad -\|B\|^2\Omega(\partial_1,\partial_2)\\ &\quad -\Omega(\left(\bar\nabla_k B_{k1}\right)^N,\partial_2)-\Omega(\partial_1,\left(\bar\nabla_k B_{k2}\right)^N)\\ &\quad -\bar\nabla_{B_{kk}}\Omega(\partial_1,\partial_2).\end{aligned}$$

using$-\|B\|^2$ to denote a negative number.

Now we still have left the terms

$$\bar{\nabla}_k B_{k1} \cdot n_p$$

The following version of the Codazzi equation can easily verified to hold on pseudo-Riemmanian manifold

Proposition 4.4. *(Codazzi Equation) Let η be a normal vector. Then,*

$$\bar{\nabla}_X B(Y,Z,\eta) - \bar{\nabla}_Y B(X,Z,\eta) = Rm_h\left(X,Y,Z,\eta\right).$$

In our case, this gives us

$$\bar{\nabla}_k B(\partial_1,\partial_k,n_p) - \bar{\nabla}_1 B(\partial_k,\partial_k,n_p) = Rm_h\left(\partial_k,\partial_1,\partial_k,n_p\right)$$

that is

$$\bar{\nabla}_k B(\partial_1,\partial_k,n_p) = \bar{\nabla}_1 B(\partial_k,\partial_k,n_p) + Rm_h\left(\partial_k,\partial_1,\partial_k,n_p\right). \tag{4.5}$$

Now

$$\begin{aligned}\bar{\nabla}_k B(\partial_1,\partial_k,n_p) &= \partial_k\left(B_{1k}\cdot n_p\right) - B(\bar{\nabla}_{\partial_k}\partial_1,\partial_k,n_p) - B(\partial_1,\bar{\nabla}_{\partial_k}\partial_k,n_p) - B(\partial_1,\partial_k,\bar{\nabla}_k n_p)\\ &= \bar{\nabla}_{\partial_k} B_{1k}\cdot n_p + B_{1k}\cdot\bar{\nabla}_{\partial_k} n_p - B(\bar{\nabla}_{\partial_k}\partial_1,\partial_k,n_p)\\ &\quad - B(\partial_1,\bar{\nabla}_{\partial_k}\partial_k,n_p) - B(\partial_1,\partial_k,\bar{\nabla}_{\partial_k} n_p).\end{aligned}$$

So we can use the Codazzi formula (4.5) to get

$$\begin{aligned}\bar{\nabla}_k B_{1k}\cdot n_p &= \bar{\nabla}_k B(\partial_1,\partial_k,n_p) - B_{1k}\cdot\bar{\nabla}_{\partial_k} n_p + B(\bar{\nabla}_{\partial_k}\partial_1,\partial_k,n_p)\\ &\quad + B(\partial_1,\bar{\nabla}_{\partial_k}\partial_k,n_p) + B(\partial_1,\partial_k,\bar{\nabla}_{\partial_k} n_p)\\ &= \bar{\nabla}_{\partial_1} B(\partial_k,\partial_k,n_p) + Rm_h\left(\partial_k,\partial_1,\partial_k,n_p\right)\\ &\quad - B_{1k}\cdot\bar{\nabla}_{\partial_k} n_p + B(\bar{\nabla}_{\partial_k}\partial_1,\partial_k,n_p) + B(\partial_1,\bar{\nabla}_{\partial_k}\partial_k,n_p) + B_{k1}\cdot\bar{\nabla}_{\partial_k} n_p)\\ &= \partial_1 B(\partial_k,\partial_k,n_p) - B(\bar{\nabla}_1\partial_k,\partial_k,n_p) - B(\partial_k,\bar{\nabla}_1\partial_k,n_p) - B(\partial_k,\partial_k,\bar{\nabla}_1 n_p)\\ &\quad + Rm_h\left(\partial_k,\partial_1,\partial_k,n_p\right) + B(\bar{\nabla}_k\partial_1,\partial_k,n_p) + B(\partial_1,\bar{\nabla}_k k,n_p).\end{aligned}$$

