A GEOMETRIC FLOW TOWARDS HAMILTONIAN STATIONARY SUBMANIFOLDS

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Abstract. In this paper, we introduce a geometric flow for Lagrangian submanifolds in a Kähler manifold that stays in its initial Hamiltonian isotopy class and is a gradient flow for volume. The stationary solutions are the Hamiltonian stationary Lagrangian submanifolds. The flow is not strictly parabolic but it corresponds to a fourth order strictly parabolic scalar equation in the cotangent bundle of the submanifold via Weinstein's Lagrangian neighborhood theorem. For any compact initial Lagrangian immersion, we establish short-time existence, uniqueness, and higher order estimates when the second fundamental forms are uniformly bounded up to time T.

1. Introduction

The objective of this paper is to introduce a fourth order flow of Lagrangian submanifolds in a Kähler manifold as a gradient flow of volume within a Hamiltonian isotopy class and establish basic properties such as short-time existence, uniqueness, and extendibility with bounded second fundamental form.

Our setting includes a Kähler manifold (M^{2n}, h, ω, J) with symplectic form ω and compatible Kähler metric h satisfying $h(JV, W) = \omega(V, W)$ where J is a complex structure, and a given compact Lagrangian immersion $\iota : L^n \to M^{2n}$. We propose to find $F : L \times [0, T) \to M^{2n}$ satisfying

$$\frac{dF}{dt} = J\nabla \operatorname{div}(JH) \tag{1.1}$$

$$F(\cdot, 0) = \iota(\cdot) \tag{1.2}$$

where H is the mean curvature vector of $L_t = F(\cdot, t)$ in M and $\nabla, \operatorname{div}$ are along L_t in the induced metric from h.

The stationary solutions of the above evolution equation are the so-called Hamiltonian stationary submanifolds, which are fourth order generalizations of special Lagrangians, and exist in more abundance. Within a Hamiltonian isotopy class it is possible for a compact Lagrangian submanifold of $\mathbb{C}^n$ to minimize volume (for example, a Clifford torus). Meanwhile, compact special Lagrangian submanifolds of $\mathbb{C}^n$ do not exist. Recall that two manifolds are *Hamiltonian isotopic* if they can be joined by the flow generated by a vector field of the form $J\nabla f$ for a function f.

We will show that the initial value problem (1.1) - (1.2):

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(1) stays within the Hamiltonian isotopy class - that is, the flow is generated by a vector field of the form $J\nabla f$,
(2) is the gradient flow of volume with respect to an appropriate metric, so decreases volume along the flow,
(3) enjoys short time existence given smooth initial conditions, and
(4) continues to exist as long as a second fundamental form bound is satisfied.

The flow (1.1) is degenerate parabolic but not strictly parabolic. Our proof of existence and uniqueness involves constructing global solutions (locally in time) using Weinstein's Lagrangian neighbourhood theorem, which results in a nice fourth order parabolic *scalar* equation. This equation has a good structure, satisfying conditions required in [MM12] so we can conclude uniqueness and existence within a given Lagrangian neighbourhood. We then argue that the flows described by solutions to the scalar fourth order equations are in correspondence with the normal flows of the form (1.1) leading to uniqueness and extendability provided the flow remains smooth.

After proving existence and uniqueness using Weinstein neighborhoods, we turn to local Darboux charts to prove higher regularity from second fundamental form bounds. It's not immediately clear how to extract regularity from arbitrary Weinstein neighbourhoods as the submanifolds move, but using a description of Darboux charts in [JLS11] we can fix a finite set of charts and perform the regularity theory in these charts. Unlike mean curvature flow, there is no maximum principle for fourth order equations. We deliver a regularity theory using Sobolev spaces. Smoczyk [**?**, Theorem 1.9] has shown the basic fact that mean curvature flow preserves the Lagrangian property, by a maximum principle argument, while our flow evolves within a Hamiltonian class.

The newly introduced flow already exhibits nice properties in special cases:

(1) For Calabi-Yau manifolds, Harvey and Lawson [HL82] showed that, for a Lagrangian submanifold, the Lagrangian angle θ generates the mean curvature via

$$H = J\nabla\theta.$$

In this case (1.1) becomes

$$\frac{dF}{dt} = -J\nabla\Delta\theta,$$

while θ satisfies the pleasant fourth order parabolic equation

$$\frac{d\theta}{dt} = -\Delta_g^2\theta$$

where g is the induced metric on L_t.

(2) The case $n = 1$ involves curves in $\mathbb{C}$. Hamiltonian isotopy classes bound a common signed area. Higher order curvature flows have been studied and are called polyharmonic heat flow or curve diffusion flow (cf. [PW16], [EI05], [May01]), dealing with evolution in the form $\gamma' = (-1)^{p-1}\kappa_{s^{2p}}$ where $\kappa_{s^{2p}}$ is the $2p$-th order derivative of the curvature κ of a plane curve γ with respect to the arclength s. For $p = 0$ it is the standard curve shortening flow. The flow discussed here corresponds to $p = 1$.

The case $n = 1$ arises in material science [Mul99]. For more work on the pure side, see [Whe13] and the many references therein. In [Whe13] it is shown that curves near a circle have flows which exist forever and converge to a round circle, as well as a short-time existence theorem that requires only L^2 initial curvature. It is not difficult to argue that for immersed figure 8 type curves singularities must develop in finite time. It is curious to know whether there exist embedded curves which develop singularities in finite time (cf. [EI05], [PW16], [May01]).

2. Gradient Flow

We begin by setting up the equation as a formal gradient flow over an L^2 metric space. Given an embedded Lagrangian submanifold $L^n \subset M^{2n}$, we can consider Hamiltonian deformations of L, these will be flows of vector fields $J\nabla f$ for scalar functions f on M. At $x \in L$, the normal component of $J\nabla f$ is given by $J\nabla_L f$ where $\nabla_L f$ is the gradient of f as a function restricted to L; conversely, given any smooth function f on an embedded L, we can extend f to a function on M so that $\nabla_L f$ is not changed and is independent of the extension of f (see section 2.1 below).

In other words, given a family of C^1 functions $f(\cdot, t)$ along L, one can construct a family of embeddings

$$F : L \times (-\varepsilon, \varepsilon) \to M \tag{2.1}$$

satisfying

$$\frac{d}{dt} F(x,t) = J\nabla f(x,t) \tag{2.2}$$

and conversely, given any path (2.1) within a Hamiltonian isotopy class, there will be a function f so that (possibly after a diffeomorphism to ensure the deformation vector is normal) the condition (2.2) is satisfied.

Let $\mathcal{I}_{L_0}$ be the set of smooth manifolds that are Hamiltonian isotopic to L_0. The (smooth) tangent space at any $L \in \mathcal{I}_{L_0}$ is parameterized via

$$T_L\mathcal{I}_{L_0} = \left\{ f \in C^\infty(L) : \int_L f dV_g = 0 \right\}. \tag{2.3}$$

We use the L^2 metric on $T_L\mathcal{I}_{L_0}$: For $JX_i = \nabla f_i \in T_L\mathcal{I}_{L_0}, i = 1, 2$

$$\langle X_1, X_2 \rangle = \int_L f_1 f_2 \, dV_g. \tag{2.4}$$

With volume function given by

$$\mathrm{Vol}(L) = \int_L dV_g$$

the classical first variation formula gives

$$d\,\mathrm{Vol}(L)(W) = -\int_L \langle W, H \rangle dV_g$$

where W is the deformation vector field and H is the mean curvature vector. In the situation where allowable deformation vectors are of the form $W = J\nabla f$, we get

$$\begin{aligned} d\,\mathrm{Vol}(L)(f) &= -\int_L \langle J\nabla f, H\rangle dV_g \\ &= \int_L \langle \nabla f, JH\rangle dV_g \\ &= -\int_L f \,\mathrm{div}\,(JH)\, dV_g \end{aligned}$$

using the fact that J is orthogonal, then integrating by parts. Note that $-\,\mathrm{div}\,(JH)$ belongs to $T_L\mathcal{I}_{L_0}$ since it integrates to 0 on L. Therefore, it is the gradient of the volume function with respect to the metric. Thus a (volume decreasing) gradient flow for volume would be a path satisfying

$$\frac{dF}{dt} = J\nabla \,\mathrm{div}\,(JH)\,. \tag{2.5}$$

Remark 2.1. The metric (2.3) is not the usual L^2 metric for deformations of a submanifold, which would measure the length of the tangent vector by $\int |J\nabla f|^2\, dV_g$. It is better suited than the standard metric on vector fields. Suppose instead we take the "standard" L^2 metric on deformation fields:

$$d\,\mathrm{Vol}(L)(J\nabla f) = -\int_L \langle J\nabla f, H\rangle dV_g$$

The gradient with respect to this metric would be $J\nabla\eta$ for some $\eta \in C^\infty(L)$ such that

$$-\int_L \langle J\nabla f, H\rangle = \int_L \langle J\nabla f, J\nabla\eta\rangle.$$

While the Lagrangian angle θ (in the Calabi-Yau case) does produce this gradient locally, typically θ is not globally defined on L. So instead, we must find a function η solving the equation

$$\Delta\eta = -\,\mathrm{div}(JH)$$

which one can solve uniquely up to additive constants since L is compact and $\nabla\eta = -JH + X$ (divergence free vector field on L). A gradient flow would be $dF/dt = H - JX$ but there is no canonical way to determine X.

It is also worth noting that gradient flow with respect to the L^2 metric (sometimes called $\mathcal{H}^{-1}$) is not new: It has been used for example in mechanics to describe the flow of curves [Fif00].

2.1. **Related definitions of Hamiltonian deformations.** Traditionally, Hamiltonian isotopies are defined as flows of the entire manifold along the direction of a time-dependent vector field $J\nabla f$ for some f a smooth function on M. Two submanifolds are Hamiltonian isotopic if the one submanifold is transported to the other via the isotopy. In order to use the description (2.3), we note the following standard result:

Lemma 2.2. *For a smooth flow of embedded Lagrangian submanifolds satisfying*

$$\frac{dF}{dt} = J\nabla f(\cdot, t) \tag{2.6}$$

for some function f defined on L for each t, there exists a function $\tilde{f}$ on $M \times [0, 1]$ that defines a Hamiltonian isotopy on M and determines the same Hamiltonian isotopy of the submanifolds.

Conversely, given a global Hamiltonian isotopy determined by $\tilde{f}$, the function $\tilde{f}$ restricted to L_t determines a flow of the form (2.6), possibly up to reparameterization by diffeomorphisms of L.

Proof. The function f is defined on a smooth compact submanifold of $M \times [0, 1]$. We can use the Whitney extension theorem to extend a smooth function off this set in which the normal derivatives vanish. Thus along the Lagrangian submanifolds, $J\nabla f = J\nabla \tilde{f}$.

Conversely, given any function $\tilde{f}$ its gradient decomposes into the normal and tangential parts on the Lagrangian submanifold. By the Lagrangian condition, $J\nabla^T \tilde{f}$ is normal and $J\nabla^\perp \tilde{f}$ is tangential with the latter component describing merely a reparameterization of L. So the flow is completely determined by the component $J\nabla^T \tilde{f}$ which is determined by the restriction to L. □

2.2. **Immersed Lagrangian submanifolds and their Hamiltonian deformations.** Along the evolution equation (1.1), it is feasible that a submanifold which is initially embedded will become merely immersed. Thus we would like the equation to behave well even when the submanifold is immersed.

Weinstein's Lagrangian neighborhood Theorem for immersed Lagrangian submanifolds [EM02, Theorem 9.3.2] states that any Lagrangian immersion $F_0 : L \to M$ extends to an immersion Ψ from a neighborhood of the 0-section in T^*L to M with $\Psi^*\omega_M = \omega_{\text{can}}$.

Sections of the cotangent bundle T^*L are clearly embedded as graphs over the 0-section of T^*L which is identified with L, so by factoring the immersion through T^*L, we get immersed submanifolds in M, in particular, immersed Lagrangian submanifolds in M for sections defined by closed 1-forms on L.

Even though the deformation of an immersed manifold is not properly Hamiltonian (that is, velocity vector $J\nabla f$ determined by a global function f on M) one can define deformations by using a function f defined on the submanifold, and $J\nabla f$ makes sense within T^*L as pullback by the immersion. For example, the figure 8 is not problematic because the two components of a neighborhood of the crossing point can have different velocity vectors; these are separated within the cotangent bundle.

2.3. **The evolution equation in terms of θ.** By [HL82], for a Lagrangian submanifold L in a Calabi-Yau manifold (M^n, ω, J, Ω) with a covariant constant holomorphic n-form Ω, the mean curvature of L satisfies $H = J\nabla\theta$ where

$$\Omega|_L = e^{i\theta} d\,\mathrm{Vol}_L\,.$$

Now (2.5) leads to

$$\frac{dF}{dt} = J\nabla \operatorname{div}(JH) = J\nabla \operatorname{div}(JJ\nabla\theta) = -J\nabla\Delta\theta.$$

Differentiating the left-hand side (cf. [Woo20, Prop 3.2.1]):

$$\begin{aligned}
\frac{d}{dt}\Omega|_L &= \frac{d}{dt}(F_t^*\Omega) = F_0^*\mathcal{L}_{-J\nabla\Delta\theta}\Omega \\
&= F_0^* d(\iota_{-J\nabla\Delta\theta}\Omega) = d\left(F_0^*(\iota_{-J\nabla\Delta\theta}\Omega)\right) \\
&= d\left(F_0^* i(\iota_{-\nabla\Delta\theta}\Omega)\right) \\
&= d\left(ie^{i\theta} dVol_L(-\nabla\Delta\theta, \cdot, ..., \cdot)\right) \\
&= -d\left(ie^{i\theta} * d\Delta\theta\right) \\
&= \left(e^{i\theta} * d\Delta\theta - ie^{i\theta} d(*d\Delta\theta)\right).
\end{aligned}$$

Then differentiating the right hand side:

$$\frac{d}{dt}e^{i\theta}d\operatorname{Vol}_L = e^{i\theta}\frac{d}{dt}d\operatorname{Vol}_L + ie^{i\theta}\frac{d\theta}{dt}d\operatorname{Vol}_L.$$

Comparing the imaginary parts (after multiplying by $e^{-i\theta}$) of the above two gives

$$\frac{d\theta}{dt} = -\Delta^2_{g(t)}\theta. \tag{2.7}$$

3. Existence and Uniqueness via a scalar equation on a Lagrangian Neighborhood

The system of equations (1.1) is not strictly parabolic as given. Our approach is to make good use of the Lagrangian property, in particular, by setting up the equation as a scalar, uniformly parabolic equation via Weinstein's Lagrangian neighborhood theorem. For the convenience of using common terminologies, we make our discussion for embeddings but the conclusions hold for immersions in view of subsection 2.2.

