# NOTES ON MINING AND ASICS MARKETS

MICAH WARREN

Abstract. We discuss Bitcoin mining markets and the market for ASICs. We model the economic pairing between sellers and buyers via optimal transport, and argue that if the supply is sufficiently constrained, there will be a unique equilibrium.

## 1. Introduction

These notes attempt to give a brief background to the fundamental market concepts that drive Bitcoin mining. We start by introducing some classical models, and then apply these directly to the naturally occurring economics in Bitcoin mining. The full picture depends on at least 4 markets, and indirectly on many more markets beyond that. Perhaps the most significant factor is the price of Bitcoin, which is driven by independent forces. (This separation is by design, some would argue, and serves to distinguish proof-of-work from proof-of-stake.) The central market is the production of the hashes, which are converted to Bitcoin via proof-of-work and the difficulty adjustment. But because Bitcoin requires sophisticated ASICs to mine, the mining market depends heavily on the market for the ASICs themselves. In 2021, the capacity to produce ASICs is heavily competitive and depends on the ability of ASIC producers to gain access to chip-making facilities. These foundries are in hot demand: ASIC production is in competition with worldwide markets for other technologies, such as iPhones. In this chapter we present various models that may be useful for analyzing mining as an economic market. The market for hash production is fairly simple and straightforward, thanks to Bitcoin's simple rules and the difficulty adjustment. This will be covered in section 2.

The market for special equipment is a bit more complicated, and will be discussed in section 3 . It is exceeding difficult to present a grand unifying theory, and also somewhat pointless as the reality is complicated, in particular when involving predictions, which are tough to make, especially about the future. The goal is develop some toy models giving an idea of how the market should work. Section 4 will offer some simple applications: Incentives for "green" mining and the profitability of intermittent mining.

## 2. Hashing as a market

In the first section we deal with miners with a fixed allocation of equipment and study what drives them to mine.

### 2.1. Cournot Oligopoly Model.

We begin introducing a market model describing the mining production. As has been noted [4], the Bitcoin mining market is one of the most perfectly competitive markets that occur in nature. There are no transportation costs for hashes, the barriers to entry are relatively small, and the products (hashes) are identical. These dynamics will lead to a competitive market with many participants.

Rather than simply observing that Bitcoin mining is a perfectly competitive market, we will go ahead and derive this fact by taking a limit of a classical model, the Cournot oligopoly model. That is, we are going to consider the model for oligopolistic markets, but take the number of firms to be quite large. As the number of firms become large, the dynamics approach those of perfectly competitive markets. This will give us an opportunity to explore the dynamics and also the motivations for and against collusion.

A discussion of mining as an oligopoly market is similar to the standard discussion found in many economic textbooks. Mining has the feature that the price is not determined by the supply of hashes, but the share of revenue is. As we will see these two effects combine to create a model in which revenue is inversely proportional to the supply. We will introduce the classical model, and then describe how this should be tweaked to apply to to mining.

#### 2.1.1. *Classical model.*

In a classical oligopolistic market, there will be a number of firms $\{F_i\}$ producing a certain good . Each producer $\{F_i\}$ incurs a cost $c_i$ per unit of the good produced, and sells to consumers a price $P$. Each producer produces output quantity $q_i$. For this particular model, we assume that price is determined by where supply and demand meet, that is, there is a function $P(q)$ which describes the price when the total quantity of goods (over all the firms) is $q$. The function $P(q)$ should be decreasing: more quantity produced means lower prices. We also assume that there is one price at which the good is sold across the market (no price differentiation.)

For each firm, we define a profit function

$$\pi_i(q_1, .., q_n) = P(q)q_i - c_i q_i$$

which represents the market price times the quantity produced by firm $i$, minus the cost of producing this quantity.

This represents an $n$-player game: Each player chooses a quantity $q_i$ and receives a payoff $\pi_i$ which depends on the choices $\{q_j\}$ .