Summing over k and noting that B evaluates the tangential components of $\bar{\nabla}_{\partial_k}\partial_k$, which vanish at the point, we have

$$\begin{aligned}\sum_k \bar{\nabla}_k B_{1k}\cdot n_p &= \sum_k \partial_1 B(\partial_k,\partial_k,n_p) - B(\partial_k,\partial_k,\bar{\nabla}_1 n_p) + Rm_h\left(\partial_k,\partial_1,\partial_k,n_p\right)\\ &= \partial_1(H\cdot n_p) - H\cdot\bar{\nabla}_1 n_p + \sum_k Rm_h\left(\partial_k,\partial_1,\partial_k,n_p\right)\\ &= \bar{\nabla}_1 H\cdot n_p + Rm_h\left(\partial_2,\partial_1,\partial_2,n_p\right)\end{aligned}$$

in the last line, using symmetries of Riemannian tensor along with fact that $n=2$. Thus we can represent the normal component as

$$\left(\sum_k \bar{\nabla}_k B_{1k}\right)^N = \left(\bar{\nabla}_1 H\cdot n_p + Rm_h\left(\partial_2,\partial_1,\partial_2,n_p\right)\right) n^{ps} n_s.$$

So now we have

$$\begin{aligned}\sum_k \Omega(\left(\bar{\nabla}_k B_{k1}\right)^N,\partial_2) &= \Omega(\left(\bar{\nabla}_1 H\cdot n_p + Rm_h\left(\partial_2,\partial_1,\partial_2,n_p\right)\right) n^{ps} n_s,\partial_2)\\ &= \Omega(\left(\bar{\nabla}_1\left(\bar{\nabla} f\right)^N\cdot n_p + Rm_h\left(\partial_2,\partial_1,\partial_2,n_p\right)\right) n^{ps} n_s,\partial_2)\end{aligned}$$

Now note that

$$\partial_1\left(\left(\bar\nabla f\right)^N\cdot n_p\right)=\partial_1\left(\left(\bar\nabla f\right)\cdot n_p\right)$$
$$=\bar\nabla^2 f(n_p,\partial_1)+\left(\bar\nabla_{\partial_1}n_p\right)f$$

So

$$\bar\nabla_1 H\cdot n_p=\partial_1\left(\left(\bar\nabla f\right)^N\cdot n_p\right)-\left(\bar\nabla f\right)^N\cdot\bar\nabla_1 n_p$$
$$=\bar\nabla^2 f(n_p,\partial_1)+\left(\bar\nabla_{\partial_1}n_p\right)f-\left(\bar\nabla f\right)^N\cdot\bar\nabla_1 n_p$$
$$=\bar\nabla^2 f(n_p,\partial_1)-\left(\bar\nabla f\right)^T\cdot\bar\nabla_1 n_p$$

that is

(4.6)

$$\sum_k\Omega(\left(\bar\nabla_k B_{k1}\right)^N,\partial_2)=\Omega(\left(\bar\nabla^2 f(n_p,\partial_1)-\left(\bar\nabla f\right)^T\cdot\bar\nabla_1 n_p+R\left(2,1,2,n_p\right)\right)n^{ps}n_s,\partial_2).$$

Now we go back and plug (4.6) in

$$\sum_k\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\Omega(\partial_1,\partial_2)-\nabla_k\nabla_k\Omega(\partial_1,\partial_2)$$
$$=-2\left\{\bar\nabla_k\Omega(B_{k1},\partial_2)+\bar\nabla_k\Omega(\partial_1,B_{k2})\right\}-2\Omega(B_{k1},B_{k2})$$
$$-\left\|B\right\|^2\Omega(\partial_1,\partial_2)$$
$$-\Omega(\left(\bar\nabla^2 f(n_p,\partial_1)-\left(\bar\nabla f\right)^T\cdot\bar\nabla_1 n_p+Rm_h\left(\partial_2,\partial_1,\partial_2,n_p\right)\right)n^{ps}n_s,\partial_2)$$
$$-\Omega(\partial_1,\left(\bar\nabla^2 f(n_p,\partial_2)-\left(\bar\nabla f\right)^T\cdot\bar\nabla_2 n_p+Rm_h\left(\partial_1,\partial_2,\partial_1,n_p\right)\right)n^{ps}n_s)$$
$$-\bar\nabla_{B_{kk}}\Omega(\partial_1,\partial_2).$$

This finishes the proof of Lemma 4.3.