3.1. **Accompanying flow of scalar functions.** Let L be an embedded compact Lagrangian submanifold in a symplectic manifold (M, ω). By Weinstein's Lagrangian neighborhood [Wei71, Corollary 6.2] theorem, there is a diffeomorphism Ψ from a neighborhood $U \subset T^*L$ of the 0-section (identified with L) to a neighborhood $V \subset M$ of L such that $\Psi^*\omega = d\lambda_{can}$ and Ψ restricts to the identity map on L.

Let $\varphi(x,t)$ be a smooth function on $L \times [0, \delta)$ with $\varphi(\cdot, 0) = 0$. Then $d\varphi$ is a t-family of exact (hence closed) 1-forms on L hence a family of sections of T^*L and each is a graph over the 0-section. The symplectomorphism Ψ yields a t-family of Lagrangian submanifolds L_t in M near L:

$$F = \Psi(x, d\varphi(x,t)) = \Psi\left(x, \frac{\partial\varphi}{\partial x^k}dx^k\right). \tag{3.1}$$

Proposition 3.1. *Suppose that $d\varphi(x,t)$ is an exact section describing an evolution of Lagrangian submanifolds which satisfy the equation*

$$\left(\frac{dF}{dt}\right)^{\perp} = J\nabla\,\mathrm{div}\,(JH)\,. \tag{3.2}$$

Then there is a function G (depending on Ψ) such that φ satisfies

$$\frac{\partial\varphi}{\partial t} = -g^{ap}g^{ij}\frac{\partial^4\varphi}{\partial x^a\partial x^j\partial x^i\partial x^p} + G(x, D\varphi, D^2\varphi, D^3\varphi).$$

The coordinate free expression is

$$\frac{\partial\varphi}{\partial t} = \mathrm{div}\, J\Delta_g\Psi(x, d\varphi).$$

Proof. Taking (x,v) for coordinates of T^*L, let y^α be coordinates in M, $\alpha = 1, ..., 2n$. This gives a frame

$$e_i := \frac{\partial F}{\partial x^i} = \frac{\partial\Psi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha} + \frac{\partial\Psi^\alpha}{\partial v^k}\frac{\partial^2\varphi}{\partial x^i\partial x^j}\delta^{jk}\frac{\partial}{\partial y^\alpha} = \Psi_i + \varphi_{ij}\delta^{jk}\Psi_{k+n} \tag{3.3}$$

where

$$\Psi_i := D_{x^i}\Psi = \frac{\partial\Psi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}$$
$$\Psi_{j+n} := D_{v^j}\Psi = \frac{\partial\Psi^\alpha}{\partial v^j}\frac{\partial}{\partial y^\alpha}.$$

Letting also

$$F_i^\alpha := \frac{\partial\Psi^\alpha}{\partial x^i} + \frac{\partial\Psi^\alpha}{\partial v^k}\frac{\partial^2\varphi}{\partial x^i\partial x^j}\delta^{jk}$$

we have

$$e_i = F_i^\alpha\frac{\partial}{\partial y^\alpha}.$$

Now suppose

$$h = h_{\alpha\beta}dy^a dy^\beta$$

is the Riemannian metric on M. We are also assuming $\omega(V,W) = h(JV,W)$. We compute the induced metric g from the immersion:

(3.4)

$$\begin{aligned} g_{ij} &= h(\partial_i F, \partial_j F) \\ &= \left(\frac{\partial\Psi^\alpha}{\partial x^i}\frac{\partial\Psi^\beta}{\partial x^j} + \sum_k\left(\frac{\partial\Psi^\alpha}{\partial x^i}\frac{\partial\Psi^\beta}{\partial v^k}\frac{\partial^2\varphi}{\partial x^j\partial x^k} + \frac{\partial\Psi^\alpha}{\partial x^j}\frac{\partial\Psi^\beta}{\partial v^k}\frac{\partial^2\varphi}{\partial x^i\partial x^k}\right) + \sum_{k.l}\frac{\partial\Psi^\alpha}{\partial v^k}\partial^2_{ik}\varphi\frac{\partial\Psi^\beta}{\partial v^l}\partial^2_{jl}\varphi\right)h_{\alpha\beta}. \\ &= h(\Psi_i + \varphi_{ik}\Psi_{k+n}, \Psi_j + \varphi_{jl}\Psi_{l+n}) \\ &= h(\Psi_i, \Psi_j) + \sum_k\Big(\varphi_{ik}h(\Psi_{k+n}, \Psi_j) + \varphi_{jk}h(\Psi_i, \Psi_{k+n})\Big) + \sum_{k,l}\varphi_{ik}\varphi_{jl}h(\Psi_{k+n}, \Psi_{l+n}). \end{aligned}$$

Since $\Psi : T^*L \to M$ is a symplectomorphism with $\Psi^*\omega = dx \wedge dv$, we have

$$\begin{aligned}\delta_{ij} &= dx \wedge dv(\partial/\partial x^i, \partial/\partial v^j) = \Psi^*\omega\,(\partial/\partial x^i, \partial/\partial v^j) \\ &= h(J\Psi_*(\partial/\partial x^i), \Psi_*(\partial/\partial v^j)) = h(J\Psi_i, \Psi_{j+n}).\end{aligned} \tag{3.5}$$

Similarly

$$h(J\Psi_{i+n}, \Psi_{j+n}) = \omega(\Psi_*(\partial/\partial v^i), \Psi_*(\partial/\partial v^j)) = 0. \tag{3.6}$$

Now as F describes a Lagrangian manifold, (summing repeated indices below)

(3.7)
$$\begin{aligned}0 &= \omega(e_i, e_j) = h(J\Psi_i + \partial_k\varphi_i J\Psi_{k+n}, \Psi_j + \partial_l\varphi_j\Psi_{l+n}) \\ &= h(J\Psi_i, \Psi_j) + \partial_l\varphi_j h(J\Psi_i, \Psi_{l+n}) + \partial_k\varphi_i h(J\Psi_{k+n}, \Psi_j) + \partial_k\varphi_i\partial_l\varphi_j h(J\Psi_{k+n}, \Psi_{l+n}) \\ &= h(J\Psi_i, \Psi_j) - \varphi_{jk}h(\Psi_i, J\Psi_{k+n}) + \varphi_{ik}h(J\Psi_{k+n}, \Psi_j) + \varphi_{ik}\varphi_{jl}h(J\Psi_{k+n}, \Psi_{l+n}) \\ &= h(J\Psi_i, \Psi_j) + \varphi_{ik}\varphi_{jl}h(J\Psi_{k+n}, \Psi_{l+n}) \qquad \text{by (3.5)} \\ &= h(J\Psi_i, \Psi_j).\end{aligned}$$

Now, $\{\Psi_i, J\Psi_j : 1 \le i, j \le n\}$ is a basis for the ambient tangent space at a point in the image of F. So is $\{\Psi_i, \Psi_{j+n} : 1 \le i, j \le n\}$ (as Ψ is a local diffeomorphism). We represent the latter vectors by

$$\Psi_{i+n} = a^{ij}\Psi_j + b^{ij}J\Psi_j. \tag{3.8}$$

Computing the pairing $h\left(J\Psi_j, \Psi_{i+n}\right)$ using (3.5) on the left and (3.8) on the right yields $b^{ij} = h^{ij}$ as the inverse of the positive definite matrix $h_{ij} := h(\Psi_i, \Psi_j)$. Now recalling (3.1)

$$\frac{\partial F}{\partial t} = \left(\frac{\partial}{\partial t}\frac{\partial\varphi}{\partial x^k}\right)\frac{\partial\Psi}{\partial v^k},$$

project onto the normal space:

$$\begin{aligned}\left(\frac{\partial F}{\partial t}\right)^{\perp} &= \frac{\partial\varphi_t}{\partial x^k}h(\Psi_{k+n}, Je_p)Je_q g^{pq} \\ &= \frac{\partial\varphi_t}{\partial x^k}h(\Psi_{k+n}, J\Psi_p + \varphi_{pj}J\Psi_{j+n})Je_q g^{pq} \\ &= \frac{\partial\varphi_t}{\partial x^k}h(\Psi_{k+n}, J\Psi_p)Je_q g^{pq} \\ &= \frac{\partial\varphi_t}{\partial x^k}\delta_{kp}Je_q g^{pq} \qquad \text{by (3.5)} \\ &= \frac{\partial\varphi_t}{\partial x^k}\left(Je_q g^{kq}\right) \\ &= J\nabla\varphi_t.\end{aligned}$$

Let $H = H^m Je_m$ for the Lagrangian L. As JH is tangential its divergence on L is

$$\begin{aligned}\operatorname{div}(JH) &= -\operatorname{div}\left(H^m e_m\right) \\ &= -g^{ab} h(\nabla_{e_a}\left(H^m e_m\right), e_b) \\ &= -g^{ab} h\left(\frac{\partial}{\partial x^a} H^m e_m + H^m \Gamma^p_{am} e_p, e_b\right) \\ &= -g^{ab} \frac{\partial}{\partial x^a} H^m g_{mb} - g^{ab} H^m \Gamma^p_{am} g_{pb} \\ &= -\frac{\partial}{\partial x^a} H^a - H^m \Gamma^a_{am}\end{aligned} \tag{3.9}$$

where the Christoffel symbols are for the induced metric g. The components of H are given by

$$H^a = h\Big(H, Je_p\Big) g^{ap} = h\left(g^{ij}\left(\frac{\partial^2 F^\beta}{\partial x^i \partial x^j} + F^\alpha_i F^\gamma_j \tilde{\Gamma}^\beta_{\alpha\gamma}\right)\frac{\partial}{\partial y^\beta}, Je_p\right) g^{ap}. \tag{3.10}$$

Now differentiate components of (3.3)

$$\frac{\partial^2 F^\beta}{\partial x^i \partial x^j} = \frac{\partial \Psi^\beta}{\partial x^j \partial x^i} + \frac{\partial^3 \varphi}{\partial x^j \partial x^i \partial x^k}\delta^{kl}\frac{\partial \Psi^\beta}{\partial v^l} + \frac{\partial^2 \varphi}{\partial x^i \partial x^k}\delta^{kl}\frac{\partial^2 \Psi^\beta}{\partial x^j \partial v^l}.$$

Plug in (3.3) to get

$$\begin{aligned}
&h\left(\frac{\partial^2 F^\beta}{\partial x^i \partial x^j}\frac{\partial}{\partial y^\beta}, Je_p\right) = \omega\left(e_p, \frac{\partial^2 F^\beta}{\partial x^i \partial x^j}\frac{\partial}{\partial y^\beta}\right) \\
=&\omega\left(\frac{\partial \Psi^\delta}{\partial x^p}\frac{\partial}{\partial y^\delta} + \frac{\partial^2 \varphi}{\partial x^p \partial x^q}\delta^{qm}\frac{\partial \Psi^\delta}{\partial v^m}\frac{\partial}{\partial y^\delta}, \frac{\partial \Psi^\beta}{\partial x^j \partial x^i}\frac{\partial}{\partial y^\beta} + \frac{\partial^3 \varphi}{\partial x^j \partial x^i \partial x^k}\delta^{kl}\frac{\partial \Psi^\beta}{\partial v^l}\frac{\partial}{\partial y^\beta} + \frac{\partial^2 \varphi}{\partial x^i \partial x^k}\delta^{kl}\frac{\partial^2 \Psi^\beta}{\partial x^j \partial v^l}\frac{\partial}{\partial y^\beta}\right) \\
=&\frac{\partial^3 \varphi}{\partial x^j \partial x^i \partial x^k}\delta^{kl}\omega\left(\frac{\partial \Psi^\delta}{\partial x^p}\frac{\partial}{\partial y^\delta}, \frac{\partial \Psi^\beta}{\partial v^l}\frac{\partial}{\partial y^\beta}\right) + \frac{\partial^2 \varphi}{\partial x^p \partial x^q}\delta^{qm}\frac{\partial^3 \varphi}{\partial x^j \partial x^i \partial x^k}\delta^{kl}\omega\left(\frac{\partial \Psi^\delta}{\partial v^m}\frac{\partial}{\partial y^\delta}, \frac{\partial \Psi^\beta}{\partial v^l}\frac{\partial}{\partial y^\beta}\right) \\
&+ F^\delta_p\left(\frac{\partial \Psi^\beta}{\partial x^j \partial x^i} + \frac{\partial^2 \varphi}{\partial x^i \partial x^k}\delta^{kl}\frac{\partial^2 \Psi^\beta}{\partial x^j \partial v^l}\right)\omega_{\delta\beta}\circ F \\
=&\frac{\partial^3 \varphi}{\partial x^j \partial x^i \partial x^k}\delta^{kl}\Psi^*\omega\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial v^l}\right) + \frac{\partial^2 \varphi}{\partial x^p \partial x^q}\delta^{qm}\frac{\partial^3 \varphi}{\partial x^j \partial x^i \partial x^k}\delta^{kl}\Psi^*\omega\left(\frac{\partial}{\partial v^m}, \frac{\partial}{\partial v^l}\right) \\
&+ F^\delta_p\left(\frac{\partial \Psi^\beta}{\partial x^j \partial x^i} + \frac{\partial^2 \varphi}{\partial x^i \partial x^k}\delta^{kl}\frac{\partial^2 \Psi^\beta}{\partial x^j \partial v^l}\right)\omega_{\delta\beta}\circ F \\
=&\frac{\partial^3 \varphi}{\partial x^j \partial x^i \partial x^k}\delta^{kl}\delta_{pl} + F^\delta_p\left(\frac{\partial \Psi^\beta}{\partial x^j \partial x^i} + \frac{\partial^2 \varphi}{\partial x^i \partial x^k}\delta^{kl}\frac{\partial^2 \Psi^\beta}{\partial x^j \partial v^l}\right)\omega_{\delta\beta}\circ F.
\end{aligned}$$