To determine the Nash equilibrium, we use multi-variable calculus. The space of all possible choices of $\{q_j\}$ will be

$$\mathbb{R}_n^+ = \{(q_1, .., q_n) \mid q_i \geq 0 \text{ for all } i\}.$$

If we assume that the costs $c_i > 0$ and the price $P(q) \to 0$ when $q \to \infty$, the positive values for the profit function $\pi_i$ must occur inside a compact region, and must occur in the interior, provided each firm is making a profit. (If that firm will not make a profit, we ignore this firm and reassign the number $n$ to a smaller number.). Suppose that $P(q)$ is a differentiable function. Then if $q_i$ is an interior value that maximizes $\pi_i$ while holding each $\{q_j, j \neq i\}$ constant, we will need

$$\frac{\partial \pi_i}{\partial q_i}(q_1, ..., q_n) = 0.$$

Computing, we check that this is

$$\frac{dP}{dq}(q)q_i + P(q) - c_i = 0. \tag{2.1}$$

So for a Nash equilbrium, we need to find $q_1, q_2, ..., q_n$ and $q = q_1 + ... + q_n$ such that (2.1) is satified.

**Exercise 1.** *Let*

$$P(q) = 100 - \frac{q}{3}$$

*and assume there are two firms producing goods, with costs* 4 *and* 5 *per unit, respectively. Find the Nash equilibrium. (Hint: (2.1) is a 2×2 system of linear equations).*

2.2. **A model for the mining market.** In order to apply the general model to mining, we need a reasonable description of how price (meaning the amount paid out per hash, not the price of Bitcoin) depends on the output. The price of Bitcoin itself is not determined by the supply of hashes, rather the price is determined by the independent market of supply and demand for Bitcoin. Bitcoin's price $P_B$ will be considered to be a variable that does not depend on the number of hash created by the miners. Nonetheless, the price paid per hash has a very specific expression, thanks to the difficulty adjustment:

$$P(q) = \chi \frac{P_B}{q}$$

where

$$\begin{aligned} P_B &= \text{Price of a bitcoin in dollars} \\ q &= \text{total hashes produced by all miners in a ten minute period} \\ \chi &= \text{Block reward, in units of bitcoins.} \end{aligned}$$

2.3. **Linear cost model.** In our simplest model, we will assume that the miners are paying a fixed per unit cost. This is unlikely for large demand, but reasonable for smaller demand, as we will discuss, but it will provide the basis for a toy model.

Each producer produces $q_i$ hashes at a cost $c_i q_i$. The profit is determined by

$$\pi_i = \chi \frac{q_i}{q} P_B - c_i q_i.$$

We are assuming a difficulty-adjusted view. This model isn't accurate in the very short term, but zooming out beyond two weeks it will be true. For this reason we also assume that $P_B$ nor $q$ is going to changed drastically over a two weeks period.

Applying (2.1)

$$-\chi \frac{P_B}{q^2} q_i + \chi \frac{P_B}{q} - c_i = 0. \tag{2.2}$$

Now at a Nash equilibrium, this will be true for each $i$ and we may sum:

$$\sum_i \left( -\chi \frac{P_B}{q^2} q_i \right) + \sum_i \chi \frac{P_B}{q} - \sum_i c_i = 0$$

that is

$$(n-1)\chi \frac{P_B}{q} = \sum_i c_i.$$

So

$$q = \frac{\chi P_B (n-1)}{\sum_i c_i}.$$

The total quanity $q$ uniquely determined, we can then solve for each $q_i$ using (2.2):

$$\begin{aligned} q_i &= \frac{\chi \frac{P_B}{q} - c_i}{\chi \frac{P_B}{q^2}} \\ &= \frac{\chi P_B (n-1)}{\sum_j c_j} \left( 1 - \frac{(n-1)c_i}{\sum_j c_j} \right). \end{aligned} \tag{2.3}$$

Notice that if all costs $c_i = c$ are equal, this reduces to

$$q_i = \frac{\chi P_B (n-1)}{n^2 c}$$

and profits will be

$$\pi_i = \frac{1}{n^2} \chi P_B. \tag{2.4}$$

**Conclusion 1.** *If we assume that each producer has the same fixed cost per hash, profits decrease on the order of $\frac{1}{n^2}$, where $n$ is the number of miners.*

Inspecting (2.3) we see that the condition for profit to be positive for miner $i$ can be formulated as

$$\sum_{i=1}^{n} c_j \geq c_i(n-1)$$

that is

$$c_i \leq \frac{n}{n-1} \text{average}\{c_j\}.$$

This suggests that if this model holds true, for $n$ large, there is a very narrow band of per units costs that will remain profitable.