4.0.2. *More computations.* Still at the point $(x_0,\bar x_0)$, we have

Lemma 4.5.

$$Rm(\partial_2,\partial_1,\partial_2,n_1)=R_{\bar 1 2 2\bar 1}\left(\frac{\lambda_1}{\lambda_2}\right)^2-R_{1\bar 2\bar 2 1}\left(\frac{\lambda_2}{\lambda_1}\right)^2$$
$$Rm(\partial_2,\partial_1,\partial_2,n_2)=2\left[R_{2\bar 1\bar 2 2}\left(\frac{\lambda_1}{\lambda_2}\right)+R_{\bar 2 1\bar 2 2}\left(\frac{\lambda_2}{\lambda_1}\right)\right]$$
$$Rm(\partial_1,\partial_2,\partial_1,n_1)=-2R_{1\bar 2 1\bar 1}\frac{\lambda_2}{\lambda_1}-2R_{\bar 1 2\bar 1 1}\frac{1}{\lambda_2}\lambda_1$$
$$Rm(\partial_1,\partial_2,\partial_1,n_2)=R_{1\bar 2\bar 2 1}\left(\frac{\lambda_2}{\lambda_1}\right)^2-R_{2\bar 1\bar 1 2}\left(\frac{\lambda_1}{\lambda_2}\right)^2$$

where

$$R_{\bar 1 2 2\bar 1}=Rm_h(\bar e_1,e_2,e_2,\bar e_1),\ \textit{etc.}$$

Proof. Note that in the Kim-McCann metric [KM10, Lemma 4.1], these computations become massively simplified by the fact that the only curvature terms that do not vanish are those with two barred and two unbarred indices. There will be six nontrivial terms in each expression. Straightforward computations using the symmetries of the curvature tensor yield the results. □

Lemma 4.6. *At the point* $(x_0, \bar{x}_0)$ *we have*

$$\begin{aligned}
\Omega(\partial_1, n_1) &= 0 \\
\Omega(\partial_1, n_2) &= \frac{1}{\lambda_1\lambda_2}\left(-\lambda_1^2\lambda_2^2 + \lambda_1^2 - \lambda_2^2 + 1\right) \\
\Omega(\partial_2, n_1) &= -\frac{1}{\lambda_1\lambda_2}\left(-\lambda_1^2\lambda_2^2 - \lambda_1^2 + \lambda_2^2 + 1\right) \\
\Omega(\partial_2, n_2) &= 0 \\
\Omega(\partial_1, \partial_2) &= \frac{1}{\lambda_1\lambda_2}\left(\lambda_1^2\lambda_2^2 + \lambda_1^2 + \lambda_2^2 + 1\right) \\
\Omega(n_1, n_2) &= \frac{1}{\lambda_1\lambda_2}\left(\lambda_1^2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 + 1\right)
\end{aligned}$$

Proof. Calculation. □

Now combining the above two lemmas, we get

Corollary 4.7. *As above,*

$$\begin{aligned}
&2\sum_p \left[\Omega(Rm(\partial_2, \partial_1, \partial_2, n_p)n_p, \partial_2) + \Omega(\partial_1, Rm(\partial_1, \partial_2, \partial_1, n_p)n_p)\right] \\
&= 4\frac{1}{\lambda_1\lambda_2}\left(\lambda_2^2 - \lambda_1^2\right)\left\{R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2 - R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right\}.
\end{aligned}$$

Proof.

$$\begin{aligned}
&2\sum_p \left[\Omega(Rm(\partial_2, \partial_1, \partial_2, n_p)n_p, \partial_2) + \Omega(\partial_1, Rm(\partial_1, \partial_2, \partial_1, n_p)n_p)\right] \\
&= 2Rm(\partial_2, \partial_1, \partial_2, n_1)\frac{1}{\lambda_1\lambda_2}\left(-\lambda_1^2\lambda_2^2 - \lambda_1^2 + \lambda_2^2 + 1\right) \\
&+ 2Rm(\partial_1, \partial_2, \partial_1, n_2)\frac{1}{\lambda_1\lambda_2}\left(-\lambda_1^2\lambda_2^2 + \lambda_1^2 - \lambda_2^2 + 1\right) \\
&= 2\left[R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2 - R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right]\frac{1}{\lambda_1\lambda_2}\left(-\lambda_1^2\lambda_2^2 - \lambda_1^2 + \lambda_2^2 + 1\right) \\
&+ 2\left[R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2 - R_{2\bar{1}\bar{1}2}\left(\frac{\lambda_1}{\lambda_2}\right)^2\right]\frac{1}{\lambda_1\lambda_2}\left(-\lambda_1^2\lambda_2^2 + \lambda_1^2 - \lambda_2^2 + 1\right) \\
&= 4\frac{1}{\lambda_1\lambda_2}\left(\lambda_2^2 - \lambda_1^2\right)\left\{R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2 - R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right\}.
\end{aligned}$$