Now also

$$h\left(\frac{\partial}{\partial y^\beta}, Je_p\right) = \omega\left(F^\delta_p \frac{\partial}{\partial y^\delta}, \frac{\partial}{\partial y^\beta}\right) = F^\delta_p \omega_{\delta\beta}\circ F.$$

Combining (3.10) and the above

$$H^a = g^{ij}g^{ap}\left(\frac{\partial^3\varphi}{\partial x^j\partial x^i\partial x^k}\delta^{kl}\delta_{pl} + F_p^\delta\left(\frac{\partial\Psi^\beta}{\partial x^j\partial x^i} + \frac{\partial^2\varphi}{\partial x^i\partial x^k}\delta^{kl}\frac{\partial^2\Psi^\beta}{\partial x^j\partial v^l}\right)\omega_{\delta\beta}\circ F\right)$$
$$+ g^{ij}g^{ap}F_i^\alpha F_j^\gamma\tilde{\Gamma}_{\alpha\gamma}^\beta F_p^\delta\omega_{\delta\beta}\circ F.$$

Thus using the expression we derived in (3.9)

(3.11)

$$\operatorname{div}(JH) = -g^{ap}g^{ij}\frac{\partial^4\varphi}{\partial x^a\partial x^j\partial x^i\partial x^p} - \left(\frac{\partial}{\partial x^a}\left(g^{ij}g^{ap}\right)\right)\frac{\partial^3\varphi}{\partial x^j\partial x^i\partial x^p}$$
$$- \frac{\partial}{\partial x^a}\left(g^{ij}g^{ap}\right)F_p^\delta\left[\left(\frac{\partial\Psi^\beta}{\partial x^j\partial x^i} + \frac{\partial^2\varphi}{\partial x^i\partial x^k}\delta^{kl}\frac{\partial^2\Psi^\beta}{\partial x^j\partial v^l}\right) + F_i^\alpha F_j^\gamma\tilde{\Gamma}_{\alpha\gamma}^\beta\right]\omega_{\delta\beta}\circ F - H^m\Gamma_{am}^a.$$

Now recalling (3.4), we see the metric components g_{ab} involve second order derivatives in terms of φ, thus Γ_{ij}^k are third order. So each term above after the first term is at most third order. □

3.2. **Short time existence.**

Proposition 3.2. *Given an initial smooth immersion of a compact $L \to M$, there exists a solution to (1.1,1.2) for some short time.*

Proof. Choose a Weinstein neighborhood containing L. Now suppose we have φ which satisfies the fourth order equation

$$\begin{aligned} \varphi_t &= -g^{ap}g^{ij}\frac{\partial^4\varphi}{\partial x^a\partial x^j\partial x^i\partial x^p} + G(x, D\varphi, D^2\varphi, D^3\varphi) = \operatorname{div}(JH) \\ \varphi(\cdot, 0) &= 0. \end{aligned} \tag{3.12}$$

Then the immersions F generated from $\varphi(x,t)$ satisfy

$$\left(\frac{\partial F}{\partial t}\right)^\perp = J\nabla\varphi_t = J\nabla\operatorname{div}(JH).$$

As the normal component satisfies the appropriate equation, we may compose with diffeomorphisms to get a flow (see Claim 3.3 below) such that

$$\frac{\partial F}{\partial t} = J\nabla\operatorname{div}(JH). \tag{3.13}$$

Now the equation (3.12) is precisely of the form of $2p$ order quasilinear parabolic equation studied in [MM12]. By [MM12, Theorem 1.1] we have short time existence for the solution to (3.12), thus we have short-time existence for the flow (3.13). □

3.3. **Uniqueness.** We start with a standard observation.

Claim 3.3. *Suppose that $F : L \times [0,T) \to M$ is a family of immersions satisfying*

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = N(x,t)$$

for some vector field $N(x,t)$ which is normal to the immersed submanifold $F(\cdot,t)(L)$. There exists a unique family of diffeomorphisms $\chi_t : L \to L$ such that

$$\frac{\partial}{\partial t} F(\chi_t(x),t) = N(\chi_t(x),t) \quad \textit{and} \quad \chi_0 = Id_{|L}.$$

Proof. Given the flow exists, the given velocity field will decompose orthogonally into normal and tangential components:

$$\frac{\partial F}{\partial t} = N(x,t) + T(x,t)\,.$$

Consider the time-dependent vector field on L

$$V(x,t) = -D_L F(x,t)^{-1} T(x,t)$$

By the Fundamental Theorem on Flows, (cf. [Lee13, Theorem 9.48]) there is a unique flow on L starting at the identity and satisfying

$$\frac{\partial}{\partial t}\chi_t(x) = V(\chi_t(x),t).$$

Composing this flow with the original flow F yields the result. □

Theorem 3.4. *The solution to the initial value problem* (1.1) - (1.2) *is unique. More precisely, if F_1 and F_2 are two solutions of* (1.1) *such that $F_1(x,t_0) = F_2(x,t_0')$ for some t_0, t_0' and all $x \in L$, then*

$$F_1(x,t_0+\tau) = F_2(x,t_0'+\tau)$$

for all τ in an open neighborhood of 0 *where both sides above are defined.*

Proof. Without loss of generality, we take $t_0 = t_0' = 0$. Let $L = F_1(\cdot,0) = F_2(\cdot,0)$ and $\Psi : U \subset T^*L \to V \subset M$ be a Lagrangian neighborhood mapping.

First, we show that the normal flow of $F_i(\cdot,t)$ is given in the neighborhood Ψ by the graph of an exact section $d\varphi_i(\cdot,t)$ where φ_i solves a problem of the form (3.12). To this end, note that for τ in the domain, the path $\{F_i(\cdot,t), t \in [0,\tau]\}$ is a Hamiltonian isotopy between $F_i(\cdot,0)$ and $F_i(\cdot,\tau)$. Being a Hamiltonian isotopy is invariant under the symplectomorphism Ψ, so the sections $\Psi^{-1}(F_i(\cdot,0))$ and $\Psi^{-1}(F_i(\cdot,\tau))$ are Hamiltonian isotopic. By [Wei71, Corollary 6.2], Lagrangian submanifolds that are near to the 0-section are given as graphs of closed sections of the cotangent bundle. As the flow is smooth, for small times the Lagrangian submanifolds are near enough to be described by closed sections. According to [MS17, Proposition 9.4.2], these sections are exact, that is

$$\Psi^{-1} \circ F_i(\cdot,\tau)(L) = \{d\varphi_i(x,\tau) : x \in L\}\,.$$

In other words,

$$\Psi(\{d\varphi_i(x,\tau) : x \in L_i\}) = F_i(\cdot,\tau)(L)$$

meaning that for each τ

$$\Psi \circ d\varphi_i : L \to T^*L \to M$$

is a Lagrangian immersion, which may have reparameterized the base. In particular, the flow F_i determines a flow of scalar functions φ_i, which recovers the same family of submanifolds at F_i (up to reparameterization) as do $\Psi \circ d\varphi_i(\cdot, t)$. By Proposition 3.1, the scalar equation (3.12) holds on φ_i as does the initial condition $d\varphi_i \equiv 0$. It follows that φ_1 and φ_2 both satisfy the same equation (3.12) and have the same initial condition, so $\varphi_1 = \varphi_2 + C$ for some constant C. Thus the flows F_1 and F_2 are the same. □

Theorem 3.4 allows for seamless extension of the flow: While the Weinstein's Lagrangian neighborhood may only exist around L_0, if another Lagrangian neighborhood of L_0 extends the flow, the two flows patch together smoothly.

4. Higher order estimates based on curvature bounds

The goal of this section is to show that a solution with uniformly bounded second fundamental form over $[0, T)$ enjoy estimates of all orders and can be extended.

Theorem 4.1. *Suppose that the flow (1.1) exists on* $[0, T)$ *and the second fundamental form has a uniform bound on* $[0, T)$*. Then the flow converges smoothly as* $t \to T$ *so can be extended to* $[0, T + \varepsilon)$ *for some* $\varepsilon > 0$*.*

To prove this theorem, it is essential in our approach to establish a-priori estimates from the integral estimates derived from the following differential inequality:

Proposition 4.2. *Suppose that F is a solution to (1.1) on* $[0, T)$ *for a compact Lagrangian submanifold L inside a compact M. Suppose the second fundamental form has a uniform bound K. There exists C depending on K, the ambient geometry of M and* $\mathrm{Vol}(L_0)$ *such that for all $k \geq 2$*

$$\frac{d}{dt}\int_L \left|\nabla^{k-1}A\right|^2 dV_g(t) \leq C\int_L \left|\nabla^{k-1}A\right|^2 dV_g(t) + C\sum_{l=0}^{k-2}\int_L \left|\nabla^{l}A\right|^2 dV_g(t). \tag{4.1}$$

A Weinstein neighborhood map determines the equation the flow must satisfy, and we could derive estimates of all orders based on this particular equation. However, the flow is expected to leave a given neighborhood after some time, and we will need to take a new neighborhood. We would need to know the speed of the flow to patch estimates from one neighborhood to another, but this requires knowing the size of the Weinstein neighborhoods around the Lagrangian submanifolds at different times.

We require charts with uniform geometric estimates. To obtain these we appeal to uniform local Darboux coordinates given in [JLS11]. These charts are local but are given with uniform geometric bounds. The short-time existence of the flow is already determined by the global Weinstein neighborhoods; we write the flow in these Darboux charts as a scalar equation from which we derive integral estimates for derivatives of any order.

4.1. **Uniform Darboux charts.** We record [JLS11, Prop.3.2 and Prop.3.4] on existence of Darboux coordinates with estimates on a symplectic manifold. Let $\pi : \mathcal{U} \to M$ be the $U(n)$ frame bundle of M. A point in $\mathcal{U}$ is a pair (p, v) with $\pi(p, v) = p \in M$ and $v : \mathbb{R}^{2n} \to T_pM$ an isomorphism satisfying $v^*(\omega_p) = \omega_0$ and $v^*(h|_p) = h_0$ (the standard metric on $\mathbb{C}^n$). The right action of $U(n)$ on $\mathcal{U}$ is free: $\gamma(p, v) = (p, v \circ \gamma)$ for any $\gamma \in U(n)$.

Proposition 4.3 (Joyce-Lee-Schoen)**.** *Let (M, ω) be a real $2n$-dimensional symplectic manifold without boundary, and a Riemannian metric h compatible with ω and an almost complex structure J. Let $\mathcal{U}$ be the $U(n)$ frame bundle of M. Then for small $\varepsilon > 0$ we can choose a family of embeddings $\Upsilon_{p,v} : B^{2n}_\varepsilon \to M$ depending smoothly on $(p, v) \in U$, where B^{2n}_ε is the ball of radius ε about 0 in $\mathbb{C}^n$, such that for all $(p, v) \in U$ we have*

(1) $\Upsilon_{p,v}(0) = p$ *and* $d\Upsilon_{p,v}|_0 = v : \mathbb{C}^n \to T_pM$;
(2) $\Upsilon_{p,v\circ\gamma}(0) \equiv \Upsilon_{p,v} \circ \gamma$ *for all* $\gamma \in U(n)$;
(3) $\Upsilon^*_{p,v}(\omega) \equiv \omega_0 = \frac{\sqrt{-1}}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$; *and*
(4) $\Upsilon^*_{p,v}(h) = h_0 + O(|z|)$.

Moreover, for a dilation map $\mathbf{t} : B^{2n}_R \to B^{2n}_\varepsilon$ given by $\mathbf{t}(z) = tz$ where $t \le \varepsilon/R$, set $h^t_{p,v} = t^{-2}(\Upsilon_{p,v} \circ \mathbf{t})^*h$. Then it holds

(5) $\|h^t_{p,v} - h_0\|_{C^0} \le C_0 t \quad$ *and* $\quad \|\partial h^t_{p,v}\|_{C^0} \le C_1 t,$

where norms are taken w.r.t. h_0 and ∂ is the Levi-Civita connection of h_0.