In reality mining costs may change depending on how many hashes are produced: A firm may have more efficient machines that are turned on first, and less efficient machines turned on to meet demand.

2.3.1. *Uniqueness of $n$.* The quantities determined by (2.3) are well-defined provided the values $\{c_j\}$ are known; necessarily this requires knowledge of how many and which firms are mining profitably. The computation above assumed that all miners are mining profitably: The $q_i$ value must not be negative. In the situation where one or more of the producers is predicted to produce a negative quantity, this firm should be excluded, and the computation should be redone, with only the firms that can produce profitably.

It is a reasonable question to ask: Are there different combinations of mining firms that will be able to mine profitably to the exclusion of other firms? Excluding one firm from the computation should change the profitability for all firms.

It turns out that for linear costs, there is a unique maximal subset of firms that can mine profitably (proof is left as exercises.) That is, given a set of firms $S$ who are interested in mining, there will be a unique subset $S' \subset S$ such that 1) The Nash equilibrium determined when only firms mining from $S'$ provides non-negative profits for all members of $S'$, and 2) For any larger set $S''$ strictly containing $S'$, the Nash equilibrium (2.3) will prescribe negative quantities to some miners.

**Exercise 2.** *Show that if the firms are ordered by costs $c_1, ..., c_n$, then if there is a Nash equilibrium that is profitable for the $k$th firm, it must be profitable for all firms $i \leq k$.*

**Exercise 3.** *Show that if the firms are ordered by costs $c_1, ..., c_n$, and if the equations (2.3) computed when only the firms $1, ..., k$ participate prescribe a negative value for the $k$th firm, these equations will also prescribe a negative value for the $(k+1)$st firm if the first $k+1$ firms are included.*

**Exercise 4.** *Use induction on Exercise 3 together with the results of Exercise 2 to conclude that there is a unique maximal profitable subset of mining firms.*

This phenomenon will be a recurring theme when computing Nash equilibria: We may attempt to compute the equilibrium using the assumption that marginal profit is zero, and that the quantity is within the bounds of the problem. If we find after doing the computation that this assumption fails, we may remove a player and try again - the equations should all have changed a bit when we try again.

2.4. **Monopolist pricing and collusion.** The monopolist hashing model is drastically different. If there is single miner producing hashes, they will maximize the function

$$\pi = \frac{q}{q}\chi P_B - cq.$$

The upper bound on values that can be obtained here is $\chi P$, which will be approached as the hash output approaches 0. So a monopolist who is the only player in a hash market would expend as little positive energy as possible, and still claim 100% of the rewards. In practice, this is unlikely; even the slightest bit of competition for the mining market would change this dynamic.

Note that the reward for collusion in the mining market is extremely high, but the reward for defecting is also extremely high: If the set of miners is fixed and cartelized, they could in theory, agree to reduce hashrates all the way to near zero, and split the proceeds. Costs would go to near zero, but the total revenues would be the same and profits would be large. However, the nature of the market allows anyone to sweep in and claim a windfall of profit by defecting from (or simply never agreeing to) such an arrangement.

**Example 2.1.** *Suppose that there are 5 miners, each with identical costs to produce hashes. Each will make a profit of* $\chi P_B/25$. *If all 5 where to collude and produce very small number of hashes, each could claim profits of* $\chi P_B/5$. *If 3 collude to exclude the other 2, and proceed to produce very few hashes they could claim profits of* $\chi P_B/3$. *This latter scenario requires that the other 2 have given up and dropped out.*

2.5. **Nonlinear cost models.** For Bitcoin mining, a linear cost at all scales is unlikely. Specialized equipment is costly to purchase. Less efficient equipment (say from 4 years ago) is less costly and may be more available.