□

Lemma 4.8.

$$\|B\|^2\,\Omega(\partial_1, \partial_2) + 2\Omega(B(\partial_k, \partial_1), B(\partial_k, \partial_2)) \geq \frac{2}{\lambda_1\lambda_2}\|B\|^2.$$

Proof. Taking

$$h_{ijs} = (B(\partial_i, \partial_j)\cdot n_s)$$

$$\begin{aligned}
\|B\|^2 &= \sum_{i,j,s} h_{ijs}^2 \\
2\Omega(B(\partial_k,\partial_1),B(\partial_k,\partial_2)) &= \sum_{k,s,p} 2h_{k1s}h_{k2p}\Omega(n_s,n_p) \\
&= \sum_k 2h_{k11}h_{k22}\Omega(n_1,n_2) + \sum_k 2h_{k12}h_{k21}\Omega(n_2,n_1) \\
&= \sum_k 2\left(h_{k11}h_{k22}-h_{k12}h_{k21}\right)\frac{1}{\lambda_1\lambda_2}\Omega(n_1,n_2) \\
&= \sum_k 2\left(h_{k11}h_{k22}-h_{k12}h_{k21}\right)\frac{1}{\lambda_1\lambda_2}\left\{(\lambda_1-1)(\lambda_2-1)\right\} \\
&\leq \frac{1}{\lambda_1\lambda_2}\sum_k \left(h_{k11}^2+h_{k22}^2+h_{k12}^2+h_{k21}^2\right)|(\lambda_1-1)(\lambda_2-1)| \\
&= \frac{1}{\lambda_1\lambda_2}\sum_{k,s,p} h_{ksp}^2\,|(\lambda_1-1)(\lambda_2-1)|
\end{aligned}$$

On the other hand

$$\|B\|^2\,\Omega(\partial_1,\partial_2) = \sum_{i,j,s} h_{ijs}^2\left\{\frac{1}{\lambda_1\lambda_2}(\lambda_1+1)(\lambda_2+1)\right\}.$$

It follows that

(4.7)

$$-\|B\|^2\,\Omega(\partial_1,\partial_2) - 2\sum_k \Omega(B(\partial_k,\partial_1),B(\partial_k,\partial_2)) \leq \frac{1}{\lambda_1\lambda_2}\left\{|(\lambda_1-1)(\lambda_2-1)|-|(\lambda_1+1)(\lambda_2+1)|\right\}\|B\|^2 < 0.$$

Now we take two cases. First case, $\lambda_2 \leq 1$. In this case we have

$$\frac{1}{\lambda_1\lambda_2}\left\{|(\lambda_1-1)(\lambda_2-1)|-|(\lambda_1+1)(\lambda_2+1)|\right\} = -\frac{2(1+\lambda_1\lambda_2)}{\lambda_1\lambda_2} < -2\frac{1}{\lambda_1\lambda_2}.$$

In the other case $\lambda_1 > \lambda_2 > 1$ we have

$$\frac{1}{\lambda_1\lambda_2}\left\{|(\lambda_1-1)(\lambda_2-1)|-|(\lambda_1+1)(\lambda_2+1)|\right\} = -\frac{2(\lambda_1+\lambda_2)}{\lambda_1\lambda_2} < -2\frac{1}{\lambda_1\lambda_2}.$$

□

Finally, we will need the following.

