MINIMAL LAGRANGIAN SUBMANIFOLDS OF WEIGHTED KIM-MCCANN METRICS.

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Abstract. We explore the regularity theory of optimal transport maps for costs satisfying a Ma-Trudinger-Wang condition, by viewing the graphs of the transport maps as maximal Lagrangian surfaces with respect to an appropriate pseudo-Riemannian metric on the product space. We recover the local regularity theory in two-dimensional manifolds.

1. Introduction

When giving an alternate formulation of the Ma-Trudinger-Wang regularity theory, Kim and McCann [KM10] defined pseudo-Riemmanian metrics with signature (n,n) on the product space $M \times \bar{M}$. This was followed [KMW10] by the observation that the graph of the optimal transportation map is a volume maximizing n-dimensional submanifold (of codimension n), with respect to a conformal modification of the metric in [KM10]. In this note, we explore the idea that one can directly derive regularity theory purely from the property of being maximal. If the metric is of Kim-McCann type, a calibrated Lagrangian submanifold is either the graph of a solution to an optimal transportation problem, or, if the manifold has topology, could be the graph of a Lie solution to the optimal transportation problem [War11]. Here, to keep things simple, we restrict our attention to the case when $n = 2$.

Recall the setting of Kim-McCann [KM10] where metrics on the product space $M \times \bar{M}$ are locally given by

$$h = \begin{pmatrix} 0 & h_{i\bar{j}}(x,\bar{x}) \\ h_{i\bar{j}}^T(x,\bar{x}) & 0 \end{pmatrix} \tag{1.1}$$

where

$$h_{i\bar{j}}(x,\bar{x}) = -\frac{1}{2}\partial_{\bar{j}}\partial_i c(x,\bar{x}) \tag{1.2}$$

for some cost function $c : M \times \bar{M} \to \mathbb{R}$. We assume here and in the sequel that c is twice differentiable and satisfies the (A2) condition on $\mathcal{N} \subset M \times \bar{M} \backslash C$ where C is a measure zero set which we call the 'cut locus'. In particular, $h_{i\bar{j}}(x,\bar{x})$ is non-degenerate on $\mathcal{N}$, where h defines an (n,n) signature metric.

The underlying optimal transportation problem is to find the map between the measures defined by bounded mass densities $\rho, \bar{\rho}$ that minimizes the total cost:

$$\int_M c(x, T(x))\rho(x)dx$$

over the space of maps T satisfying

$$T_{\#}\rho(x)dx = \bar{\rho}(\bar{x})d\bar{x}.$$

In [KMW10] it was illustrated that when taking the conformal metric

$$\tilde{h} = \left[\frac{\rho(x)\bar{\rho}\,(\bar{x})}{\det\left(-\partial_{\bar{j}}\partial_i c(x,\bar{x})\right)}\right]^{1/n} \begin{pmatrix} 0 & h_{i\bar{j}}\,(x,\bar{x}) \\ h^T_{i\bar{j}}\,(x,\bar{x}) & 0 \end{pmatrix} \tag{1.3}$$

graphs of optimal transportion plans T are calibrated maximal submanifolds with respect to this metric [KMW10, Theorem 1.1]. The conformal modification of the Kim-Mcann metric can sometimes lead to technical computational issues. So here we point out that a weighted approach can be convenient: Consider instead the weighted manifold

$$\left(M \times \bar{M}, h, \left(\frac{\rho(x)\bar{\rho}(\bar{x})}{\det\left(-\partial_{\bar{j}}\partial_i c(x,\bar{x})\right)}\right)^{1/2}\right). \tag{1.4}$$

One can check that the volume of an n-submanifold with this weight is the same as the volume of the n-submanifold in the conformal setting defined by (1.3). Thus the minimal surfaces are the same in either setting. However, instead of the minimal surface equation occuring in the setting (1.3), in the latter setting (1.4), the manifolds will be locally defined by the weighted minimal surface equation

$$H + (\nabla f)^N = 0 \tag{1.5}$$

where

$$e^{-f} = \left(\frac{\rho(x)\bar{\rho}(\bar{x})}{\det\left(-\partial_{\bar{j}}\partial_i c(x,\bar{x})\right)}\right)^{1/2}.$$

Here we offer a proof-of-concept that the regularity theory can be derived geometrically, obtaining yet another approach for the Ma-Trudinger-Wang regularity theory. The original paper [MTW05] presents a maximum principle argument applied to the Monge-Ampère equation, while the approach by Loeper [Loe09] is a careful analysis of the cost function which requires less a priori regularity. The setting of Kim-McCann offers a solid pseudo-Riemannian geometric formulation of the approach offered by Loeper, where they identify the MTW condition as a cross-curvature condition on the metric h, namely

$$Rm_h(e_i, e_{\bar{j}}, e_{\bar{j}}, e_i) > 0$$

whenever $e_i \in T_pM, e_{\bar{j}} \in T_{\bar{p}}\bar{M}$ and $h(e_i, e_{\bar{j}}) = 0$. The cross-curvature positivity is preserved under conformal changes [KMW10, Remark 4.2] so if it is present in the Kim-McCann metric (1.1) it will also be present in the metric (1.3).

To significantly simplify the computation we restrict to the case of compact ambient manifolds of the form $M \times \bar{M}$ with metric of the form (1.1) where both manifolds have dimension $n = 2$. While we focus on the case where the metric is of the Kim-McCann form (1.2), it should be possible to loosen this to more general pseudo-Riemmanian submanifolds satisfying suitable curvature conditions. Indeed, our approach is motivated by the paper [LS11] where general regularity results follow from curvature conditions.