It's difficult to model this precisely, but we will consider a simple toy example.

**Example 2.2.** *Consider a firm that invests continually over time, purchasing equipment every month that consumes the same amount of energy, but that becomes exponentially more efficient due to Moore's law. If we run time backwards, the function*

$$O(t) = E_0 e^{-\lambda t}$$

*measures the output of the equipment acquired $t$ years ago, where $E_0$ is the efficiency of equipment produced today. Note that the first hashes one produces will incur a cost $\frac{1}{E_0}$. In order to properly model the cost as a function of total output, we need to solve for $t$ in terms of total $q$. This is best done by first computing an expression for $q$ in terms of $t$ :*

$$q(t) = \int_0^t E_0 e^{-\lambda s} ds = \frac{1}{\lambda} E_0 (1 - e^{-\lambda t}).$$

*Inverting, some computations yield*

$$t = \frac{1}{\lambda} \ln \left( \frac{E_0}{E_0 - \lambda q} \right).$$

*Now because we are assuming that each time interval's worth of equipment consumes the same amount of energy, $t$ is a constant times the cost used to produce $q$. In other words, up to some constants, we expect*

$$\begin{aligned} c(q) &= C \ln \left( \frac{E_0}{E_0 - \lambda q} \right) \\ &= C \ln \left( \frac{1}{1 - \theta q} \right). \end{aligned}$$

In general, we can assume that the marginal cost $c_i(q_i)$ to produces hashes increases with $q_i$. We observe

$$\text{Total cost to produce } q_i \text{ hashes} = \int_0^{q_i} c_i(s) ds.$$

The profit function for each miner is a concave function in output, as can be seen by differentiating. Starting with

$$\pi_i = \chi \frac{q_i}{\sum_{j \neq i} q_j + q_i} P_B - \int_0^{q_i} c_i(s) ds, \tag{2.5}$$

we take the first derivative,

$$\frac{\partial}{\partial q_i} \pi_i = \chi \frac{\sum_{j \neq i} q_j}{\left( \sum_{j \neq i} q_j + q_i \right)^2} P_B - c_i(q_i),$$

and the second

$$\frac{\partial^2}{\partial q_i^2} \pi_i = -2\chi \frac{\sum_{j \neq i} q_j}{\left( \sum_{j \neq i} q_j + q_i \right)^3} P_B - \frac{d}{dq_i} c_i(q_i). \tag{2.6}$$

Clearly if the marginal cost $c_i(q_i)$ to produce hashes is non-decreasing, this quantity above is negative and the function is strictly concave. We say a function $f(x)$ is strictly concave, if for each $x, y$ and $t \in (0,1)$ we have

$$tf(x) + (1-t)f(y) < f(tx + (1-t)y).$$

If the function $f$ is twice differentiable, this is equivalent to the condition that $f''(x) < 0$ for all $x$ except perhaps isolated points at which $f''(x) = 0$.

2.6. **Nash Equilibrium in the mining market.** We recall a Theorem due to Rosen in 1965 [5].

**Theorem 2.1.** *Suppose that the strategies of $n$ players are described by a convex and compact (closed and bounded) subset of Euclidean space $(q_1, .., q_n) \in \mathbb{R}^n$. If the payoff functions $\pi_i$ are continuous and strictly concave in the variable $q_i$, a unique Nash equilibrium will exist.*

This theorem, combined with (2.6) tells us there is a unique Nash equilibrium for miners with increasing marginal cost functions.

**Remark 2.1.** The theorem guarantees uniqueness or a fixed $n$-player game with concave payoff: It applies even when we include miners who will be non-profitable; Their outputs will be zero: We take our compact convex region to be a cube

$$\{(q_1, ..., q_n) \mid 0 \leq q_i \leq R\}$$

for some $R >>$ any feasible hashrate. This allows some values of $q_i$ to take the value 0.

We will prove existence directly, as it gives us an opportunity to use the Brouwer fixed point theorem.