Lemma 4.9. *For some universal constants C*

$$|2\nabla_k\Omega(B_{k1},\partial_2) - 2\nabla_k\Omega(\partial_1,B_{k2})| \leq \frac{2}{\lambda_1\lambda_2}\|B\|^2 + \frac{\lambda_1\lambda_2}{2}C\left(\frac{1}{\lambda_1^2}+\lambda_1^2\right)\left(\frac{1}{\lambda_2^2}+\lambda_2^2\right)$$

$$\left|\sum_k -\bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\Omega(\partial_1,\partial_2)\right| \leq C\left[\left(\frac{1}{\lambda_1^2}+\lambda_1^2\right)\left(\frac{1}{\lambda_2^2}+\lambda_2^2\right)\right]^{1/2}\left[\frac{1}{\lambda_1^2}+\lambda_1^2+\frac{1}{\lambda_2^2}+\lambda_2^2\right]$$

$$\nabla_{B(k,k)}\Omega(\partial_1,\partial_2) \leq C\sqrt{\frac{1}{\lambda_1^2}+\lambda_1^2}\sqrt{\frac{1}{\lambda_2^2}+\lambda_2^2}$$

$$\Omega(\left(\bar{\nabla}^2 f(n_p,\partial_1) - \left(\bar{\nabla} f\right)^T \cdot \bar{\nabla}_1 n_p\right) n^{ps} n_s, \partial_2) \leq C\left(\frac{1}{\lambda_1^2} + \lambda_1^2\right)\frac{1}{\lambda_1\lambda_2}\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\right)$$
$$\Omega(\partial_1, \left(\bar{\nabla}^2 f(n_p,\partial_2) - \left(\bar{\nabla} f\right)^T \cdot \bar{\nabla}_2 n_p\right) n^{ps} n_s) \leq C\left(\frac{1}{\lambda_2^2} + \lambda_2^2\right)\frac{1}{\lambda_1\lambda_2}\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\right)$$

Proof. Note that the size of the components of the $\nabla_k\Omega$ itself are controlled by a priori bounds on the derivatives of Ω and the length of the vector ∂_k. As

$$|\partial_1| = \sqrt{\frac{1}{\lambda_1^2} + \lambda_1^2}$$

we have

$$2\nabla_k\Omega(B_{k1},\partial_2) \leq 2C\sqrt{\frac{1}{\lambda_1^2} + \lambda_1^2}\,\|B\|\sqrt{\frac{1}{\lambda_2^2} + \lambda_2^2} \leq \frac{2}{\lambda_1\lambda_2}\|B\|^2 + \frac{\lambda_1\lambda_2}{2}C\left(\frac{1}{\lambda_1^2} + \lambda_1^2\right)\left(\frac{1}{\lambda_2^2} + \lambda_2^2\right).$$

Next

$$\left|\bar{\nabla}_{\partial_1}\bar{\nabla}_{\partial_1}\Omega(\partial_1,\partial_2)\right| \leq \left|D^2\Omega\right|\left|\partial_1\right|^3\left|\partial_2\right| \leq C\left(\frac{1}{\lambda_1^2} + \lambda_1^2\right)^{3/2}\left(\frac{1}{\lambda_2^2} + \lambda_2^2\right)^{1/2} = C\left[1 + \frac{\lambda_1^2}{\lambda_2^2}\right]\lambda_1\lambda_2$$

or

$$\left|\bar{\nabla}_{\partial_2}\bar{\nabla}_{\partial_2}\Omega(\partial_1,\partial_2)\right| \leq \left|D^2\Omega\right|\left|\partial_1\right|\left|\partial_2\right|^3 \leq C\left(\frac{1}{\lambda_1^2} + \lambda_1^2\right)^{1/2}\left(\frac{1}{\lambda_2^2} + \lambda_2^2\right)^{3/2} \leq C\left[1 + \frac{\lambda_1^2}{\lambda_2^2}\right]\frac{1}{\lambda_1\lambda_2}.$$

Next, using the minimal surface equation (1.5) we have

$$\begin{aligned}\bar{\nabla}_{B(k,k)}\Omega(\partial_1,\partial_2) &= -\bar{\nabla}_{\left(\bar{\nabla} f\right)^N}\Omega(\partial_1,\partial_2)\\ &\leq |Df|\,|D\Omega|\sqrt{\frac{1}{\lambda_1^2} + \lambda_1^2}\sqrt{\frac{1}{\lambda_2^2} + \lambda_2^2}.\end{aligned}$$

Next, when considering terms of the form $\Omega(\left(\bar{\nabla}^2 f(n_p,\partial_1) - \left(\bar{\nabla} f\right)^T \cdot \bar{\nabla}_1 n_p\right) n^{ps} n_s, \partial_2)$ note that we only need consider (by Lemma 4.6) terms with $s = p = 1$.