Proposition 4.4. *Suppose that M is a compact symplectic manifold with a compatible Riemannian metric h. Suppose that L is a compact Lagrangian submanifold of M with second fundamental form bounded above by K and volume bounded above by V_0. Given $c_n > 0$, there exists an $r_0 = r_0(K, c_n) > 0$ and a finite cover of L by Darboux charts Υ_{p_i,v_i}:$B^{2n}_{r_0} \to M$ centered at points p_i on L such that*

(1) *The connected component of $L \cap B^{2n}_{r_0}$ containing p_j is represented by a graph $\left(x, d\varphi^{(j)}\right)$ over $B^{2n}_{r_0} \cap \mathbb{R}^n \times \{0\}$ for some potential $\varphi^{(j)}$.*
(2) *The tangent plane at each point of this connected component satisfies a closeness condition with respect to the planes $v_i(\mathbb{R}^n \times \{0\})$:*

$$\max_{\substack{|e|_g=1,\, e\in T_pL \\ |\nu|_{\delta_0}=1, \nu\in\{0\}\times\mathbb{R}^n}} e \cdot \nu < c_n \tag{4.2}$$

where the dot product is in the euclidean metric δ_0, and c_n is a small universal constant (say $c_n = \frac{1}{10\sqrt{n}}$) chosen so that quantities such as the volume element and coordinate expression for h are bounded by universal constants.
(3) *The ambient metric h is very close to the euclidean metric, that is $\|h - \delta_0\| < c_n$ for some c_n (can be the same c_n as in (2) above).*
(4) *The submanifold L is covered by the charts obtained by restricting these charts to $B^{2n}_{r_0}(p_j)$*

(5) *The number of such points* $\left\{p_j\right\}$ *satisfies*

$$N(K, V_0) \leq \frac{C(n)V_0}{r_0^n(K, c_n)}. \tag{4.3}$$

Proof. At each point $p \in L$ we take a Darboux chart $\Upsilon_{p,v}$ as described above with that $T_pL = \mathbb{R}^n \times \{0\}$ in the given chart. Note that after some fixed re-scalings, we can assert via Proposition 4.3 that $\Upsilon_{p,v}$ exists on $B^{2n}_{\varepsilon_0}$ and satisfies any near euclidean metric conditions we choose to prescribe, including the closeness condition: $|h - \delta_0| < c$. Now we may apply Proposition 5.1 which asserts existence of a ball $B^n_{r_0}(p) \subset \mathbb{R}^n \times \{0\}$ over which L is representable as a graph, with (4.2) holding. The quantity r_0 will depend on K.

Consider the compact immersed submanifold L as a metric space (L, d). Taking a finite cover of metric balls $B_{r_0/4}(p)$ for $p \in L$ and applying Vitalli's covering Lemma, we conclude that there is a subset of these points $\{p_j\}$ so that $L = \cup_i B_{3_0/4}(p_i)$ and $B_{r_0/4}(p_i)$ are mutually disjoint. By (4.2), $L \cap B_{3r_0/4}(p_i)$ is in the image of a graph given by Proposition 5.1. In particular, the disjoint $B_{r_0/4}(p_i)$'s have a minimum total volume $\omega_n c^n r_0^n$. The bound (4.3) on the number of balls follows. As $\{B_{3r_0/4}(p_i)\}$ covers L and each of these balls is contained in a graph over $B^n_{r_0}$, we take the set of the graphs as the cover. □

The scalar functions from the the exact sections of T^*L are globally defined on L via the abstract Weinstein map Ψ. We have utilized them to establish short-time existence and uniqueness for our geometric flow of F. However, for higher order a-priori estimates, we need to set up the flow equation in a Darboux chart with estimates on the metric as described above. Fortunately, each $\Upsilon_{p,v}$ is a symplectomorphism, which takes gradient graphs $(x, d\varphi)$ to Lagrangian submanifolds, so the computations in section 3.1 can be repeated verbatim, with $\Upsilon_{p,v}$ in place of Ψ. In particular, in each chart, the flow is determined by an equation

$$\varphi_t = -g^{ap} g^{ij} \frac{\partial^4 \varphi}{\partial x^a \partial x^j \partial x^i \partial x^p} + G(x, D\varphi, D^2\varphi, D^3\varphi). \tag{4.4}$$

Remark 4.5. A precise computation in Darboux coordinates of the expression (3.10) gives

$$\begin{aligned} h\left(H, Je_p\right) = &\; g^{ij}\varphi_{pij} + g^{ij}\tilde{\Gamma}^{p+n}_{ij} + g^{ij}\varphi_{kj}\delta^{km}\tilde{\Gamma}^{p+n}_{i,m+n} + g^{ij}\varphi_{ki}\delta^{km}\tilde{\Gamma}^{p+n}_{m+n,j} \\ &+ g^{ij}\varphi_{ki}\varphi_{lj}\delta^{km}\delta^{lr}\tilde{\Gamma}^{p+n}_{m+n,r+n} - g^{ij}\tilde{\Gamma}^q_{ij}\varphi_{pq} - g^{ij}\varphi_{kj}\delta^{km}\tilde{\Gamma}^q_{i,m+n}\varphi_{pq} \\ &- g^{ij}\varphi_{ki}\delta^{km}\tilde{\Gamma}^q_{m+n,j}\varphi_{pq} - g^{ij}\varphi_{ki}\varphi_{lj}\delta^{km}\delta^{lr}\tilde{\Gamma}^q_{m+n,r+n}\varphi_{pq} \end{aligned}$$

where $\tilde{\Gamma}^q_{ij}$ are Christoffel symbols in the ambient metric (M, h). Considering that each expression of the form g^{ab} is a smooth function in terms of $D^2\varphi$ with dependence on zero order of h and each $\tilde{\Gamma}^\beta_{ij}$ expression depends on Dh and h, one may conclude (after computing $\operatorname{div}(JH)$ as in (3.11)) that G can be written as a sum of expressions that are

(1) quadratic in $D^3\varphi$ and smooth in $D^2\varphi, h$ in a predetermined way
(2) linear in $D^3\varphi$, smooth in $D^2\varphi, h, Dh$ in a predetermined way

(3) smooth in $D^2\varphi, h$ and linear in D^2h in a predetermined way
(4) smooth in $D^2\varphi, h, Dh$ in a predetermined way.

This allows us to make a claim that there is uniform control on the important quantities involved in the equation we are solving.

Proposition 4.6. *Suppose that L is a compact Lagrangian manifold with volume V_0 and evolves by* (1.1) *on* $[0, T)$*. If the norm of second fundamental A of L_t satisfies $|A|_{g(t)} \leq K$ for $t \in [0, T)$, then after a fixed rescaling on M there is a finite set of Darboux charts such that*

(1) *The submanifold is covered by graphs over $B_1^{2n} \cap \mathbb{R}^n \times \{0\}$.*
(2) *The submanifold is graphical over $B_5^{2n} \cap \mathbb{R}^n \times \{0\}$ in each chart.*
(3) *The slope bound (4.2) holds over $B_5^n(0)$.*
(4) *The flow (1.1) is governed by (4.4) locally in these charts.*
(5) *For each chart, the G from (4.4) satisfies a uniform bound on any fixed order derivatives of G (in terms of all four arguments, not with respect to x coordinate before embedding.)*
(6) *The number of charts is controlled*

$$N(K, V_0) \leq C(K)\frac{V_0}{r_0^n(K)}. \tag{4.5}$$

Proof. Rescale M so that $r_0 = 5$. Then the expression G becomes predictably controlled by Remark 4.5. Choosing a cover with interior balls, as in the proof of Proposition 4.4, determines the necessary number of balls. □

4.2. **Localization.** Let L_t evolve by (1.1) with time $t \in [0, T)$, and assume $|A|_{g(t)} \leq K$ for all L_t. Our goal is to establish integral bounds for $\left|\nabla^l A\right|^2_{g(t)}$, which only depend on k, K, M and the initial volume V_0 of L_0. To derive the differential inequality (4.1) at any time t_0, we use Proposition 4.6 and express geometric quantities $g, A, \nabla^l A$, etc., in the (no more than N) Darboux charts in terms of $\varphi(x, t)$ for $x \in B_5^n$. By compactness of L and smoothness of the flow, the flow will continue to be described by graphs of $d\varphi(x, t)$ in this open union of N charts for $t \in [t_0, t_1)$ for some $t_1 > t_0$.

To be precise, each of the Darboux charts in Lemma 4.6 has a product structure; we may assume that each chart contains coordinates $B_4^n(0) \times B_2^n(0)$ so that L is graphical over $B_5^n(0)$ and further that the collection of $B_1^n(0) \times B_1^n(0)$ covers L. Now we may fix once and for all a function η which is equal to 1 on $B_1^n(0) \times B_1^n(0)$ and vanishes within $B_2^n(0) \times B_2^n(0)$. For a given chart Υ^α (here $\alpha \in \{1, ..N\}$ indexes our choice of charts) we call the function η_α. This function will have uniformly bounded dependence on the variables x and y in the chart.

Now once these η_α are chosen, we may then define a partition of unity for the union of charts which form a tubular neighborhood of L, which will restrict to a partition of unity for small variations of L:

$$\rho_\alpha^2 := \frac{\eta_\alpha^2}{\sum \eta_\alpha^2}. \tag{4.6}$$

By compactness of the unit frame bundle and the smoothness of the family of charts defined in Proposition 4.3, the transition functions between charts will have bounded derivatives to any order. Thus, in a fixed chart, where a piece of L is represented as $\{(x, d\varphi(x)) : x \in B_1(0)\}$, the dependence of ρ_α^2 will be uniformly controlled in terms of these variables, so there is a uniform pointwise bound

$$\left|D_x^2 \rho_\alpha^2\right| \leq C(D^3\varphi, D^2\varphi, D\varphi, x) \tag{4.7}$$

were this dependence is at most linear on $D^3\varphi$. We will be using the x coordinates as charts for L.

Note also that, if we have a uniform bound on $\frac{d}{dt}D\varphi$ and $\frac{d}{dt}D^2\varphi$ we can conclude a positive lower bound on $t_1 - t_0$; the flow will be described by graphs of $\varphi(x, t)$ in these N charts, and the condition (4.2) will be satisfied for a slightly larger c_n' (say $c_n' = \frac{1}{5\sqrt{n}}$ instead of $c_n = \frac{1}{10\sqrt{n}}$).

4.2.1. *Expression for metric and second fundamental form.* In the Darboux charts for M, the manifold L is expressed graphically over the x coordinate via

$$x \mapsto F(x) = (x, d\varphi(x))\,.$$

Thus we have a tangential frame:

$$e_i = \partial_{x^i} F = E_i + \varphi_{ik}\delta^{km}E_{m+n} \tag{4.8}$$

with

$$g_{ij} = h_{ij} + \varphi_{ik}\delta^{km}\varphi_{jl}\delta^{lr}h_{(m+n)(l+n)} + \varphi_{jl}\delta^{lr}h_{(i)(l+n)} + \varphi_{ik}\delta^{km}h_{(m+n)(j)}.$$

Recalling (2) and (3) in Proposition 4.4 we may assume that the expression of h in these coordinates is very close to δ_{ij} and that $D^2\varphi$ is not large.

Differentiating the components of the induced metric gives

$$\begin{aligned}\partial_{x^p} g_{ij} = \text{function of } &(x, D\varphi)\\ &+ \text{Terms involving up to three factors of } D^2\varphi \text{ but no higher}\\ &+ \text{Terms involving up to two factors of } D^2\varphi \text{ and one factor of } D^3\varphi.\end{aligned}$$

Lemma 4.7. *In a Darboux chart, using the coordinate basis (4.8) for the tangent space and $\{Je_l\}$ for the normal space, the covariant derivatives of the second fundamental form and of the potential φ are related by*

$$\nabla^{k-1}A = D^{k+2}\varphi + S_k \tag{4.9}$$

where S_1 is a smooth controlled function depending on the chart, h and $D^2\varphi$, S_2 depends also on $D^3\varphi$ and for $k \geq 3$:

1. *Each S_k is a sum of of multilinear forms of $D^4\varphi, ..., D^{k+1}\varphi$*
2. *The coefficients of these forms are functions of $\left(x, D\varphi, D^2\varphi, D^3\varphi\right)$*
3. *The total sum of the derivatives of $D^3\varphi$ that occur in a given term is no more than $k - 2$.*

(Note that (4.9) is interpreted as literal equality of the symbols in the choice of basis, not simply "up to a smooth function")

Proof. Starting with $k = 1$, differentiate in the ambient space

$$\tilde{\nabla}_{e_i} e_j = \tilde{\Gamma}^{\beta}_{ji} E_\beta + \varphi_{jmi}\delta^{mk} E_{n+k} + \varphi_{jm}\delta^{mk}\tilde{\Gamma}^{\beta}_{n+k,i} E_\beta + \varphi_{jm}\varphi_{ri}\delta^{mk}\delta^{rl}\tilde{\Gamma}^{\beta}_{n+k,n+l} E_\beta .$$

Using e_j and Je_k as frame and normal frame,

$$\begin{aligned} A_{ijl} &= \langle \tilde{\nabla}_{e_i} e_j, Je_l \rangle \\ &= \omega(e_l, \varphi_{jmi}\delta^{mk} E_{n+k}) + \omega(e_l, \tilde{\Gamma}^{\beta}_{ji} E_\beta + \varphi_{jm}\delta^{mk}\tilde{\Gamma}^{\beta}_{n+k,i} E_\beta + \varphi_{jm}\varphi_{ri}\delta^{mk}\delta^{rl}\tilde{\Gamma}^{\beta}_{n+k,n+l} E_\beta) \\ &= \varphi_{jli} + S_1 \end{aligned}$$

where S_1 is a smooth function involving $D^2\varphi$ and the ambient Christoffel symbols at $(x, D\varphi)$ and recalling

$$\omega(e_l, E_\beta) = \omega(E_l + \varphi_{lk}\delta^{jk} E_{j+n}, E_\beta) = \begin{cases} \delta_{sl}, & \text{if } \beta = s + n \text{ for } s \in \{1, ..., n\} \\ -\varphi_{sl}, & \text{if } \beta = s \in \{1, ..., n\}. \end{cases}$$

Now for $k = 2$ (∇ denotes covariant derivatives on the submanifold):

$$(\nabla A)_{pijl} = \partial_p A_{ijl} - A\left(\nabla_{e_p} e_i, e_j, e_l\right) - A\left(e_i, \nabla_{e_p} e_j, e_l\right) - A\left(e_i, e_j, \nabla_{e_p} e_l\right).$$