**Theorem 2.2** (Brouwer Fixed Point Theorem)**.** *Suppose that $F : K \to K$ is a continuous function, and $K \subset \mathbb{R}^n$ is a convex and compact (closed and bounded) subset of Euclidean space. Then $F$ has at least one fixed point, namely an $x^*$ such that $F(x^*) = x^*$.*

To use this theorem, we define the following function

$$F(q_1, q_2, ..., q_n)$$
$$= (\arg\max \pi_1 (*, q_2, ..., q_n), \arg\max \pi_2 (q_1, *, q_3, .., q_n), ..., \arg\max \pi_n (q_1, q_2, ..., *)).$$

Here

$$\arg\max \pi_1 (q_1, *, ..., q_n) = \{q_2 : \text{the maximum for } s \mapsto (q_1, s, ..., q_n) \text{ is attained at } q_2\},$$

etc. By strict concavity, this $\arg\max$ function is well-defined and unique. A strictly concave function on a compact set must achieve a unique maximum.

**Exercise 5.** *Prove that the* $\arg\max$ *function is continuous as a function of all variables, provided the function is strictly concave in each variable.*

The function $F$ is continuous and $F$ must have a fixed point

$$F(q_1, q_2, ..., q_n) = (q_1, q_2, ..., q_n),$$

in other words, the best strategy given fixed $q_2, ..., q_n$ must be $q_1$. The best strategy given $q_1, q, ..., q_n$ must be $q_2$, that is, this set of strategies must represent a Nash equilibrium.

2.6.1. *Profitability.* Allowing each mining firm to produce until production capacity runs out (equivalently, $c_i(q) = +\infty$), there will be a Nash equilibrium, that is, a point $(q_1, q_2, ..., q_n)$ at satisfying

$$\chi \frac{q - q_i}{q^2} P_B - c_i(q_i) = 0 \text{ for all } i. \tag{2.7}$$

We can solve for the value of a unit of hash:

$$\chi \frac{P_B}{q} = \chi \frac{q_i}{q^2} P_B + c_i(q_i)$$

and compute the profit:

$$\begin{aligned} \pi_i &= q_i \left( \chi \frac{q_i}{q^2} P_B + c_i(q_i) \right) - \int_0^{q_i} c_i(s) ds \\ &= \chi \frac{q_i^2}{q^2} P_B + \int_0^{q_i} \left[ c_i(q_i) - c_i(s) \right] ds. \end{aligned}$$

The first term, involving the square of the firm's proportion of the pool, is reminiscent of (2.4), but allows for more general distribution of hashrates. The next term represents the surplus between the maximum that they pay for hashrate and what they pay for less expensive hashes.

Observe that solving for the $c_i(q_i)$ in (2.7) will tell us the maximum marginal cost $c_i^{\max}(q_i)$ paid by each firm. Summing as before we conclude that

$$q_i = q \left( 1 - \frac{(n-1) c_i^{\max}(q_i)}{\sum_j c_j^{\max}(q_i)} \right). \tag{2.8}$$

So we conclude the following: A miner will be able to mine profitably, provided that

$$c_i^{\max} \leq \frac{n}{n-1} \text{average} \left\{ c_j^{\max} \right\}.$$

If we take the other values $\left\{ c_j^{\max}, j \neq i \right\}$ as given, this condition is equivalent to the statement that the mining firm has access to hashing at marginal cost $c_i$ for any $c_i$ satisfying

$$c_i < \frac{n-1}{n-2} \text{average}_{j \neq i} \left\{ c_j^{\max} \right\}.$$

## 3. Market for ASICs

In the Cournot model, it is assumed that the price depends on the total quantity of goods produced. Firms that produce the goods decide how much to produce, and price is a dependent variable that will respond to the choice of quantities.

Another model, called Bertrand competition, does not assume that price is determined precisely by output, rather that firms will choose a price, and the quantity of output will increase or decrease to meet the demand which happens at that price. In short, the quantity sold by each firm is a function of the prices charged by each of the suppliers.