Recently Brendle-Léger-McCann-Rankin [BLMR23] have proven regularity for maximal surfaces in pseudo-Riemannian metrics with positive cross-curvature in general dimensions via a generalized method which applies to maximal surfaces in manifolds of general codimension. The method is similar but slightly different: here our strategy is to bound the Hodge dual of an alternating two form (following [LS11]), whereas the approach in [BLMR23] is to bound the maximum eigenvalue of a symmetric $(0, 2)$ tensor field restricted to the submanifold. Naturally, both approaches rely on covariantly differentiating the $(0, 2)$ tensor twice and using the positive curvature terms that arise via the Codazzi formula, together with the minimal surface equation. [1]

2. Prelimaries

We consider the graph

$$\Gamma = \{(x, T(x) : x \in M\} \subset M \times \bar{M}$$

for some function

$$T : M \to \bar{M}.$$

Assume that the graph Γ lies in a compact subset $\mathcal{N} \subset M \times \bar{M} \backslash C$ where C is the cut locus where c might not be smooth. Here we choose $\mathcal{N}$ to be a compact subset staying clear of the cut locus. (An important aspect of regularity theory for optimal transport maps is the "stay-away" property, that is, there is an a priori lower bound on the distance of the graph to the cut locus. This is typically argued in the process of a regularity argument - but as we are focusing on an alternative approach to the local regularity, we may steer clear of these arguments by assuming the graph stays within a compact set avoiding the cut locus.)

Recall that Γ is a Lagrangian submanifold with respect to the symplectic form given by ([KM10, 5.3])

$$h_{i\bar{j}} dx^i \wedge d\bar{x}^j,$$

and that there is a relation between this symplectic form and the metric given by

$$h(V, W) = h_{i\bar{j}} dx^i \wedge d\bar{x}^j (KV, W) \tag{2.1}$$

where

$$K : T_{(p,\bar{p})}\left(M \times \bar{M}\right) \to T_{(p,\bar{p})}\left(M \times \bar{M}\right)$$

[1] An early version of this paper was posted on the author's website, and set aside during the pandemic. In the meantime, the authors of [BLMR23] developed their much more general approach.

is the map represented as

$$K_{p,\bar{p}} = I_{T_pM} \oplus (-I)_{T_{\bar{p}}\bar{M}}$$

that is, K acts as the identity on TM while performing a sign change on $T\bar{M}$. Since Γ is a Lagrangian submanifold, where (2.1) holds, K maps the tangent space to the normal space along Γ.

Claim 2.1. *K is parallel.*

Proof. Identifying K via musical isomorphism with the symplectic form, we can check that the symplectic form is parallel:

$$\bar{\nabla}_X h(KV, W) = Xh(KV, W) - h(K\bar{\nabla}_X V, W) - h(KV, \bar{\nabla}_X W).$$

First assume that V, W are in T_pM. In this case, by [KM10, Lemma 4.1] the connection maps T_pM to T_pM, so all terms above vanish. Similarly if both V, W are in $T_{\bar{p}}\bar{M}$. Now consider the case that $V \in T_pM, W \in T_{\bar{p}}\bar{M}$.

$$\bar{\nabla}_X h(KV, W) = Xh(V, W) - h(\bar{\nabla}_X V, W) - h(V, \bar{\nabla}_X W).$$

which is just the fact that h is parallel with respect to itself. On the other hand if roles are reversed

$$\bar{\nabla}_X h(KV, W) = Xh(-V, W) - h(-\bar{\nabla}_X V, W) - h(-V, \bar{\nabla}_X W).$$

Thus $K^\flat$ is parallel. □

Given a frame for the submanifold Γ, we may define the components of the second fundamental form

$$b_{ijk} = h(\bar{\nabla}_{\partial_i}\partial_j, K\partial_k).$$

Claim 2.2. *The expression for b_{ijk} is symmetric in all three indices.*

Proof. Because K is parallel, $K^2 = I$ and $h(K\cdot, K\cdot) = -h(\cdot, \cdot)$

$$\begin{aligned} 0 &= \partial_i h(\partial_j, K\partial_k) \\ &= h(\bar{\nabla}_{\partial_i}\partial_j, K\partial_k) + h(\partial_j, K\bar{\nabla}_{\partial_i}\partial_k) \\ &= h(\bar{\nabla}_{\partial_i}\partial_j, K\partial_k) - h(K\partial_j, \bar{\nabla}_{\partial_i}\partial_k) \\ &= b_{ijk} - b_{ikj}. \end{aligned}$$

This establishes symmetry in the last two. Symmetry in the first two is well-known for the second fundamental form. □

The following is the Ma-Trudinger-Wang curvature condition as uncovered by Kim and McCann in the product manifold setting [KM10, Def. 2.3].

Definition 2.3. The metric h is strictly regular, if whenever $(x, \bar{x}) \in N$ and

$$\begin{aligned} 0 &\neq \partial_i \in T_xM \\ 0 &\neq \partial_{\bar{j}} \in T_{\bar{x}}\bar{M} \\ h_{(x,\bar{x})}&(\partial_i, \partial_{\bar{j}}) = 0 \end{aligned}$$

then the curvature of the metric h satisfies

$$Rm_h\left(\partial_i, \partial_{\bar{j}}, \partial_{\bar{j}}, \partial_i\right) > 0.$$

Note that the metric $\tilde{h}$ will also satisfy this property [KMW10, Remark 4.2].

3. Special coordinate charts at a point

We start by fixing arbitrary Riemannian metrics g_1, g_2 on $M, \bar{M}$.

Let $(p, \bar{p}) \in M \times \bar{M}$. Given $\{e_1, e_2\}$ an oriented g_1-orthonormal basis for T_pM, we may take exponential coordinates (w.r.t g_1) near p for a neighborhood of M, and then do similarly for an oriented orthonormal basis for $T_{\bar{p}}\bar{M}$, giving us a local product neighborhood $U \times \bar{U}$. In this $U \times \bar{U}$, we have

$$h = \begin{pmatrix} 0 & h_{i\bar{j}}(x, \bar{x}) \\ h^T_{i\bar{j}}(x, \bar{x}) & 0 \end{pmatrix}.$$

Now take a local change of coordinates

$$\varphi : \tilde{U} \to \bar{U}$$

so that

$$\varphi(0) = \bar{p} \tag{3.1}$$

$$D\varphi = h_{i\bar{j}}(p, \bar{p})^{-1} \tag{3.2}$$

as h_{ij} is nondegenerate. After this coordinate change, the metric at the point $(p, \bar{p})$ with respect to the basis on $U \times \tilde{U}$ becomes

$$h = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3.3}$$

Now forgetting the old vertical basis vectors here, consider the new ones $\{\bar{e}_1, \bar{e}_2\}$ that occur after transformation $D\varphi$. These are not expected to be orthonormal with respect to g_2 but satisfy

$$h(e_i, \bar{e}_j) = \delta_{ij}$$

and

$$\frac{1}{C_1} \leq ||\bar{e}_i||_{g_2} \leq C_1 \tag{3.4}$$

for some universal constant C_1 depending on bounds on $h_{i\bar{j}}(p, \bar{p})^{-1}$ and our choice of g_2.