Now we can compute the Christoffel symbols with respect to the induced metric g:

$$\nabla_{e_p} e_i = 1 * D^3\varphi + \text{lower order}$$

thus

$$\begin{aligned} (\nabla A)_{pijl} &= \varphi_{jlip} + \partial_{x^p} S_1 - A * \left(D^3\varphi\right) + \text{lower order} \\ &= \varphi_{jlip} + D^3\varphi * D^3\varphi + 1 * D^3\varphi + \text{ smooth in other arguments.} \end{aligned}$$

Here and in sequel, we use $A * B$ to denote a predictable linear combination of terms from tensors A and B, and $1 * T$ to be a predictable linear combination of T. Now

$$\begin{aligned} \nabla^2 A &= D^5\varphi + D^4\varphi * D^3\varphi + \text{lower order} \\ \nabla^3 A &= D^6\varphi + D^5\varphi * D^3\varphi + D^4\varphi * D^4\varphi + \text{lower order} \end{aligned}$$

and so forth. The result follows by inductively applying the product rule and noting

$$\nabla^{k-1} A = D(\nabla^{k-2} A) + D^3\varphi * \nabla^{k-2} A + \text{lower order}$$

by the formula for covariant derivative. □

4.3. **Integral inequalities.** We will use $\|\cdot\|_\infty$ for the supremum norm in the euclidean metric δ_0 and

$$\left|D^m\varphi\right|^2_g = g^{i_1 j_1} g^{i_2 j_2} ... g^{i_m j_m} \varphi_{i_1 ... i_m} \varphi_{j_1 ... j_m}$$

to denote the norm squared with respect to g for the locally defined m-tensor $D^m\varphi$ instead of the higher covariant derivative tensor $\nabla^m\varphi$. We find that this makes computations on the chosen Darboux chart more transparent. Note that since g is close to δ_0 on the chart (with estimates on errors)

$$\frac{|D^m\varphi|^2_g}{|D^m\varphi|^2_{\delta_0}} \in (1 - c_n, 1 + c_n)$$

and

$$(1-c_n)dx \leq dV_g \leq (1+c_n)\,dx$$

for some small c_n. Thus we may regard as equivalent estimates on integrals against dx and integrals against dV_g, provided the quantities we are integrating are nonnegative. However, if the quantity being integrated is not known to be non-negative, we have to be precise in performing estimates.

All estimates below implicitly depend on c_n, but c_n need not be tracked closely: it need not be close to zero.

4.3.1. *Interpolation inequalities.* We use Gagliardo-Nirenberg interpolation to derive integral inequalities that allow us to integrate multilinear combinations of higher derivatives of φ. For simplicity of notation, we will use C for uniform constants with dependence indicated in its arguments. For our application, we give interpolations for different range of indices.

Lemma 4.8. *Let ξ be a smooth compactly supported vector-valued function on $\mathbb{R}^n$.*

(1) *If $j_1+j_2+j_3+...+j_q=m$, then*

$$\int \left|D^{j_1}\xi * D^{j_2}\xi ... * D^{j_q}\xi\right|^2 \leq C\left\|\xi\right\|_\infty^{2q-2}\int\left|D^m\xi\right|^2.$$

(2) *If $j_1+j_2+j_3+...+j_q+j^*=2\tilde{m}$ where $j_q<\tilde{m}$ and $j^*\geq 0$, then*

$$\int \left|D^{j_1}\xi * D^{j_2}\xi ... * D^{j_q}\xi\right| \leq C\left\|\xi\right\|_\infty^{q-(2-j^*/2\tilde{m})}\left(\frac{2\tilde{m}-j^*}{2\tilde{m}}\int\left|D^{\tilde{m}}\xi\right|^2+\frac{j^*}{2\tilde{m}}\left\|\chi_{supp(\xi)}\right\|_{p^*}^{2\tilde{m}/j^*}\right).$$

(3) *If $j_1+...+j_r=2\bar{m}+1$ and all $j_i\leq\bar{m}$, then for $\varepsilon>0$*

$$\int \left|D^{j_1}\xi * D^{j_2}\xi ... * D^{j_r}\xi\right| \leq \varepsilon\int\left|D^{\bar{m}+1}\xi\right|^2+C(\varepsilon,\bar{m},\|\xi\|_\infty)\left(\int\left|D^{\bar{m}}\xi\right|^2+\int\chi_{supp(f)}\right).$$

Proof. For (1), use $p_i=\frac{m}{j_i}$ and apply the generalized Hölder's inequality

$$\int \left|D^{j_1}\xi * D^{j_2}\xi ... * D^{j_q}\xi\right|^2 \leq \left\|D^{j_1}\xi\right\|_{2p_1}^2\cdots\left\|D^{j_q}\xi\right\|_{2p_q}^2$$

and then use the Gagliardo-Nirenberg interpolation inequality (cf. [FFRS21, Theorem 1.1]) with $\theta_i=\frac{j_i}{m}$.

For (2), taking $p_i=\frac{2\tilde{m}}{j_i}$ and $p^*=\frac{2\tilde{m}}{j^*}$ if $j^*>0$, then

$$\int \left|D^{j_1}\xi * D^{j_2}\xi ... * D^{j_q}\xi\right| \leq \left\|D^{j_1}\xi\right\|_{p_1}\cdots\left\|D^{j_q}\xi\right\|_{p_q}\left\|\chi_{\mathrm{supp}(\xi)}\right\|_{p^*}.$$

Now apply the Gagliardo-Nirenberg interpolation inequality with $\theta_i=\frac{j_i}{\tilde{m}}$, applying Young's inequality if $j^*>0$.

For (3), we may split, with $a\leq\bar{m}\leq\bar{m}+1\leq b$

$$j_1+...+j_s=a$$
$$j_{s+1}+...+j_r=b.$$

Now for some p, q conjugates to be determined, let

$$p_i = \begin{cases} \frac{ap}{j_i}, & \text{if } i \in \{1, ..., s\} \\ \frac{bq}{j_i}, & \text{if } i \in \{s+1, ..., r\}. \end{cases}$$

Apply the generalized Hölder's inequality

$$\int \left| D^{j_1}\xi * D^{j_2}\xi ... * D^{j_r}\xi \right| \leq \left\| D^{j_1}\xi \right\|_{p_1} ... \left\| D^{j_q}\xi \right\|_{p_r}. \tag{4.10}$$

We have from the Gagliardo-Nirenberg interpolation inequality

$$\left\| D^{j_i}\xi \right\|_{p_i} \leq \begin{cases} \left\| D^{\bar{m}}\xi \right\|_{\frac{ap}{\bar{m}}}^{\frac{j_i}{\bar{m}}} \|\xi\|_\infty^{1-\frac{j_i}{\bar{m}}} & \text{if } i \in \{1, ..., s\} \text{ with } \theta_i = \frac{j_i}{\bar{m}} \\ \left\| D^{\bar{m}+1}\xi \right\|_{\frac{bq}{\bar{m}+1}}^{\frac{j_i}{\bar{m}+1}} \|\xi\|_\infty^{1-\frac{j_i}{\bar{m}+1}}, & \text{if } i \in \{s+1, ..., r\} \text{ with } \theta_i = \frac{j_i}{\bar{m}+1}. \end{cases}$$

Taking the product and then applying Young's inequality (for the same p, q)

$$\begin{aligned} \left\| D^{j_1}\xi \right\|_{p_1} ... \left\| D^{j_q}\xi \right\|_{p_r} &\leq \left\| D^{\bar{m}}\xi \right\|_{\frac{ap}{\bar{m}}}^{\frac{a}{\bar{m}}} \left\| D^{\bar{m}+1}\xi \right\|_{\frac{bq}{\bar{m}+1}}^{\frac{b}{\bar{m}+1}} \|\xi\|_\infty^{r-\frac{a}{m}-\frac{b}{m+1}} \\ &\leq C(\varepsilon, p, q, r, \|\xi\|_\infty) \left\| D^{\bar{m}}\xi \right\|_{\frac{ap}{\bar{m}}}^{\frac{ap}{\bar{m}}} + \varepsilon \left\| D^{\bar{m}+1}\xi \right\|_{\frac{bq}{\bar{m}+1}}^{\frac{bq}{\bar{m}+1}} \\ &= C(\varepsilon, p, q, r, \|\xi\|_\infty) \left\| D^{\bar{m}}\xi \right\|_{2\frac{a(\bar{m}+1)}{\bar{m}(a+1)}}^{2\frac{a(\bar{m}+1)}{\bar{m}(a+1)}} + \varepsilon \left\| D^{\bar{m}+1}\xi \right\|_2^2 \end{aligned} \tag{4.11}$$

where in the last line we have made the choices

$$q = \frac{2(\bar{m}+1)}{b}, \quad p = \frac{2(\bar{m}+1)}{a+1}.$$

Since $1 \leq a \leq \bar{m}$ we have

$$\frac{a(\bar{m}+1)}{\bar{m}(a+1)} \leq 1$$

and can use Hölder's and Young's inequalities to get

$$C(\varepsilon, p, q) \left\| D^{\bar{m}}\xi \right\|_{2\frac{a(\bar{m}+1)}{\bar{m}(a+1)}}^{2\frac{a(\bar{m}+1)}{\bar{m}(a+1)}} \leq C(a, \bar{m}) \left(\int \left| D^{\bar{m}}\xi \right|^2 + \int \chi_{\mathrm{supp}(f)} \right) \tag{4.12}$$

omitting the last term in the case $a = \bar{m}$. Chaining together (4.10, 4.11, 4.12) gives the result.

□

Lemma 4.9. *Let $f \in C^\infty(B_4)$ and $r_1 < r_2 \leq 4$.*

(1) *If $j_1 + ... + j_s = m$, then*

$$\int_{B_{r_1}} \left| D^{j_i} f \cdots D^{j_s} f \right|^2 \leq C(m, r_1, r_2) \|f\|_\infty^{2s-2} \sum_{j=0}^{m} \int_{B_{r_2}} \left| D^j f \right|^2.$$

(2) *If $j_1 + j_2 + j_3 + ... + j_s + j^* = 2\tilde{m}$ where $j_q < \tilde{m}$ and $j^* \geq 0$, then*

$$\int_{B_{r_1}} \left| D^{j_i} f \cdots D^{j_s} f \right| \leq C(\tilde{m}, r_1, r_2) \, \|f\|_\infty^{s-(2-j^*/2\tilde{m})} \left(\frac{2\tilde{m} - j^*}{2\tilde{m}} \sum_{j=0}^{\tilde{m}} \int_{B_{r_2}} \left| D^j f \right|^2 + \frac{j^*}{2\tilde{m}} \left\| \chi_{supp(f)} \right\|_{p^*}^{2\tilde{m}/j^*} \right).$$

(3) *If* $j_1 + ... + j_r = 2\bar{m} + 1$ *and all* $j_i \leq \bar{m}$, *then for* $\varepsilon > 0$

$$\int_{B_{r_1}} \left| D^{j_1} f * D^{j_2} f ... * D^{j_r} f \right| \leq \varepsilon \int_{B_{r_2}} \left| D^{\bar{m}+1} \xi \right|^2 + C(\varepsilon, r_1, r_2, \bar{m}, \|f\|_\infty) \left(\sum_{j=0}^{\bar{m}} \int_{B_{r_2}} \left| D^j \xi \right|^2 + 1 \right).$$

Proof. Set $\tilde{\eta} \in C_0^\infty(B_3)$ that is 1 on B_{r_1}, 0 on $B_3(0) \backslash B_{r_2}(0)$, $0 \leq \tilde{\eta} \leq 1$ and $\|\tilde{\eta}\|_{C^m} \leq C(m)$. By Lemma 4.8 (second line below)

$$\begin{aligned}
\int_{B_{r_1}} \left| D^{i_1} f \cdots D^{i_s} f \right|^2 &\leq \int_{B_{r_2}} \left| D^{i_1}(\tilde{\eta} f) \cdots D^{i_s}(\tilde{\eta} f) \right|^2 \\
&\leq \|\tilde{\eta} f\|_\infty^{2s-2} \int_{B_{r_2}} \left| D^m(\tilde{\eta} f) \right|^2 \\
&\leq C(m, r_1, r_2, \|\tilde{\eta}\|_{C^m}) \|f\|_\infty^{2s-2} \sum_{j=0}^{m} \int_{B_{r_2}} \left| D^j f \right|^2 .
\end{aligned}$$

The second and third inequalities in the statement of the Lemma follows by applying the previous Lemma in a similar way. □

The following is simple but will be used repeatedly, so we explicitly note it.