The profit function is similar, but now depends on a vector of prices $p_1, ..., p_n$ :

$$\pi_i(p_1, .., p_n) = D_i\left(p_1, .., p_n\right) p_i - \tilde{c}_i q_i.$$

Here $D_i$ represents the demand met by firm $i$ given the firms have specified these prices, and $\tilde{c}_i$ is the production cost. In the most competitive settings, the model requires that consumers will always choose the producer that sells at the lowest price. This is a bit of an extreme assumption: It assumes that the buyer of a \$129.99 product will switch to a competitor if the competitor offers the same product for \$129.98. In practice, methods of buyer or product differentiation deal with this discontinuity. However, the basic Bertrand model makes this assumption. Necessarily, this also requires the assumption that the capacity to produce is unconstrained: The firm can produce billions of dollars worth of goods on a moment's notice.

3.0.1. *Nash equilibrium with Bertrand assumptions.* Suppose that multiple firms are competing to sell the same good. Is there a Nash equilibrium in the choice of prices?

First we observe that each firm sells only at price $p_i \geq \tilde{c}_i$, otherwise they would be producing and selling at a loss. Observe next that any firm hoping to make a sale must sell goods at costs no higher than any other firm. So $p_i = p_j$ for all $i, j$ unless the firm is already resigned to not making any sales. Now if all firms are selling at the same price, they should be splitting the quantity sold. But if $p_i = p_j > \tilde{c}_i$, firm $i$ can charge price $p_i - \varepsilon > \tilde{c}_i$, (this is called epsilon-undercutting) and steal all of the customers from the other firms. An infinitesimal decrease in per unit revenue combined with a significant non-infinitesimal increase in quantity yields a much better profit. We conclude that there is only one possible Nash equilibrium, and that is that each firm with cost $\tilde{c} := \min_i \{\tilde{c}_i\}$ charge $\tilde{c}$ and each other firm either drops out completely, or charges a higher price, knowing they will expect no sales and zero profits.

The Bertrand model says that with such competitive assumptions in place, products will be sold at cost, for zero profit, and only firms that produce at the minimum cost will engage the market. In practice this is a bit extreme. Product and consumer differentiation allow for a less severe cutoff function happening near $p_1 = p_2$, allowing for Nash equilibria with positive profits. Reality also dictates that production cannot instantly toggle from 0 to millions. So this model is somewhat imperfect. However, in many cases, it is approximately correct.

Clearly, because hashrate cannot be increased arbitrarily, the Bertrand model does not apply to the market for mining. However, it may apply, with some modifications and constraints, to the market for mining equipment. We will explore this in several stages.

3.1. **Future assumptions in a model.** In this section, we set up a slightly more cumbersome model simply for the purpose of becoming aware of it before reducing to a more tractable model.

Suppose a miner produces $q_i(t)$ hashes while incurring electricity cost $c_i\left(t\right)$. In general, we would like to allow $c_i(t)$ to depend on time, because this may depend on expectations of costs into the future. The miner sees an expected profit function given by

$$\mathbb{E}\left[\pi_i\right] = \mathbb{E}\left[\int_0^\infty \left(\frac{\chi(t)P(t)}{q(t)} q_i(t) - c_i(t) q_i(t)\right)^+ dt\right]$$

Here, $P(t)$ is the price function into the future, $\chi(t)$ is the blockreward function (which will be cut in half at some point with the next four years, so does have nontrivial time-dependence) and $c_i(t)$ is the electricity cost (also possibly time-dependent) per unit hash produced. The notation $(\cdot)^+$ indicates that if the inner quantity falls below zero, mining will be halted and no loss will be incurred. Thus we can also rewrite this as

$$\mathbb{E}\left[\pi_i\right] = \mathbb{E}\left[\int_{\left\{t \mid \frac{\chi(t)P(t)}{q(t)} q_i - c_i(t) q_i \geq 0\right\}} \left(\frac{\chi(t)P(t)}{q(t)} q_i - c_i(t) q_i\right) dt\right] \tag{3.1}$$

where the expression

$$\left\{t \mid \frac{\chi(t)P_B(t)}{q(t)} q_i - c_i(t) q_i \geq 0\right\}$$

denotes the time interval on which the machine will remain profitable. We expect this to be finite and not too small. Predicting this exactly is obviously going to be somewhat difficult.