Now given a g_1-orthonormal $\{e_1, e_2\}$ basis at p and a graphical manifold Γ, let g be the induced metric from h on Γ, that is

$$g_{11} = h(e_1, e_1) \text{ etc.}$$

This can be diagonalized by an orthogonal rotation of $\{e_1, e_2\}$ so we may assume that the expression of the metric g is diagonal for our choice of $\{e_1, e_2\}$: There will be positive λ_1, λ_2 such that

$$\begin{aligned} \|e_i\|_g^2 &= \lambda_i^2, \ i = 1, 2. \\ \langle e_1, e_2 \rangle_g &= 0 \\ \langle e_i, e_j \rangle_{g_1} &= \delta_{ij} \end{aligned} \tag{3.5}$$

and WLOG $\lambda_1 \geq \lambda_2$. (Here we use e_i in two senses, identifying e_i first as an abstract tangent vector on the manifold M, which is measured by g_1, but also as a tangent vector to the submanifold Γ, which is measured by h, restricting to the metric g, which is positive definite on Γ as the submanifold Γ is spacelike).

It will be convenient to use the orthonormal basis with respect to g :

$$\begin{aligned} \partial_1 &= \frac{1}{\lambda_1} e_1 \\ \partial_2 &= \frac{1}{\lambda_2} e_2 \end{aligned}$$

so that

$$\|\partial_1\|_g = \|\partial_2\|_g = 1.$$

Using (3.5) and (3.3)

$$\begin{aligned} \partial_1 &= (\frac{1}{\lambda_1}, 0, \lambda_1, *) \\ \partial_2 &= (0, \frac{1}{\lambda_2}, *, \lambda_2) \end{aligned}$$

and then (3.3) with the fact that Γ is Lagrangian gives us (still only at the point)

$$\begin{aligned} \partial_1 &= (\frac{1}{\lambda_1}, 0, \lambda_1, 0) \\ \partial_2 &= (0, \frac{1}{\lambda_2}, 0, \lambda_2). \end{aligned} \tag{3.6}$$

At the point $(p, \bar{p})$, consider also the orthonormal basis for the normal space

$$\begin{aligned} n_1 &= (\frac{1}{\lambda_1}, 0, -\lambda_1, 0) \\ n_2 &= (0, \frac{1}{\lambda_2}, 0, -\lambda_2) \end{aligned} \tag{3.7}$$

which can be extended to a normal frame via

$$n_i := K\partial_i.$$

Now with respect to the Euclidean systems on $U \times \tilde{U}$ we have the representation

$$DT = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}. \tag{3.8}$$

4. Setup before the covariant differentiation

Our goal is to prove the following theorem, which is a special case of general work by Ma-Trudinger-Wang and Loeper.

Theorem 4.1. *Suppose that T is a smooth optimal transport plan from an oriented compact two-dimensional manifold M to an oriented compact two-dimensional manifold $\bar{M}$ with respect to a cost function c which satisfies a positive cross-curvature condition (Definition 2.3), and measures ρ and $\bar{\rho}$. Assume that the densities of $\rho, \bar{\rho}$ are smooth and bounded away from zero, and that the graph of T lies in an a priori determined compact set $\mathcal{N}$ on which c is smooth. Then with respect to any given metrics g_1 and g_2 on M and $\bar{M}$, the derivative DT satisfies an a priori bound:*

$$\|DT\| \leq C(g_1, g_2, c, \rho, \bar{\rho}, \mathcal{N})$$

where

$$\|DT\| = \max\left\{\|DT\,(V)\|_{g_2} \mid V \in T_xM, \|V\|_{g_1} = 1\right\}.$$

Choose g_1, g_2 to be arbitrary Riemannian metrics on $M, \bar{M}$. These metrics will be fixed and will serve as a gauge against which to obtain estimates. We assume that M is oriented, so on any neighborhood U we can take an oriented orthonormal (w.r.t g_1) frame $\{e_1, e_2\}$ for U. Now define

$$\Omega = \left[g_1\,(e_1, \cdot) + h_1(e_1, \cdot)\right] \wedge \left[g_1\,(e_2, \cdot) + h_1(e_2, \cdot)\right] \tag{4.1}$$

to be a two-form on $U \times \bar{M}$. One can check that this does not depend on the choice of oriented orthonormal frame $\{e_1, e_2\}$ so using a partition of unity, we may extend Ω to a well-defined form everywhere where h is defined on $M \times \bar{M}$. Our goal is to show that the maximum value of the Hodge dual of this two-form restricted to Γ is a priori bounded.