$$\begin{aligned}\Omega(\left(\bar{\nabla}^2 f(n_1,\partial_1) - \left(\bar{\nabla} f\right)^T \cdot \bar{\nabla}_1 n_1\right) n_1, \partial_2) &\leq \left(\bar{\nabla}^2 f\right)|n_1|\,|\partial_1|\frac{1}{\lambda_1\lambda_2}\left(\lambda_1^2\lambda_2^2 + \lambda_1^2 - \lambda_2^2 - 1\right)\\ &\leq C\left(\frac{1}{\lambda_1^2} + \lambda_1^2\right)\frac{1}{\lambda_1\lambda_2}\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\right)\end{aligned}$$

□

We collect the above computations in the following:

$$
\begin{aligned}
&-\sum_k \nabla_k \nabla_k \Omega(\partial_1, \partial_2) \\
&\leq C\left[\left(\frac{1}{\lambda_1^2}+\lambda_1^2\right)\left(\frac{1}{\lambda_2^2}+\lambda_2^2\right)\right]^{1/2}\left[\frac{1}{\lambda_1^2}+\lambda_1^2+\frac{1}{\lambda_2^2}+\lambda_2^2\right]-\frac{2}{\lambda_1\lambda_2}\left\|B\right\|^2. \\
&+\frac{2}{\lambda_1\lambda_2}\left\|B\right\|^2+\frac{\lambda_1\lambda_2}{2}C\left(\frac{1}{\lambda_1^2}+\lambda_1^2\right)\left(\frac{1}{\lambda_2^2}+\lambda_2^2\right)+C\sqrt{\frac{1}{\lambda_1^2}+\lambda_1^2}\sqrt{\frac{1}{\lambda_2^2}+\lambda_2^2} \\
&+4\frac{1}{\lambda_1\lambda_2}\left(\lambda_2^2-\lambda_1^2\right)\left\{R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2-R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right\} \\
&C\left(\frac{1}{\lambda_1^2}+\lambda_1^2\right)\frac{1}{\lambda_1\lambda_2}\left(\lambda_1^2\lambda_2^2+\lambda_1^2\right)
\end{aligned}
$$

Now recalling, (4.1), (2.2), (2.1), as the densities are bounded above and away from zero, with respect to the volume forms for g_1 and g_2, there is a constant C such that

$$\frac{1}{C_\rho} \leq \lambda_1\lambda_2 \leq C_\rho.$$

Using this, recalling that

$$\lambda_1 \geq \lambda_2$$

and using

$$\frac{1}{C_\rho}\lambda_1 \leq \frac{1}{\lambda_2} \leq C_\rho\lambda_1$$

we continue the above computation

$$
\begin{aligned}
&-\sum_k \nabla_k \nabla_k \Omega(\partial_1, \partial_2) \\
&\leq C(\lambda_1^4+1)+4\frac{1}{C_\rho}\left(\lambda_2^2-\lambda_1^2\right)\frac{1}{C_\rho^2}R_{\bar{1}22\bar{1}}\lambda_1^4.
\end{aligned}
$$

Now suppose that the maximum value for μ happens at a point $x_0 \in M$. By the maximum principle,using the fact that the covariant derivative commutes with the Hodge-star operator, we have

$$\sum_k \nabla_{\partial_k}\nabla_{\partial_k}\Omega(\partial_1, \partial_2) = \sum_k \partial_k\partial_k\mu \leq 0.$$

In particular

$$0 \leq C(\lambda_1^4+1)+4\frac{1}{C_\rho}\left(\lambda_2^2-\lambda_1^2\right)\frac{1}{C_\rho^2}R_{\bar{1}22\bar{1}}\lambda_1^4.$$

or

$$c_0 R_{\bar{1}22\bar{1}}\lambda_1^6 \leq C(\lambda_1^4+1)$$

for some positive c_0 bounded away from 0 and C bounded above. Because we are on the compact manifold the positive quantity $R_{\bar{1}22\bar{1}}$ has a strict minimum, and we can conclude that

$$\lambda_1 \leq C_2$$

for some constant C_2. Recalling (3.5) this proves Proposition 4.1

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