A side-remark here on discounting: Often it is unfair to compare future money to money today. Typically, money today is more valuable than money in the future. One can deal with this by explicitly discounting, say multiplying the function inside the integral by $e^{-\delta t}$ for some small $\delta$, or, we can also declare that the future price $P(t)$ and cost $c_i(t)$ has absorbed the discount. This amounts to "gauge-fixing" the price to today's dollars, i.e. a Bitcoin worth \$1.34 Million in 2045 may be counted as being worth \$780,000 in today's dollars, but in 2045. Note that discounting can involve not only loss of buying power, but the opportunity cost of money. This latter approach is simpler computationally.

Now suppose that a miner has equipment to produce hashrate $q_i$ today, and is deciding whether or not to take on new equipment. To maximize the expectation, take the derivative of (3.1) with respect to $q_i$ and ask if the future returns exceed the price of the additional units being purchased.

$$\frac{d\mathbb{E}\left[\pi_i\right]}{dq_i} = \mathbb{E}\left[\int_{\left\{t\mid \frac{\chi(t)P(t)}{q(t)}q_i - c_i(t)q_i \geq 0\right\}} \left(\frac{\chi(t)P_B(t)}{q(t)} - \frac{\chi(t)P_B(t)}{q(t)^2}q_i - c_i(t)\right) dt\right] \tag{3.2}$$

If this value is larger than $p$, the miner will be willing to pay $p$ to obtain the equipment.

Three comments: First, if the miner possesses a negligible portion of the hashrate, the second term is ignored and the integral becomes

$$\mathbb{E}\left[\int_{\left\{t\mid \frac{\chi(t)P(t)}{q(t)}q_i - c_i(t)q_i \geq 0\right\}} \left(\frac{\chi(t)P_B(t)}{q(t)} - c_i(t)\right) dt\right]. \tag{3.3}$$

Second, it follows directly from the expression that if two mining firms have the same costs but one has a larger portion of the current hashrate, the firm with the larger hash rate will have slightly less expected increase in profits and thus will have a slightly lower threshold at which a purchase would become profitable. The end result is that both costs $c_i$ and current hash $q_i$ will differentiate the miners and create a total demand curve $D(p)$ for mining units that decreases as the Bitcoin market price decreases.

Finally one may choose to ignore this second term if the firms takes the perspective that the unit will be sold to another miner in the event that miner $i$ does not procure it. In other words, (3.2) does not price in the "defensive" incentive to prevent other miners from buying mining equipment. If we take this perspective, the second term in (3.2) is ignored and we are left with (3.3). Our analysis in the sequel will use (3.2), assuming that the the firm will not purchase equipment defensively.

3.1.1. *Simplified model.* For a simpler model, which we adopt in the sequel, we need some assumptions.

Time Epoch. First, it makes sense to define a time epoch - this is meant to roughly approximate the profitability window of ASICs being purchased today, or the time over which any reasonable predictions can be made, whichever is shorter. This could be a year or two. Beyond this time period predictions about price and equipment supply availability become quite fraught and not worth coding into a hard mathematical model. We assume that decision makers have a time epoch in mind when making a decision, and are not making decision based on hunches about the price of a bitcoin or ASICs decades into the future.

We assume that $P_B(t)$ is roughly a constant $P_B$ into a near future that includes the life-span of the mining equipment. If the block reward halving is approaching, one may also take a weighted average of the block reward $\chi$. The same is true for $q(t)$ : While we will be solving for the values $q_i(t)$, rather than closely inspect the time-evolution of these values, we simply assume these remain close to constant values $q_i$ over the time epoch.

The decision-making process involves deciding how much hash rate $\Delta q_i$ that a miner will add to their arsenal. The marginal profit obtained by purchasing mining equipment capable of producing $\Delta q_i$ addition hashrate over the lifetime of the equipment is determined by the profit function

$$\pi_i = \frac{\chi P_B}{q + \sum_j \Delta q_j} \left(q_i + \Delta q_i\right) - c_i \left(q_i + e\Delta q_i\right) - p\Delta q_i \tag{3.4}$$

where

$P_B =$ price of Bitcoin in dollars averaged over the time epoch

$q =$ total hashes that will be produced by all miners with machines already acquired over the time epoch

$q_i =$ total hashes produced by miner $i$ over the time epoch, from already purchased equipment

$\Delta q_i =$ total hashes purchased by miner $i$ deployed over the time epoch

$\chi =$ Block reward in Bitcoins

$c_i =$ expected electric cost over the time epoch

$e =$ electrical inefficiency of the equipment to be purchased

$p =$ price of the equipment required to produce one unit of hash.