Lemma 4.10. *Suppose that* $r_1 < r_2$. *Then for* $\tilde{\varepsilon} > 0$

$$\int_{B_{r_1}} \left| D^{k+3} \varphi \right|^2 \leq \tilde{\varepsilon} \int_{B_{r_2}} \left| D^{k+4} \varphi \right|^2 + C(\tilde{\varepsilon}, r_1, r_2) \int_{B_{r_2}} \left| D^{k+2} \varphi \right|^2 .$$

Proof. For some $\tilde{\eta} = 1$ on B_{r_1} supported inside B_{r_2}

$$\begin{aligned}
\int_{B_{r_2}} \left| D^{k+3} \varphi \right|^2 \tilde{\eta}^2 &= - \int_{B_{r_2}} D^{k+2} \varphi * \left(\tilde{\eta}^2 D^{k+4} \varphi + 2 \tilde{\eta} D \tilde{\eta} D^{k+3} \varphi \right) \\
&\leq \tilde{\varepsilon} \int_{B_{r_2}} \tilde{\eta}^2 \left| D^{k+4} \varphi \right|^2 + \frac{1}{\tilde{\varepsilon}} C \int_{B_{r_2}} \tilde{\eta}^2 \left| D^{k+2} \varphi \right|^2 \\
&\quad + \frac{1}{2} \int_{B_{r_2}} \tilde{\eta}^2 \left| D^{k+3} \varphi \right|^2 + C \int_{B_{r_2}} |D \tilde{\eta}|^2 \left| D^{k+2} \varphi \right|^2 .
\end{aligned}$$

Thus

$$\int_{B_{r_1}} \left| D^{k+3} \varphi \right|^2 \leq 2 \tilde{\varepsilon} \int_{B_{r_2}} \tilde{\eta}^2 \left| D^{k+4} \varphi \right|^2 + \frac{2}{\tilde{\varepsilon}} C \int_{B_{r_2}} \left(\tilde{\eta}^2 + |D \tilde{\eta}|^2 \right) \left| D^{k+2} \varphi \right|^2 .$$

□

4.3.2. *Evolution inequalities.*

Proposition 4.11. *Let* $\rho_\alpha^2 \in C_0^\infty(B_2(0))$ *defined by (4.6). Working in Darboux charts, for* $\varepsilon > 0$ *we have*

$$\int_{B_2} \frac{d}{dt} \left(\left| D^{k+2} \varphi \right|_g^2 dV_g \right) \rho_\alpha^2 \leq -2 \int_{B_2} \left| D^{k+4} \varphi \right|_g^2 \rho_\alpha^2 dV_g + \varepsilon \int_{B_3} |D^{k+4} \varphi|^2 dV_g + C(k, \varepsilon, \|\varphi\|_{C^3}) \left(\sum_{m=3}^{k+2} \int_{B_3} \left| D^m \varphi \right|_g^2 dV_g + 1 \right). \tag{4.13}$$

Proof. In a Darboux chart, express $dV_g = V_g dx$. We have

$$\begin{aligned}\frac{d}{dt}\left(\left|D^{k+2}\varphi\right|_g^2 V_g\right) =& 2\left(\partial_t\varphi_{i_1\dots i_{k+2}}\right)\left(\varphi_{j_1\dots j_{k+2}} g^{i_1j_1}g^{i_2j_2}\dots g^{i_{k+2}j_{k+2}}V_g\right)\\ &+\varphi_{i_1\dots i_{k+2}}\varphi_{j_1\dots j_{k+2}}\partial_t\left(g^{i_1j_1}g^{i_2j_2}\dots g^{i_{k+2}j_{k+2}}V_g\right)\\ =& -2(g^{kl}g^{pq}\varphi_{klpq}+G)_{i_1\dots i_{k+2}}\left(\varphi_{j_1\dots j_{k+2}}g^{i_1j_1}\dots g^{i_{k+2}j_{k+2}}V_g\right)\\ &+\varphi_{i_1\dots i_{k+2}}\varphi_{j_1\dots j_{k+2}}\partial_t\left(g^{i_1j_1}g^{i_2j_2}\dots g^{i_{k+2}j_{k+2}}V_g\right).\end{aligned} \tag{4.14}$$

We count the highest order of derivatives of φ in $x^1, ..., x^n$ for each term below:

(1) g, g^{-1}, V_g are of 2nd order
(2) $\partial_t g, \partial_t g^{-1}$ and $\partial_t V_g = V_g g^{ij}\partial_t g_{ij}$ are of 6th order
(3) $(g^{kl}g^{pq}\varphi_{klpq})_{i_1\dots i_{k+2}}$ is of $(k+6)$th order and $G_{i_1\dots i_{k+2}}$ is of $(k+5)$th order.

For the sake of notation, we will use

(1) $P = P(x, D\varphi, D^2\varphi, D^3\varphi)$,
(2) $Q = P(x, D\varphi, ..., D^{k+3}\varphi)$ to be described in (4.18) and below.
(3) Bounded second order quantities are absorbed and not explicitly stated unless necessary (in particular, dV_g will be dropped when not being differentiated.)

Multiplying ρ_α^2 to localize in a chart then integrate on L, we may then perform:

(**A**) Integration by parts twice the first term in (4.14) leads to

(4.15)
$$\int_{B_2}(-2g^{kl}g^{pq}\varphi_{klpq})_{i_1\dots i_{k+2}}\left(\varphi_{j_1\dots j_{k+2}}g^{i_1j_1}\dots g^{i_{k+2}j_{k+2}}V_g\right)\rho_\alpha^2 = -2\int_{B_2}\left|D^{k+4}\varphi\right|_g^2\rho_\alpha^2\,dV_g + I$$

where

$$I = \int_{B_2} D^{k+4}\varphi * \left(D^4\varphi + P^2\right) * D^{k+2}\varphi\rho_\alpha^2 + D^{k+3}\varphi * \left(P * \rho_\alpha^2 + D\rho_\alpha^2\right) + D^{k+2}\varphi * \left(P * D\rho_\alpha^2 + D^2\rho_\alpha^2\right)\right).$$

To deal with the first term in I:
(4.16)
$$\int_{B_2} D^{k+4}\varphi * \left(D^4\varphi + P^2\right) * D^{k+2}\varphi \le \varepsilon\int_{B_2}\rho_\alpha^2|D^{k+4}\varphi|^2 + C(\varepsilon)\int_{B_2}\rho_\alpha^2\left(\left|D^{k+2}\varphi * \left(D^4\varphi + P^2\right)\right|^2\right).$$

Lemma 4.9 with $m = k$, $f = D^3\varphi$ and $r_2 = 5/2$ yields

$$\int_{B_2}\left|D^{k+2}\varphi * D^4\varphi\right|^2 \le C(D^3\varphi)\sum_{j=0}^{m}\int_{B_{5/2}}\left|D^{j+3}\varphi\right|^2.$$

Applying Lemma (4.10) to the highest order term provides a bound of (4.16) by the positive terms in (4.13) noting also that

$$\int_{B_2}\left|D^{k+2}\varphi * P^2\right|^2 \le C(D^3\varphi)\int_{B_2}\left|D^{k+2}\varphi\right|^2.$$

Next
(4.17)
$$\int_{B_2}\left|(P*\rho_\alpha^2+D\rho_\alpha^2)D^{k+4}\varphi*D^{k+3}\varphi\right|\leq\varepsilon\int_{B_2}\rho_\alpha^2|D^{k+4}\varphi|^2+\frac{1}{\varepsilon}C(P)\int_{B_2}\left|D^{k+3}\varphi\right|^2$$

recalling that $D\rho_\alpha$ is bounded by uniform constants and $D^2\varphi$. By Lemma 4.10 (choosing $\tilde{\varepsilon}\approx c\varepsilon^2$), (4.17) is bounded by

$$\varepsilon\int_{B_2}\rho_\alpha^2|D^{k+4}\varphi|^2+\varepsilon\int_{B_3}\left|D^{k+4}\varphi\right|^2+\frac{1}{\varepsilon^3}C\int_{B_3}\left|D^{k+2}\varphi\right|^2$$

which is of the correct form. Finally for I, using (4.7),

$$\int_{B_2}\left|D^{k+4}\varphi*D^{k+2}\varphi*\left(P*D\rho_\alpha^2+D^2\rho_\alpha^2\right)\right|\leq\varepsilon\int_{B_2}\rho_\alpha^2|D^{k+4}\varphi|^2+C(\varepsilon,P)\int_{B_2}\left|D^{k+2}\varphi\right|^2.$$

(**B**) Note that when applying the product rule successively to G, we will get

(1) A single highest order term which is linear in the highest order with coefficients involving at most order $D^3\varphi$.
(2) Second to highest order terms that are linear in the second highest order, may have a factor of $D^4\varphi$, all other dependence of lower order.
(3) Terms of lower order, which could be multilinearly dependent on various lower orders.

Thus

$$G_{i_1\ldots i_{k+1}}=1*D^{k+4}\varphi+Q. \tag{4.18}$$

with Q having highest order $D^{k+3}\varphi$. This is observed by iterating the following expansion: Using DG to denote a derivative in x of the composition

$$x\mapsto G(x,D\varphi(x),D^2\varphi(x),D^3\varphi(x))$$

and $\bar{D}G$ to denote derivatives in all 4 arguments of G, we have

$$DG=\bar{D}G*\left(D^4\varphi+D^3\varphi+D^2\varphi+\phi\right)$$

where ϕ is the term generated by $\bar{D}G/Dx$. Continuing

$$\begin{aligned}D^2G&=\bar{D}G*\left(D^5\varphi+D^4\varphi+D^3\varphi+D\phi*\left(D^4\varphi+D^3\varphi+D^2\varphi\right)\right)\\&\quad+\bar{D}^2G*\left(D^4\varphi+D^3\varphi+D^2\varphi+\phi\right)*\left(D^4\varphi+D^3\varphi+D^2\varphi+\phi\right)\\&\quad\ldots\end{aligned}$$

$$\begin{aligned}D^{k+1}G&=\bar{D}G*\left(D^{k+4}\varphi+D^{k+3}\varphi+D^{k+2}\varphi+\ldots\right)\\&\quad+\bar{D}^2G*\left(D^{k+3}\varphi+D^{k+2}\varphi+D^{k+1}\varphi+\ldots\right)*\left(D^4\varphi+D^3\varphi+D^2\varphi+\phi\right)\\&\quad+\bar{D}^3G*\left(D^{k+2}\varphi+\ldots\right)*\left\{\left(D^5\varphi+\ldots\right)+\left(D^4\varphi+\ldots\right)*\left(D^4\varphi+\ldots\right)\right\}\\&\quad\ldots\end{aligned} \tag{4.19}$$

Now integrate by parts:

$$\int_{B_2} G_{i_1\ldots i_{k+2}}\left(\varphi_{j_1\ldots j_{k+2}} g^{i_1 j_1}\ldots g^{i_{k+2}j_{k+2}} V_g\right)\rho_\alpha^2 = -\int_{B_2} G_{i_1\ldots i_{k+1}}\partial_{i_{k+2}}\left[\rho_\alpha^2\left(\varphi_{j_1\ldots j_{k+2}} g^{i_1 j_1}\ldots g^{i_{k+2}j_{k+2}} V_g\right)\right]$$
$$= \int_{B_2}\left(D^{k+4}\varphi * D^{k+3}\varphi\right)\rho_\alpha^2 + \int_{B_2}(D\rho_\alpha^2 + \rho_\alpha^2 P)D^{k+4}\varphi * D^{k+2}\varphi$$
$$+ \int_{B_2}\left(Q * D^{k+3}\varphi\right)\rho_\alpha^2 + \int_{B_2}(D\rho_\alpha^2 + \rho_\alpha^2 P)Q * D^{k+2}\varphi.$$

We use Peter-Paul's inequality we split into two types of terms:

$$\varepsilon\int_{B_2}\left|D^{k+4}\varphi\right|^2\rho_\alpha^2 + \varepsilon\int_{B_2} Q^2\rho_\alpha^2$$

and

$$C(\varepsilon, D\rho_\alpha)\int_{B_2}\left(\left|D^{k+3}\varphi\right|^2 + \left|D^{k+2}\varphi\right|^2\right)(P^2+P+1)\rho_\alpha^2.$$

First,

$$\int_{B_2}\left|D^{k+3}\varphi\right|^2\left(P^2+P+1\right)\rho_\alpha^2 \le C\left(D^3\varphi\right)\int_{B_2}\left|D^{k+3}\varphi\right|^2$$

is bounded by the argument in Lemma (4.10).

One can prove by induction, observing (4.18) and (4.19), that for each term in Q, the total number of derivatives of $D^3\varphi$ that arise will sum up to no more than $k+1$ (i.e. $D^{k-1}\varphi * D^6\varphi * D^5\varphi = D^{3+k-4}\varphi * D^{3+3}\varphi * D^{3+2}\varphi$, here $k-4+3+2 = k+1$.) Applying Lemma 4.9 for $f = D^3\varphi$ and $m = k+1$ to each of the squared terms gives

$$\int_{B_2} Q^2 dV_g \le C\left(||D^3\varphi||\right)\sum_{i=0}^{k+1}\int_{B_3}|D^{i+3}\varphi|^2$$

thus $\varepsilon\int_{B_2} Q^2 dV_g$ has the correct bound, by Lemma 4.10.