4.1. Jacobian condition.

Given charts, the objects $\rho, \bar{\rho}$ can be represented by measure densities, so when we write down the Jacobian equation satisfied by an optimal map (or more generally, a map whose graph is calibrated as in [KMW10]):

$$\det DT(x)\bar{\rho}(T(x)) = \rho(x)$$

we are implictly using coordinate systems on both sides to define $\det DT(x)$ and each of the densities. Given a choice of metric g_1 we can define $\rho\,(x)$ to be a well-defined density by letting

$$\rho(x) = \frac{d\rho}{dV_{g_1}}$$

and similar for $\bar{\rho}$. Fixing these metrics, if

$$0 < \frac{1}{\mu_1} \leq \rho \leq \mu_1$$
$$0 < \frac{1}{\mu_2} \leq \bar{\rho} \leq \mu_2$$

we may conclude that

$$\frac{1}{\Lambda} \leq \det DT \leq \Lambda \tag{4.2}$$

in any coordinates with $dV_{g_1}(x) = dV_{g_2}(T(x)) = 1$. By (4.2) it follows that there exists $\Lambda(h, g_1, g_2, \rho, \bar{\rho})$ (maybe slightly different from previous Λ) such that

$$\frac{1}{\Lambda} \leq \lambda_1 \lambda_2 \leq \Lambda \tag{4.3}$$

provided the measure densities are continuous and bounded away from 0 and $M, \bar{M}$ are compact.

4.2. **Maximum Principle argument.** With the metric g on Γ, we may consider the scalar function

$$w := *\Omega|_\Gamma$$

that is, the ratio

$$w = \frac{\Omega|_\Gamma(\partial_1, \partial_2)}{dV_g(\partial_1, \partial_2)}$$

for any tangent frame ∂_1, ∂_2. This function attains a maximum value at some point $(x_{\max}, T(x_{\max})) \in \Gamma$.

Claim 4.2. *Theorem 4.1 follows from an a priori bound on the function w.*

Proof. Suppose $w(x) \leq \bar{C}$. Then for any x, choosing special coordinates (3.5) (3.6), (3.7) (3.8) we get (see Claim 6.2)

$$\begin{aligned} w(x) &= \frac{\Omega(\partial_1, \partial_2)}{1} \\ &= \frac{1}{\lambda_1 \lambda_2}\left(1 + \lambda_1^2\right)(1 + \lambda_2^2) \leq \bar{C}. \end{aligned}$$

From (3.8) and (3.4) we have

$$\|DT(e_i)\|_{g_2} = \left\|\lambda_i^2 \bar{e}_i\right\|_{g_2} \leq \lambda_i^2 C_1$$

thus

$$\max\left\{\|DT(V)\|_{g_2} \mid V \in T_x M, \|V\|_{g_1} = 1\right\} \leq \Lambda \bar{C} C_1.$$

□

Theorem 4.1 will be proved as follows: Go to $(x_{\max}, T(x_{\max}))$ and apply the maximum principle. This requires an expression for the covariant derivative of Ω computed in the next section.

5. The covariant differentiation

Lemma 5.1. *Suppose that $\Gamma = (x, T(x))$ is a submanifold with mean curvature vector $\vec{H}$ and Ω is as described in (4.1). At $(x_{\max}, \bar{x}_{\max}) \in \Gamma$, with frames defined by (3.6) and (3.7) we have*

$$\begin{aligned}
\sum_k \nabla_{\partial_k}\nabla_{\partial_k}\Omega(\partial_1,\partial_2) - \sum_k \bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\Omega(\partial_1,\partial_2)&\\
&= \|B\|^2\,\Omega(\partial_1,\partial_2) + 2\sum_k \Omega(B(\partial_k,\partial_1), B(\partial_k,\partial_2))\\
&+ 2\sum_k\left\{\bar{\nabla}_{\partial_k}\Omega(B(\partial_k,\partial_1),\partial_2) + \bar{\nabla}_{\partial_k}\Omega(\partial_1, B(\partial_k,\partial_2))\right\} + \bar{\nabla}_{\vec{H}}\Omega(\partial_1,\partial_2)\\
&- \sum_p\left\{Rm_h\left(\partial_2,\partial_1,\partial_2,n_p\right)\Omega(n_p,\partial_2) + Rm_h\left(\partial_1,\partial_2,\partial_1,n_p\right)\Omega(\partial_1,n_p)\right\}\\
&- \sum_s\left\{\left(\bar{\nabla}_{\partial_1}\vec{H}\cdot n_p\right)\Omega(n_p,\partial_2) + \left(\bar{\nabla}_{\partial_2}\vec{H}\cdot n_p\right)\Omega(\partial_1,n_p)\right\}.
\end{aligned}$$

Here, $\|B\|^2$ is the positive norm of the second fundamental form B for the submanifold Γ, and, Rm_h is the curvature tensor of h.

Proof. To begin, extend tangent vectors $\{\partial_1, \partial_2\}$ by taking these to be the coordinate derivatives with respect to normal coordinates on g. Then we differentiate covariantly with respect to g :

$$\nabla_{\partial_k}\Omega(\partial_1,\partial_2) = \partial_k\Omega(\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\nabla_{\partial_k}\partial_2)$$

and again

$$\begin{aligned}
\nabla_{\partial_k}\nabla_{\partial_k}\Omega(\partial_1,\partial_2) &= \partial_k\left\{\partial_k\Omega(\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\nabla_{\partial_k}\partial_2)\right\}\\
&- \left\{\begin{array}{c}
\nabla_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\nabla_{\nabla_{\partial_k}\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\nabla_{\nabla_{\partial_k}\partial_k}\partial_2)\\
\partial_k\Omega(\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\nabla_{\partial_k}\partial_2)\\
\partial_k\Omega(\partial_1,\nabla_{\partial_k}\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\nabla_{\partial_k}\partial_2) - \Omega(\partial_1,\nabla_{\partial_k}\nabla_{\partial_k}\partial_2)
\end{array}\right\}\\
&= \partial_k\left\{\partial_k\Omega(\partial_1,\partial_2) - \Omega(\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\nabla_{\partial_k}\partial_2)\right\}\\
&- \left\{\begin{array}{c}
\partial_k\Omega(\nabla_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\left(\nabla_{\partial_k}\partial_1\right)^T,\partial_2)\\
\partial_k\Omega(\partial_1,\nabla_{\partial_k}\partial_2) - \Omega(\partial_1,(\nabla_{\partial_k}\left(\nabla_{\partial_k}\partial_2\right))^T)
\end{array}\right\}.
\end{aligned}$$

discarding terms that vanish when taking normal coordinates. (Implicity Ω is identified with its restriction to Γ; normal covariant differentiation is via the connection on normal bundle.) Now repeat with respect to the ambient connection:

$$\begin{aligned}
\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\Omega(\partial_1,\partial_2) &= \partial_k\left\{\partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\right\}\\
&- \left\{\begin{array}{c}
\bar{\nabla}_{\partial_k}\partial_k\Omega(\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_1,\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_2)\\
\partial_k\Omega(\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_1,\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2)\\
\partial_k\Omega(\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\bar{\nabla}_{\partial_k}\partial_1,\bar{\nabla}_{\partial_k}\partial_2) - \Omega(\partial_1,\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_2)
\end{array}\right\}
\end{aligned}$$