Finally we finish bounding the last term in (4.14)

$$\int\varphi_{i_1\ldots i_{k+2}}\varphi_{j_1\ldots j_{k+2}}\partial_t\left(g^{i_1 j_1}g^{i_2 j_2}\ldots g^{i_{k+2}j_{k+2}}V_g\right)\rho_\alpha^2 = \int\left(D^{k+2}\varphi * D^{k+2}\varphi * D^6\varphi\right)\rho_\alpha^2.$$

Apply Lemma 4.9 for $f = D^3\varphi$ and $\tilde{m} = k$

$$\int_{B_2}\left(D^{k+2}\varphi * D^{k+2}\varphi * D^6\varphi\right)\rho_\alpha^2 \le \varepsilon\int_{B_3}\left|D^{k+4}\varphi\right|^2 + C\left(\varepsilon, k, D^3\varphi\right)\sum_{j=3}^{k+3}\int_{B_3}\left|D^j\varphi\right|^2$$

We may then sweep away the $\int_{B_2}\left|D^{k+3}\varphi\right|^2$ term Lemma 4.10 (choosing $\tilde{\varepsilon} \approx c\varepsilon^2$) to conclude the proof. □

Proposition 4.12. *Let $\rho_\alpha^2 \in C_0^\infty(B_2(0))$. Considering the decomposition in Lemma 4.7, for $\varepsilon > 0$ we have*

$$\int_{B_2} \frac{d}{dt}\left(\left(|S_k|_g^2 + 2\langle D^{k+2}\varphi, S_k\rangle_g\right) dV_g\right)\rho_\alpha^2 \leq C(k,\varepsilon, D^3\varphi)\left(\sum_{j=3}^{k+2}\int_{B_3}\left|D^j\varphi\right|^2 + 1\right) + \varepsilon\int_{B_3}\left|D^{k+4}\varphi\right|^2.$$

Proof. Recall that

$$S_k = \left(1 + D^3\varphi\right) * (D^{k+1}\varphi + D^k\varphi * D^4\varphi + D^{k-1}\varphi * D^4\varphi * D^4\varphi + D^{k-1}\varphi * D^5\varphi + ...) \tag{4.20}$$

Differentiating with respect to t generates product rule expansions with 4 orders of derivatives added to a factor in each term, that is (modulo lower order geometrically controlled values like V_g)

(4.21)

$$\frac{d}{dt}|S_k|_g^2\, dV_g = \left(1 + D^3\varphi\right) * \left(D^{k+5}\varphi + D^{k+4}\varphi * D^4\varphi + D^k\varphi * D^8\varphi + ...\right) * S_k$$
$$\left(D^6\varphi + D^7\varphi\right) * \left(D^{k+1}\varphi + D^k\varphi * D^4\varphi + D^{k-1}\varphi * D^4\varphi * D^4\varphi + ...\right) * S_k.$$

Integrate by parts:

$$\int_{B_2}\left(D^{k+5}\varphi * \left(1 + D^3\varphi\right) * S_k\right)\rho_\alpha^2 = \int_{B_2} D^{k+4}\varphi * \left(D^4\varphi * S_k + \left(1 + D^3\varphi\right) * DS_k\right)\rho_\alpha^2$$
$$+ \int_{B_2}\left(D^{k+4}\varphi * \left(1 + D^3\varphi\right) * S_k\right) * D\rho_\alpha^2$$
$$\leq \varepsilon\int_{B_2}\left|D^{k+4}\varphi\right|^2\rho_\alpha^2 + C(\varepsilon)\int_{B_2}\left(\left|D^4\varphi * S_k\right|^2 + \left|D^3\varphi * DS_k\right|^2\right)\rho_\alpha^2$$
$$+ C(\varepsilon)\int_{B_2}\left|\left(1 + D^3\varphi\right) * S_k\right|^2 |D\rho_\alpha|^2.$$

Now apply Lemma 4.9 with $f = D^3\varphi$ and $m = k-1$

$$\int_{B_2}\left(\left|D^4\varphi * S_k\right|^2 + \left|D^3\varphi * DS_k\right|^2\right)\rho_\alpha^2 \leq C(k, D^3\varphi)\sum_{j=3}^{k+2}\int_{B_3}\left|D^j\varphi\right|^2. \tag{4.22}$$

Similarly using $\tilde{\eta}$ as in the proof of Lemma 4.9, we have

$$\int_{B_2}\left|\left(1 + D^3\varphi\right) * S_k\right|^2 |D\rho_\alpha|^2 \leq C\left(D\rho_\alpha, D^3\varphi\right)\sum_{j=3}^{k+1}\int_{B_3}\left|D^j\varphi\right|^2.$$

Continuing with the terms in (4.21)

$$\int_{B_2}\left(1 + D^3\varphi\right) * \left(D^{k+4}\varphi * D^4\varphi * S_k\right)\rho_\alpha^2 \leq \varepsilon\int_{B_2}\left|D^{k+4}\varphi\right|^2\rho_\alpha^2 + C(\varepsilon)\int_{B_2}\left|D^3\varphi * D^4\varphi * S_k\right|^2\rho_\alpha^2$$

with the latter term enjoying the same bound as (4.22). The remaining terms are of the form

$$\int\left(D^{3+j_1}\varphi * D^{3+j_2}\varphi * .. * D^{3+j_q}\varphi\right)\rho_\alpha^2$$

with $j_1 + ... + j_q \le 2k$ so

$$\int_{B_2} \left(D^{3+j_1}\varphi * D^{3+j_2}\varphi * .. * D^{3+j_q}\varphi\right)\rho_\alpha^2 \le C(m, D^3\varphi)\left(\sum_{j=3}^{k+3}\int_{B_3}\left|D^j\varphi\right|^2 + 1\right)$$

by Lemma 4.9 again. Applying Lemma 4.10 to $\int_{B_3}\left|D^{k+3}\varphi\right|^2$ completes the desired bound for the integral of the (4.21) terms.

Next

(4.23)

$$\int_{B_2}\frac{d}{dt}\left(2\langle D^{k+2}\varphi, S_k\rangle_g dV_g\right)\rho_\alpha^2 dV_g = \int_{B_2}\left(D^{k+6}\varphi * S_k\right)\rho_\alpha^2 dV_g + \int_{B_2} D^{k+2}\varphi * \frac{d}{dt}\left(S_k * V\right)\rho_\alpha^2.$$

Integrating the first term by parts twice yields

$$\begin{aligned}\int_{B_2} D^{k+6}\varphi * S_k\rho_\alpha^2 &= \int_{B_2} D^{k+4}\varphi * D^2\left(S_k * V\right)\rho_\alpha^2 + D\rho_\alpha^2 * D\left(S_k * V\right) + D^2\rho_\alpha^2 * \left(S_k * V\right)\\ &\le \varepsilon\int_{B_2}\left|D^{k+4}\varphi\right|^2\rho_\alpha^2 + C(\varepsilon)\int_{B_2}\left|D^2 S_k\right|^2\rho_\alpha^2 + C\left(\varepsilon, D^2\rho_\alpha^2\right)\int_{B_2}\left(|DS_k|^2 + |S_k|^2\right).\end{aligned}$$

Again Lemma 4.9 with $f = D^3\varphi$ and $\tilde{m} = k$ and $r_2 = 5/2$

$$\int_{B_2}\left|D^2 S_k\right|^2\rho_\alpha^2 \le C(k, D^3\varphi)\sum_{j=3}^{k+3}\int_{B_{5/2}}\left|D^j\varphi\right|^2. \tag{4.24}$$

$$\le \varepsilon\int_{B_3}\left|D^{k+4}\varphi\right|^2 + C\left(\varepsilon, k, D^3\varphi\right)\sum_{j=3}^{k+2}\int_{B_3}\left|D^j\varphi\right|^2 \tag{4.25}$$

using Lemma 4.10.

Now look at second term in (4.23). Note that

$$D^{k+2}\varphi * \frac{d}{dt}\left(S_k * V\right) = D^{k+2}\varphi * D^{k+5}\varphi * D^3\varphi + D^{k+2}\varphi * \left(D^{k+4}\varphi * D^4\varphi + ...\right).$$

The highest order term can be dealt with via integration by parts away from $D^{k+5}\varphi$ and then an iterated Peter-Paul, carefully choosing smaller ε and using Lemma 4.9. For the remaining terms, we need the third statement in Lemma 4.9 which gives

$$\begin{aligned}\int_{B_2}\left(D^{3+j_1}\varphi * D^{3+j_2}\varphi * .. * D^{3+j_q}\varphi\right)\rho_\alpha^2 &\le \varepsilon\int_{B_{5/2}}\left|D^{k+4}\varphi\right|^2\\ &+ C(\varepsilon, k, \frac{5}{2}, \|f\|_\infty)\left(\sum_{j=0}^{k}\int_{B_{5/2}}\left|D^{3+j}\varphi\right|^2 + 1\right)\end{aligned}$$

as $j_1 + ... + j_q = 2k + 1$. A final application of Lemma 4.10 to $\int_{B_{5/2}}\left|D^{k+3}\varphi\right|^2$ completes the proof. □

In Proposition 4.11 we have isolated 'good' terms $-2\int_{B_2}\left|D^{k+4}\varphi\right|^2\rho_\alpha^2$. We would like to use them to offset the 'bad' terms of the form $\varepsilon\int_{B_3}\left|D^{k+4}\varphi\right|^2 dV_g$ that occur in Propositions 4.11 and 4.12. Because the expressions for $D^{k+4}\varphi$ are different in each chart in the cover, the difficulty arises that we cannot directly beat the terms occurring on a larger ball by terms on a smaller ball of different charts, even when the smaller balls cover the larger ball. To make an argument that the bad terms in a larger ball of one chart are offset by the good terms in a smaller ball in a different chart requires bounding the bad terms by a global, well-defined geometric quantity involving derivatives of the second fundamental form, modulo a lower order difference. This is the point of the following lemma.

Lemma 4.13. *Take a finite cover of charts Υ^α, each over $B_4(0)$ and partition of unity ρ_α^2 in $B_2(0)$ in each respective chart as described by (4.6). Then*

$$\sum_\alpha \int_{B_3} \left|D^{k+4}\varphi\right|^2 dV_g \leq 2N \sum_{m=0}^{k+1} \int_L \left|\nabla^{k+1-m} A\right|^2 dV_g + C.$$

Proof. Note that from Lemma 4.7

$$\left|D^{k+4}\varphi\right|_g^2 \leq 2\left|\nabla^{k+1} A\right|_g^2 + 2\left|S_{k+2}\right|_g^2.$$

Let $\iota = \frac{1}{k+2}$. Then we have, taking $\tilde{\eta} = 1$ on each $B_3(0)$, with $\tilde{\eta} \in C_c^\infty(B_{3+\iota}(0))$

$$\begin{aligned}\int_{B_3} \left|D^{k+4}\varphi\right|^2 dV_g &\leq 2\int_{B_3} \left(\left|\nabla^{k+1}A\right|^2 + \left|S_{k+2}\right|_g^2\right) dV_g \\ &\leq 2\int_{B_3} \left|\nabla^{k+1}A\right|^2 dV_g + 2\int_{B_{3+\iota}} \left|S_{k+2}\right|_g^2 \tilde{\eta}^2 dV_g \\ &\leq 2\int_{B_3} \left|\nabla^{k+1}A\right|^2 dV_g + C\left(\sum_{m=3}^{k+3} \int_{B_{3+\iota}} \left|D^m\varphi\right|^2 dV_g + 1\right)\end{aligned}$$

by Lemma 4.9. Iterating this argument, using

$$\int_{B_{3+\iota}} \left|D^{k+3}\varphi\right|^2 dV_g \leq 2\int_{B_{3+\iota}} \left|\nabla^k A\right|^2 dV_g + C\left(\sum_{m=3}^{k+2} \int_{B_{3+2\iota}} \left|D^m\varphi\right|^2 dV_g + 1\right)$$

and so forth, for a total of $k+1$ steps, we have by using $\left|D^3\varphi\right| \leq |A| + C$ that

$$\int_{B_3} \left|D^{k+4}\varphi\right|^2 dV_g \leq 2\sum_{m=0}^{k+1} \int_{B_{3+\frac{k+1}{k+2}}} \left|\nabla^{k+1-m}A\right|^2 \tilde{\eta}^2 dV_g + C.$$

Now for any set of functions $\tilde{\eta}_\alpha$ who are 1 on $B_r \subset B_4$ on each chart Υ^α, we can bound

$$\sum_\alpha \int_{B_4} \left|\nabla^m A\right|^2 \tilde{\eta}_\alpha^2 dV_g \leq \max_{x\in L}\left(\sum_\alpha \tilde{\eta}_\alpha^2(x)\right) \int_L \left|\nabla^m A\right|^2 dV_g \leq N \int_L \left|\nabla^m A\right|^2 dV_g.$$

It follows that

$$\sum_\alpha \int_{B_3} \left|D^{k+4}\varphi\right|^2 dV_g \leq 2N \sum_{m=0}^{k+1} \int_L \left|\nabla^{k+1-m} A\right|^2 dV_g + C.$$

□

4.4. **Proof of the main theorem.**

Proof of Proposition 4.2. At a fixed time t_0 we may take the ambient charts $\{\Upsilon^\alpha\}$ for a tubular neighborhood of L and subordinate partition of unity $\left\{\rho_\alpha^2\right\}$ which restrict to charts (via the x coordinate) for L with the same partition of unity.