Computing the difference, using definition of second fundamental form

$$B(\partial_k,\partial_j) = \bar{\nabla}_{\partial_k}\partial_j - \nabla_{\partial_k}\partial_j$$

we get

(5.1)

$$\nabla_{\partial_k}\nabla_{\partial_k}\Omega(\partial_1,\partial_2)-\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\Omega(\partial_1,\partial_2)=2\partial_k\left\{\Omega(B(\partial_k,\partial_1),\partial_2)+\Omega(\partial_1,B(\partial_k,\partial_2))\right\}$$
$$+\Omega(\bar{\nabla}_{\partial_k}\left(\nabla_{\partial_k}\partial_1\right)^T-\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_1-\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_1,\partial_2)$$
$$+\Omega(\partial_1,(\nabla_{\partial_k}\left(\nabla_{\partial_k}\partial_2\right))^T-\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_2-\bar{\nabla}_{\bar{\nabla}_{\partial_k}\partial_k}\partial_2)$$
$$-2\Omega\left(B(\partial_k,\partial_1),B(\partial_k,\partial_2)\right)+B(\partial_k,\partial_k)\Omega(\partial_1,\partial_2).$$

Summing over k, recalling that we have chosen an orthonormal basis at the point

$$\vec{H}=B(\partial_k,\partial_k).$$

Noting also that in normal coordinates (introducing $B_{k1}:=B(\partial_k,\partial_1)$ etc., as short-hand)

$$\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\partial_1-\nabla_{\partial_k}\nabla_{\partial_k}\partial_1=\bar{\nabla}_{\partial_k}\left\{\nabla_{\partial_k}\partial_1+B_{k1}\right\}-\nabla_{\partial_k}\nabla_{\partial_k}\partial_1$$
$$=\bar{\nabla}_{\partial_k}\left\{\Gamma^l_{k1}\partial_l+B_{k1}\right\}-\nabla_{\partial_k}\left(\Gamma^l_{k1}\partial_l\right)$$
$$=\left\{\partial_k\Gamma^l_{k1}\partial_l+\Gamma^l_{k1}\left(\nabla_{\partial_k}\partial_l+B(k,l)\right)+\bar{\nabla}_{\partial_k}B_{k1}\right\}-\left(\partial_k\Gamma^l_{k1}\partial_l+\Gamma^l_{k1}\nabla_k\partial_l\right)$$
$$=\bar{\nabla}_{\partial_k}B_{k1}. \tag{5.2}$$

Now in the first line of (5.1) we may use

$$\partial_k\Omega(B_{k1},\partial_2)=\bar{\nabla}_{\partial_k}\Omega(B_{k1},\partial_2)+\Omega\left(\bar{\nabla}_{\partial_k}B_{k1},\partial_2\right)+\Omega\left(B_{k1},\bar{\nabla}_{\partial_k}\partial_2\right)$$

together with (5.2) to get

$$\nabla_{\partial_k}\nabla_{\partial_k}\Omega(\partial_1,\partial_2)-\bar{\nabla}_{\partial_k}\bar{\nabla}_{\partial_k}\Omega(\partial_1,\partial_2)=2\bar{\nabla}_{\partial_k}\Omega(B_{k1},\partial_2)+2\bar{\nabla}_{\partial_k}\Omega(\partial_1,B_{k2})$$
$$+\bar{\nabla}_H\Omega(\partial_1,\partial_2)+2\Omega\left(B_{k1},\bar{\nabla}_{\partial_k}\partial_2\right)$$
$$+\Omega\left(\bar{\nabla}_{\partial_k}B_{k1},\partial_2\right)+\Omega(\partial_1,\bar{\nabla}_{\partial_k}B_{k2}). \tag{5.3}$$

Now using the alternating nature of Ω and tangential and normal decomposition, we have

$$\Omega(\bar{\nabla}_{\partial_k}B_{k1},\partial_2)=\Omega\left(\left(\bar{\nabla}_{\partial_k}B_{k1}\right)^T+\left(\bar{\nabla}_{\partial_k}B_{k1}\right)^N,\partial_2\right)$$
$$=\Omega\left(\left(\left(\bar{\nabla}_{\partial_k}B_{k1}\right)\cdot\partial_1\right)\partial_1,\partial_2\right)+\Omega\left(\left(\bar{\nabla}_{\partial_k}B_{k1}\right)^N,\partial_2\right)$$
$$=-B_{k1}\cdot B_{k1}\Omega(\partial_1,\partial_2)+\Omega\left(\left(\bar{\nabla}_{\partial_k}B_{k1}\right)^N,\partial_2\right).$$

Here we have used

$$0=\bar{\nabla}_{\partial_k}\left(B_{k1}\cdot\partial_j\right)$$
$$=\bar{\nabla}_{\partial_k}B_{k1}\cdot\partial_j+\left(B_{k1}\cdot\bar{\nabla}_{\partial_k}\partial_j\right)$$
$$=\bar{\nabla}_{\partial_k}B_{k1}\cdot\partial_j+B_{k1}\cdot B_{kj}.$$

Now let

$$\|B\|^2=-\left(B_{k1}\cdot B_{k1}+B_{k2}\cdot B_{k2}\right)$$

where the positive sign is appropriate as B_{kj} are time-like vectors, and (5.3) becomes

$$2\bar{\nabla}_{\partial_k}\Omega(B_{k1},\partial_2) + 2\bar{\nabla}_{\partial_k}\Omega(\partial_1,B_{k2}) + \bar{\nabla}_H\Omega(\partial_1,\partial_2) + 2\Omega\left(B_{k1},\bar{\nabla}_{\partial_k}\partial_2\right) + \|B\|^2\,\Omega(\partial_1,\partial_2) + \Omega\left(\left(\bar{\nabla}_{\partial_k}B_{k1}\right)^N,\partial_2\right) + \Omega\left(\left(\bar{\nabla}_{\partial_k}B_{k1}\right)^N,\partial_2\right). \tag{5.4}$$

The Codazzi equation holds in pseudo-Riemannian manifolds: If X, Y, Z are vectors on Γ and and η a normal vector, then

$$\bar{\nabla}_X\left(B(Y,Z)\right)\cdot\eta - \bar{\nabla}_Y\left(B(X,Z)\right)\cdot\eta = Rm_h\left(X,Y,Z,\eta\right).$$

In our case, this gives at $(x_{\max},\bar{x}_{\max})$

$$\left(\bar{\nabla}_{\partial_k}B_{1k}\right)\cdot\eta = \left(\bar{\nabla}_{\partial_1}B_{kk}\right)\cdot\eta + Rm_h\left(\partial_k,\partial_1,\partial_k,\eta\right)$$

thus

$$\left(\bar{\nabla}_{\partial_k}B_{1k}\right)^N = \left(\bar{\nabla}_{\partial_1}H\cdot n_p + Rm_h\left(\partial_2,\partial_1,\partial_2,n_p\right)\right)n^{pq}n_q$$

where we are using the negative definite $n_{pq} := n_p\cdot n_q$ (which is just $\left(-\delta_{pq}\right)$ at $(x_{\max},\bar{x}_{\max})$). So now we have

$$\Omega\left(\left(\bar{\nabla}_k B_{k1}\right)^N,\partial_2\right) = \Omega\left(\left(\bar{\nabla}_{\partial_1}H\cdot n_p + Rm_h\left(\partial_2,\partial_1,\partial_2,n_p\right)\right)n^{pq}n_q,\partial_2\right). \tag{5.5}$$

Substituting this into (5.4), using the fact that n^{pq} is negative definite provides the expression in the statement of the Lemma. □

6. Curvature and other computations with eigenvalues

First we need:

Claim 6.1. *At* $(x_{\max},\bar{x}_{\max})\in\Gamma$, *with frames defined by (3.6) and (3.7) we have*

$$\begin{aligned} Rm_h(\partial_2,\partial_1,\partial_2,n_1) &= -Rm_h(\partial_1,\partial_2,\partial_1,n_2)\\ &= R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2 - R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2 \end{aligned}$$

where

$$R_{\bar{1}22\bar{1}} = Rm_h(E_{\bar{1}},E_2,E_2,E_{\bar{1}}),\ \textit{etc.}$$

for $E_i\in T_pM$ *and* $E_{\bar{j}}\in T_p\bar{M}$.

Proof. Note that in the Kim-McCann metric [KM10, Lemma 4.1], these computations become massively simplified by the fact that the only curvature terms that do not vanish are those with two barred and two unbarred indices, so there will be at most six nontrivial terms in the expression for $Rm(\partial_2,\partial_1,\partial_2,n_1)$. Straightforward computations using the symmetries of the curvature tensor and (3.6), (3.7) yield the result. □

Claim 6.2. *At the point* $(x_{\max}, \bar{x}_{\max})$ *we have*

$$\begin{aligned}
\Omega\left(\partial_1, n_1\right) &= 0 \\
\Omega\left(\partial_1, n_2\right) &= \frac{1}{\lambda_1\lambda_2}\left(1+\lambda_1^2\right)(1-\lambda_2^2) \\
\Omega\left(\partial_2, n_1\right) &= -\frac{1}{\lambda_1\lambda_2}\left(1-\lambda_1^2\right)(1+\lambda_2^2) \\
\Omega\left(\partial_2, n_2\right) &= 0 \\
\Omega\left(\partial_1, \partial_2\right) &= \frac{1}{\lambda_1\lambda_2}\left(1+\lambda_1^2\right)(1+\lambda_2^2) \qquad (6.1)\\
\Omega\left(n_1, n_2\right) &= \frac{1}{\lambda_1\lambda_2}\left(1-\lambda_1^2\right)(1-\lambda_2^2).
\end{aligned}$$

Proof. Straightforward calculation noting that at the point

$$\begin{aligned}
\left(dx^i + dy^i\right)\partial_j &= \frac{1}{\lambda_i}\delta_{ij}\left(1+\lambda_i^2\right) \\
\left(dx^i + dy^i\right)n_j &= \frac{1}{\lambda_i}\delta_{ij}\left(1-\lambda_i^2\right).
\end{aligned}$$

□

Now combing the above two lemmas, we get

Corollary 6.3. *As above,*

$$\begin{aligned}
&\Omega(Rm_h(\partial_2, \partial_1, \partial_2, n_p)n_p, \partial_2) + \Omega(\partial_1, Rm_h(\partial_1, \partial_2, \partial_1, n_p)n_p) \\
&= 2\frac{1}{\lambda_1\lambda_2}\left(\lambda_2^2 - \lambda_1^2\right)\left\{R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2 - R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right\}.
\end{aligned}$$

Proof. As $\Omega\left(\partial_i, n_i\right) = 0$ we have

$$\begin{aligned}
&\Omega\left(Rm_h(\partial_2, \partial_1, \partial_2, n_p)n_p, \partial_2\right) + \Omega\left(\partial_1, Rm_h(\partial_1, \partial_2, \partial_1, n_p)n_p\right) \\
&= Rm_h(\partial_2, \partial_1, \partial_2, n_1)\frac{1}{\lambda_1\lambda_2}\left(1-\lambda_1^2\right)(1+\lambda_2^2) \\
&+ Rm_h(\partial_1, \partial_2, \partial_1, n_2)\frac{1}{\lambda_1\lambda_2}\left(1+\lambda_1^2\right)(1-\lambda_2^2).
\end{aligned}$$