Differentiate

$$
\begin{aligned}
\frac{d}{dt}\int_L \left|\nabla^{k-1}A\right|_g^2 dV_g &= \int_L \frac{d}{dt}\left(\left|\nabla^{k-1}A\right|_g^2 dV_g\right)\\
&= \int_{B_2}\left(\sum_\alpha \rho_\alpha^2\right)\frac{d}{dt}\left[\left(\left|D^{k+2}\varphi\right|_g^2 + |S_k|_g^2 + 2\langle D^{k+2}\varphi, S_k\rangle_g\right) dV_g\right]\\
&= \sum_\alpha \int_{B_2}\frac{d}{dt}\left(\left|D^{k+2}\varphi\right|_g^2 dV_g\right)\rho_\alpha^2\\
&\quad + \sum_\alpha \int_{B_2}\frac{d}{dt}\left[\left(|S_k|_g^2 + 2\langle D^{k+2}\varphi, S_k\rangle_g\right) dV_g\right]\rho_\alpha^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{d}{dt}\int_L \left|\nabla^{k-1}A\right|_g^2 dV_g &\le -2\sum_\alpha \int_{B_2}\left|D^{k+4}\varphi\right|_g^2 \rho_\alpha^2 dV_g + \varepsilon\sum_\alpha \int_{B_3}|D^{k+4}\varphi|^2 dV_g\\
&\quad + \sum_\alpha C(k,\varepsilon,\|\varphi\|_{C^3})\left(\sum_{m=3}^{k+2}\int_{B_3}\left|D^m\varphi\right|_g^2 dV_g + 1\right)
\end{aligned}
\tag{4.26}
$$

by Propositions 4.11 and 4.12. Now apply Lemma 4.13

$$
\begin{aligned}
\sum_\alpha \int_{B_3}|D^{k+4}\varphi|^2 dV_g &\le \left(NC\sum_{m=0}^{k+1}\int_L |\nabla^m A|^2 dV_g + C\right)\\
&= NC\sum_{m=0}^{k+1}\int_L |\nabla^m A|^2\left(\sum_\alpha \rho_\alpha^2\right) dV_g + C\\
&= NC\sum_{m=0}^{k+1}\sum_\alpha \int_{B_2}|\nabla^m A|^2 \rho_\alpha^2 dV_g\\
&\le NC\sum_{m=0}^{k+1}\sum_\alpha \int_{B_2} 2\left(\left|D^{m+3}\varphi\right|^2 + |S_{m+1}|_g^2\right)\rho_\alpha^2 dV_g\\
&= 2NC\sum_\alpha \int_{B_2}\left(\left|D^{k+4}\varphi\right|^2 + |S_{k+2}|_g^2\right)\rho_\alpha^2 dV_g\\
&\quad + 2NC\sum_{m=0}^{k}\sum_\alpha \int_{B_2} 2\left(\left|D^{m+3}\varphi\right|^2 + |S_{m+1}|_g^2\right)\rho_\alpha^2 dV_g.
\end{aligned}
$$

Note that from Lemma 4.10

$$
2\int_{B_2}\left|D^{k+3}\varphi\right|^2 \rho_\alpha^2 dV_g \le \frac{1}{4NC}\int_{B_3}|D^{k+4}\varphi|^2 dV_g + C\left(N,\left|D\rho_\alpha^2\right|\right)\int_{B_3}|D^{k+2}\varphi|^2 dV_g.
$$

Note also Lemma 4.9, recalling (4.20), then Lemma 4.10 on the highest order resulting term gives

$$
\begin{aligned}
2\int_{B_2} |S_{k+2}|_g^2 \rho_\alpha^2 dV_g &\le C\sum_{m=3}^{k+3}\int_{B_{5/2}} \left|D^m\varphi\right|_g^2 + C \\
&\le \frac{1}{4NC}\int_{B_3} |D^{k+4}\varphi|^2 dV_g + C\left(N, \left|D\rho_\alpha^2\right|\right)\sum_{m=3}^{k+2}\int_{B_3}\left|D^m\varphi\right|_g^2 + C.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_\alpha \int_{B_3} |D^{k+4}\varphi|^2 dV_g \le 4NC\sum_\alpha\int_{B_2}\left|D^{k+4}\varphi\right|^2\rho_\alpha^2 dV_g + 8NC\sum_{m=3}^{k+2}\sum_\alpha\int_{B_2}\left|D^m\varphi\right|^2\rho_\alpha^2 dV_g \\
+ 8NC\sum_{m=1}^{k+1}\sum_\alpha\int_{B_2}|S_m|_g^2\rho_\alpha^2 dV_g. \qquad (4.27)
\end{aligned}
$$

Choosing $\varepsilon < (2NC)^{-1}$ in (4.26) in light of (4.27) we have

$$
\frac{d}{dt}\int_L \left|\nabla^{k-1}A\right|_g^2 dV_g \le C(N, k, \|\varphi\|_{C^3})\left(\sum_{m=3}^{k+2}\int_{B_3}\left|D^m\varphi\right|_g^2 + \sum_{m=1}^{k+1}\int_{B_3}|S_m|_g^2 + 1\right).
$$

Applying Lemma 4.9 to the $\int |S_m|_g^2$ terms and then Lemma 4.13 to the $|D^m\varphi|_g^2$ terms yields the result. □

Proof of Theorem 4.1. Suppose now that F is a solution to (1.1) with $|A| \le K$ on $[0, T)$. Starting with

$$
\int_L |A|^2\, dV_g(t) \le K\,\mathrm{Vol}\,(L) \le C
$$

we may apply Proposition 4.2 and apply differential inequalities: continuing with

$$
\frac{d}{dt}\int_L |\nabla A|^2\, dV_g(t) \le C\int_L |\nabla A|^2\, dV_g(t) + C\int_L |A|^2\, dV_g(t)
$$

and so forth, obtaining bounds of the form

$$
\int_L \left|\nabla^{k-1}A\right|^2 dV_g(t) \le C(k, K, F_0, T) \qquad (4.28)
$$

for arbitrary k.

Now at any $t_0 \in [0, T)$ we may take a cover Υ^α as described in Proposition 4.6. By Lemma 4.13 and (4.28) we have

$$
\left\|D^k\varphi\right\|_{L^2(B_3)} \le C(k, K, F_0, T) \qquad (4.29)
$$

for all k, in every chart. By Sobolev embedding theorems, we have Hölder bounds on $D^k\varphi$ over B_2 for each chart. In particular, there will be uniform bounds on $\frac{d}{dt}D\varphi$ and $\frac{d}{dt}D^2\varphi$ which control the speed of the flow in the chart and the rate of change of the slope the manifold L_t makes with respect to the tangent plane at the origin in the chart. We conclude then the manifolds L_t will continue to be described by the set of charts taken at t_0 for $t < \max\{T, t_0 + \tau\}$ for some positive τ with an apriori lower bound. (Perhaps we take c_n slightly larger in (4.2)). By choosing t_0 near T we are assured that these fixed charts describe the flow for all values $t \in [t_0, T)$.

Now observe that with fixed speed bounds, the paths $x \mapsto F(x, t)$ of the normal flow are Lipschitz and hence the normal flow extends to a well-defined continuous map

$$F : L \times [0, T] \to M. \tag{4.30}$$

We claim that $F(\cdot, T)$ is a smooth immersion. While within a chart, the vertical maps

$$\bar{F}(x) := (x, d\varphi(x, t)) \tag{4.31}$$

converge in every Hölder norm to a smooth map at T, we still must argue that the charts given by the x coordinates do not collapse as $t \to T$. This can be argued locally, using coordinates on L_{t_0}. For any given $x \in L_{t_0}$ we may choose a chart such that $x \in B_1(0) \subset B_3(0)$. We are already assuming F is an immersion at t_0 so this coordinate chart gives us a coordinate chart for the abstract smooth manifold L. For $t > t_0$ the normal flow F is given by

$$F(x, t) = (\chi_t(x), d\varphi(\chi_t(x), t)) \tag{4.32}$$

for some local diffeomorphism $\chi_t(x) : B_1(0) \to B_2(0)$ from Claim 3.3, provided that t_0 is chosen close enough to T such that

$$\chi_t(x) \in B_2(0) \text{ for all } x \in B_1(0) \text{ and } t \in [t_0, T).$$

This choice of t_0 is possible given that $\chi_t(x)$ is controlled by the normal projection of $\frac{d\bar{F}}{dt}$ and the inverse $(d\bar{F})^{-1}$, for $\bar{F}$ defined by (4.31), both of which are universally controlled given (4.2) and (4.29).

Now because (4.31) is uniformly smooth, it can be extended smoothly to $[t_0, T + \delta)$, as well as the normal flow associated to this extension. Applying Claim 3.3 (note that we may extend the flow outside B_3 in a nice way which doesn't affect the behavior in $B_2(0)$) we get a smooth diffeomorphism χ_T. For $x \in B_1(0)$ we can compute the normal flow F,

$$F(x, T) = (\chi_T(x), d\varphi(\chi_T(x), T))$$

which is a smooth extension of (4.32) to T, by the uniform estimates on φ. Now $F(x, T)$ is a smooth immersion from $B_1(0)$ because χ_T is a diffeomorphism. As x was chosen arbitrarily, we conclude the continuous extension of F defined in (4.30) must be a smooth immersion from L at T.

We may now restart the flow by Proposition 3.2 with initial immersion $F(x, T)$. The time derivatives of the new flow and F agree to any order at T. Therefore the new flow is a smooth extension of F to $[0, T + \varepsilon)$ for some $\varepsilon > 0$. Moreover, Theorem 3.4 asserts that this is the only smooth extension. □

5. Appendix

5.1. **Submanifold with bounded second fundamental form A.** It is a known and frequently used fact that when $|A|$ is bounded then the submanifold can be written as a graph over a controlled region in its tangent space. We provide a proof below for any dimension and codimension.

Proposition 5.1. *Let L^k be a compact manifold embedded in a compact Riemannian manifold (M^{k+l}, g). Suppose that the second fundamental form of L satisfies $|A| \leq K$ for some constant $K > 0$. Then L is locally a graph of a vector-valued function over a ball $B_r(0) \subset T_pL$ in a normal neighbourhood of $p \in L$ in M and $r > C(M, g)(K + 1)^{-1}$ for some constant $C(M, g) > 0$.*

Proof. **Step 1.** Bound the injectivity radius of L from below in terms of K. Assume M is isometrically embedded in some euclidean space. For the embedding $F : L^k \xrightarrow{f} M^{k+l} \xrightarrow{\varphi} \mathbb{R}^{k+n}$, denote its second fundamental form by $\tilde{A}$ and note that

$$|\tilde{A}| \leq C(|A| + 1) \leq C(K + 1)$$

where C only depends on the isometric embedding φ. Let $\gamma : \mathbb{S}^1 \to L$ be a shortest geodesic loop based at a point $p \in L$ which is parametrized by arc-length s. Suppose $\gamma(0) = \gamma(a), \gamma'(0) = \gamma'(a)$. Take a hyperplane P in $\mathbb{R}^{k+n}$ such that P intersects γ at a point p orthogonally. There is a point $q \in \gamma$ where γ meets P again at first time. The angle between the unit vectors $\gamma'(p)$ and $\gamma'(q)$ in $\mathbb{R}^{k+n}$ is at least $\frac{\pi}{2}$. Therefore

$$\left|\gamma'(p) - \gamma('q)\right| \geq \sqrt{2}.$$

Since $F \circ \gamma : \mathbb{S}^1 \xrightarrow{\gamma} L \xrightarrow{F} \mathbb{R}^{k+l}$ factors through L where γ is a geodesic, we have (cf. [ES64], [EL78] for the notation of the second fundamental form $\nabla d\phi$ of a mapping ϕ between Riemannian manifolds),

$$\nabla d(F \circ \gamma) = dF \circ \nabla(d\gamma) + \nabla d(F)(d\gamma, d\gamma) = \nabla d(F)(d\gamma, d\gamma).$$

Since the Christoffel symbols of $\mathbb{S}^1$ and of $\mathbb{R}^{n+k}$ are 0 we have

$$\nabla d(F \circ \gamma) = (F \circ \gamma)''.$$

Therefore

$$(F \circ \gamma)'' = \tilde{A}(F)(\gamma', \gamma').$$

Integrating along the portion of γ from p to q, we get

$$\sqrt{2} \leq \left|\gamma'(p) - \gamma'(q)\right| \leq \int_p^q \left|(F \circ \gamma)''\right| ds \leq C(K + 1)\, a.$$

We conclude that that the length a has a lower bound $C/(K + 1)$.

From the Gauss equations and $|\tilde{A}| < C(K + 1)$, the sectional curvatures of L are bounded above by $C^2(K + 1)^2$. We conclude $\mathrm{inj}(L) \leq C(K + 1)^{-1}$ [Pet06, p.178].

Step 2. Take a normal neighbourhood $U \subset L$ around a given point $p \in L$ and assume U is contained in a normal neighbourhood V of M at p. We will use $C(g)$ for constants only depending on the ambient geometry of (M, g). Now, on V we will use $\delta = \langle \cdot, \cdot \rangle_{\mathbb{R}^{k+l}}$, to measure length of various geometric quantities already defined in (V, g). First,

$$|A|_\delta \leq C(g)|A|_g \leq C(g)K.$$

Identify T_pL with $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^{k+l}$. Let $e_1(x), ..., e_k(x)$ be the orthonormal frame on U obtained by parallel transporting an orthonormal frame $e_1(0), ..., e_k(0)$ at T_pL along the *unique* radial geodesic $r_x(s)$ in (U, f^*g) from 0 to an arbitrary point $x \in L$, and let $e_{1+k}(0), ..., e_{l+k}(0)$ be the orthonormal frame of $(T_pL)^\perp$. Integrating along $\gamma_x(s)$ leads to

$$\begin{aligned}
\left|\langle e_i(x), e_{j+l}(0)\rangle\right| &= \left|\langle e_i(x), e_{j+l}(0)\rangle - \langle e_i(0), e_{j+l}(0)\rangle\right| \\
&= \left|\int_0^{|x|} \frac{d}{ds}\langle e_i(\gamma_x(s)), e_{j+l}(0)\rangle ds\right| \\
&= \left|\int_0^{|x|} \langle e_i'(s), e_{j+l}(0)\rangle ds\right| \\
&\leq \int_0^{|x|} \left|\langle \nabla^g_{\partial_r} e_i, e_{j+l}(0)\rangle\right| ds \\
&= \int_0^{|x|} \left|\langle A(\partial_r, e_i) + \nabla^L_{\partial_r} e_i, e_{j+l}(0)\rangle\right| ds \\
&\leq C(g)K\,|x|
\end{aligned}$$

as $\nabla^L_{\partial_r} e_i = 0$ on L. Therefore, there exists $r_0 = C(g)K^{-1}$ (where $C(g)$ may differ from the one above) such that for any $x \in B_{r_0}(0)$ the projection of each $e_i(x)$ in each fixed normal direction $e_{j+l}(0)$ is at most $c_n/\sqrt{l}$ and the norm of the projection is no more than some universal constant $c_{n,l}$ that we get to choose. It is known that such T_xL projects bijectively to T_pL. Therefore, locally around any $x \in B_{r_0}(0)$, implicit function theorem asserts that U can be written as a graph over a ball in T_xL, hence as a graph over a ball in T_pL from the projection. The graphing functions over the fixed reference plane T_PL must coincide on the overlap of any pair of such balls. This yields a *global* graphing function $\mathcal{F}$ over $B^n_{r_0}(p) \subset T_pL$. Moreover, $|D\mathcal{F}| \leq C(g, l)$ because $D\mathcal{F}$ is close to T_xL which is close (measured in l) to T_pL via the projection. $\square$

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