From Claim 6.1 the expression becomes

$$\begin{aligned}
&= \frac{1}{\lambda_1\lambda_2}\left[R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2 - R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right]\left\{\begin{array}{c}\left(1-\lambda_1^2\right)(1+\lambda_2^2) \\ -\left(1+\lambda_1^2\right)(1-\lambda_2^2)\end{array}\right\} \\
&= 2\frac{1}{\lambda_1\lambda_2}\left(\lambda_2^2 - \lambda_1^2\right)\left\{R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2 - R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right\}.
\end{aligned}$$

□

Claim 6.4. *Suppose that $\lambda_2 \le 1$. Then for some constant C_2 depending on $\|Df\|_{g_1\times g_2}$, g_1, g_2 and Λ*

$$2\Omega\left(B(\partial_k,\partial_1),B(\partial_k,\partial_2)\right) \ge -\frac{\|B\|^2}{3}\frac{(1+\lambda_1^2)(1+\lambda_1^2)}{\lambda_1\lambda_2} - C_2(1+\lambda_1^4)$$

Proof. The minimal surface equation

$$\vec{H} = -(\nabla f)^N$$

can be expressed in normal coordinates as the following

$$\begin{aligned} b_{111} + b_{221} &= -\nabla f \cdot n_1 \\ b_{112} + b_{222} &= -\nabla f \cdot n_2, \end{aligned}$$

By Claim 2.2 we then have

$$b_{k22} = -\nabla f \cdot n_k - b_{k11}.$$

Now

$$\begin{aligned} 2\Omega\left(B(\partial_k,\partial_1),B(\partial_k,\partial_2)\right) &= \sum_{k,s,p} 2b_{k1s}b_{k2p}\Omega(n_s,n_p) \\ &= \sum_k 2b_{k11}b_{k22}\Omega(n_1,n_2) + \sum_k 2b_{k12}b_{k21}\Omega(n_2,n_1) \\ &= \sum_k 2\left(b_{k11}b_{k22} - b_{k12}b_{k21}\right)\frac{1}{\lambda_1\lambda_2}(\lambda_1^2-1)(\lambda_2^2-1) \\ &= \sum_k 2\left(b_{k11}\left(-\nabla f\cdot n_k - b_{k11}\right) - b_{k12}^2\right)\frac{1}{\lambda_1\lambda_2}(\lambda_1^2-1)(\lambda_2^2-1) \\ &= 2\sum_k b_{k11}\left(-\nabla f\right)\cdot n_k\frac{1}{\lambda_1\lambda_2}(\lambda_1^2-1)(\lambda_2^2-1) \\ &- 2\sum_k\left(b_{k11}^2 + b_{k12}^2\right)\frac{1}{\lambda_1\lambda_2}(\lambda_1^2-1)(\lambda_2^2-1) \end{aligned}$$

$$\ge -\left(\frac{1}{\beta}\sum_k \|B\|^2\left|\nabla f\cdot n_k\right|^2 + \frac{\beta}{4}\right)\cdot\frac{1}{\lambda_1\lambda_2}(\lambda_1^2+1)(\lambda_2^2+1) \tag{6.2}$$

Where we may drop the term in the second-to-last line using $(\lambda_1^2 - 1) < 0$.

Next, recalling that in our product system coordinates (with respect to Eulidean metric)

$$\|\partial_k\|_{g_1\times g_2} = \|n_k\|_{g_1\times g_2} \le \sqrt{\frac{1}{\lambda_k^2} + C_1^2\lambda_k^2}$$

we may use (4.3) to conclude that

$$\begin{aligned} |\partial_1|, |n_1| &\le \sqrt{\Lambda + C_1^2\lambda_1^2} \\ |\partial_2|, |n_2| &\le \sqrt{\Lambda\lambda_1^2 + C_1^2\Lambda}. \end{aligned}$$

or

$$|\partial_k| \leq C_0\sqrt{1+\lambda_1^2}. \tag{6.3}$$

Then evaluating

$$\nabla f\cdot n_k = Df(n_k) \leq C_0\sqrt{1+\lambda_1^2}\,\|Df\|_{g_1\times g_2} \leq C_2'(1+\lambda_1^2)^{1/2}$$

for some C_2' that depends on $\|Df\|_{g_1\times g_2}$, C_0 and Λ. Thus letting $\beta = 3\left(C_2'\right)^2(1+\lambda_1^2)$ in (6.2) gives the result. □

Claim 6.5. *For some constants C_3, C_4 depending on $\left\|D^2\Omega\right\|_{g_1\times g_2}$ and Λ.*

$$2\left(\bar\nabla_{\partial_k}\Omega(B_{k1},\partial_2)+\bar\nabla_{\partial_k}\Omega(\partial_1,B_{k2})\right) \geq -\frac{\|B\|^2}{3}\frac{(1+\lambda_1^2)(1+\lambda_2^2)}{\lambda_1\lambda_2} - C_3\left(1+\lambda_1^4\right)$$

and

$$\sum_k \bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\Omega(\partial_1,\partial_2) \geq -C_4(1+\lambda_1^4).$$

Proof. Recalling (6.3)

$$\begin{aligned}\left|\bar\nabla_{\partial_k}\Omega(B_{k1},\partial_2)\right| &\leq \|D\Omega\|_{g_1\times g_2}\|\partial_k\|_{g_1\times g_2}|b_{k1s}|\,\|n_s\|_{g_1\times g_2}\|\partial_2\|_{g_1\times g_2}\\ &\leq \|D\Omega\|_{g_1\times g_2}\sum_{k,s}|b_{k1s}|\left(C_0\left(1+\lambda_1^2\right)^{1/2}\right)^3\\ &\leq \frac{\|B\|^2}{12}\frac{\left(1+\lambda_1^2\right)(1+\lambda_2^2)}{\lambda_1\lambda_2} + 48\,\|D\Omega\|^2_{g_1\times g_2}C_0^6\frac{\lambda_1\lambda_2}{(1+\lambda_2^2)}\left(1+\lambda_1^2\right)^2.\end{aligned}$$

Noting that $\frac{\lambda_1\lambda_2}{(1+\lambda_2^2)}$ is controlled, and repeating the same computation for $\left|\bar\nabla_{\partial_k}\Omega(\partial_1,B_{k2})\right|$ gives the first inequality.

For the next inequality we may directly compute

$$\left|\sum_k \bar\nabla_{\partial_k}\bar\nabla_{\partial_k}\Omega(\partial_1,\partial_2)\right| \leq 2\left\|D^2\Omega\right\|_{g_1\times g_2}\left[C_0\sqrt{1+\lambda_1^2}\right]^4.$$

□

Finally, we bound terms involving the mean curvature

Claim 6.6. *For some constants C_5, C_6*

$$\nabla_{\vec H}\Omega(\partial_1,\partial_2) \geq -C_5\left(1+\lambda_1^4\right)$$

$$-\sum_s\left\{\left(\bar\nabla_{\partial_1}\vec H\cdot n_1\right)\Omega(n_1,\partial_2)+\left(\bar\nabla_{\partial_2}\vec H\cdot n_2\right)\Omega(\partial_1,n_2)\right\} \geq -\frac{\|B\|^2}{3}\frac{(1+\lambda_1^2)(1+\lambda_2^2)}{\lambda_1\lambda_2} - C_6\left(1+\lambda_1^4\right).$$

Proof. Note that

$$\begin{aligned}(\nabla f)^N &= ((\nabla f)\cdot n_s)\,n^{sp}n_p\\ &= (df(n_s))\,n^{sp}n_p,\end{aligned}$$

thus we have (recall (6.3))

$$\begin{aligned}\left\|\vec{H}\right\|_{g_1\times g_2} &= \left\|(\nabla f)^N\right\|_{g_1\times g_2}\\ &\leq |df(n_s)|\left\|n_p\right\|_{g_1\times g_2}\\ &\leq \|Df\|_{g_1\times g_2} C_0^2\left(1+\lambda_1^2\right).\end{aligned}$$

Then

$$\left|\nabla_{\vec{H}}\Omega(\partial_1,\partial_2)\right| \leq \|Df\|_{g_1\times g_2}\|D\Omega\|_{g_1\times g_2} C_0^4\left(1+\lambda_1^2\right)^2.$$

Next, note that

$$\begin{aligned}\left(\bar{\nabla}_{\partial_1}\vec{H}\cdot n_1\right) &= \bar{\nabla}_{\partial_1}\left(\bar{\nabla} f\right)^N\cdot n_1\\ &= \left(\bar{\nabla}_{\partial_1}\left(\bar{\nabla} f\right) - \bar{\nabla}_{\partial_1}\left(\bar{\nabla} f\right)^T\right)\cdot n_1\\ &= \bar{\nabla}^2 f\left(\partial_1, n_1\right) - \sum_k\left(\partial_k f\right) b_{1k1}\\ &\geq -\left\|D^2 f\right\|_{g_1\times g_2} C_0^2\left(1+\lambda_1^2\right) - \frac{\|B\|^2}{6} - 2\|Df\|^2_{g_1\times g_2} C_0^2\left(1+\lambda_1^2\right).\end{aligned}$$

The inequality follows from adding the terms and using

$$\left|\Omega\left(\partial_1, n_2\right)\right|, \left|\Omega\left(n_1,\partial_2\right)\right| \leq \frac{1}{\lambda_1\lambda_2}\left(1+\lambda_1^2\right)(1+\lambda_2^2).$$

□

7. Finish It

We return to the maximum principle argument.

Proof of Theorem 4.1. Pick an $x_{\max}$ in Γ where the function w is maximized. The scalar function $w = \frac{\Omega}{dV_g}$ is simply the Hodge dual of $\Omega|_\Gamma$. Covariant differentiation commutes with the Hodge operator, so

$$\frac{\nabla_{\partial_k}\nabla_{\partial_k}\Omega\left(\partial_1,\partial_2\right)}{dV_g\left(\partial_1,\partial_2\right)} = \left(\frac{\Omega}{dV_g}\right)_{kk} \leq 0.$$

Assume that $\lambda_2 \leq 1$; If not, then $\lambda_1 \leq \Lambda$ and we immediately have a bound on w.

Applying Lemma 5.1, and all the claims in the previous section, we have

$$\begin{aligned}0 \geq \left(\frac{\Omega}{dV_g}\right)_{kk} &\geq -\left(C_2+C_3+C_4+C_5+C_6\right)(1+\lambda_1^4)\\ &+ 2\frac{1}{\lambda_1\lambda_2}\left(\lambda_1^2-\lambda_2^2\right)\left\{R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1}{\lambda_2}\right)^2 - R_{1\bar{2}\bar{2}1}\left(\frac{\lambda_2}{\lambda_1}\right)^2\right\}\\ &\geq -C_{10}\left(1+\lambda_1^4\right) + 2\frac{1}{\Lambda}\left(\lambda_1^2-\Lambda\right)\left(R_{\bar{1}22\bar{1}}\left(\frac{\lambda_1^2}{\Lambda}\right)^2 - R_{1\bar{2}\bar{2}1}\right).\end{aligned}$$

Now the components $R_{\bar{1}22\bar{1}}$ are with respect to the vectors that are uniformly bounded and bounded away from zero in our gauge metrics, so $R_{\bar{1}22\bar{1}}$ has a known lower bound. We may conclude that

$$\lambda_1 \leq C_{20}$$

and thus

$$w(x_{\max}) = \frac{\left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)}{\lambda_1\lambda_2} \leq \bar{C} = 2\Lambda(1+C_{20}^2).$$

Considering Claim 4.2, Theorem 4.1 is proved. □

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