We're introducing "electrical inefficiency" $e$ here: This is a unit of energy per hash that describes the electricity required to create one unit of hash with a specified type of ASIC. For example, if we take the electric cost $c$ to be given in \$/`kW-h`, we may take $e$ to be in units of `kW-h/TH`, then the cost to produce a hash with this electricity cost and this equipment, will be $ec$, now in units of $/TH.

3.2. **Monopolistic model for ASICs.** Suppose that there is only one supplier of ASICs, which has electrical inefficiency $e$. Miners need to buy from this supplier in order to produce a competitive hashrate. The supplier decides to price the units at $p$.

We will attempt to construct a demand function $D(p)$ in terms of price as follows. For each given $p$ find a Nash equilibrium describing the miners choice of additional units to purchase $\{\Delta q_i\}$. One can check that the function (3.4) $\pi_i(\Delta q_1, ..., \Delta q_n)$ is concave in the variables $\Delta q_i$, thus the Nash equilibrium will exist and be unique by arguments very similar to those found in the example in section in (2.6). We differentiate each profit function

$$\frac{\partial \pi_i}{\partial \Delta q_i} = \frac{\chi P_B}{\left(q + \sum_j \Delta q_j\right)} - \frac{\chi P_B}{\left(q + \sum_j \Delta q_j\right)^2}(q_i + \Delta q_i) - c_i e - p$$

set these to zero

$$\chi P_B \left(q - q_i + \sum_j \Delta q_j - \Delta q_i\right) = (c_i e + p)\left(q + \sum_j \Delta q_j\right)^2$$

and sum over $i$

$$\chi P\left((n-1)q + (n-1)\Delta q\right) = \sum_i (c_i e + p)\left(q + \Delta q\right)^2$$

leading to

$$q + \Delta q = \frac{\chi P_B(n-1)}{\sum_i (c_i e + p)}.$$

This gives us a demand function in terms of price:

$$D(p) = \Delta q = \frac{\chi P_B(n-1)}{np + \sum_i c_i} - q. \tag{3.5}$$

Now the profit made by a monopolist firm selling the equipment with production cost $\tilde{c}_p$ will be

$$D(p)(p - \tilde{c}_p) = \left(\frac{\chi P_B(n-1)}{np + \sum_i c_i} - q\right)(p - \tilde{c}_p),$$

and the monopolist's price will be determined by maximizing this profit function, provided that the supplier does not have capacity constraints. If the optimal price results in a demand $D\left(p_{\max}\right)$ which is larger than the capacity of the supplier, the supplier is wise to increase price until $D(p_{\max})$ decreases to the capacity that can be satisfied.

**3.3. Duopolisitic unconstrained markets for ASICs.** Suppose there are two or more suppliers for ASICs. If we take the (somewhat unrealistic) Bertrand type assumption that the market is unconstrained, we find the same dynamics. Provided that one supplier has the ability to $\varepsilon$-undercut the other, the only feasible Nash equilibrium will happen when each supplier is making zero profits. In a Nash equilibrium, producers will sell equipment at cost or not at all.

**3.4. Duopolisitic constrained markets for ASICs and the Edgeworth paradox.** A more interesting question is what happens when the supply is constrained. To illustrate the problem, consider a made-up demand function given by

$$D(p) = \frac{1000}{p+8} - 50$$

If a monopolist is unconstrained and incurs a production cost of 1 per unit, they would maximize the function

$$\pi(p) = \left(\frac{1000}{p+8} - 50\right)(p-1)$$

which is maximized when

$$p \approx 5.4164$$
$$D(p) \approx 24.536.$$

If the production capacity $C$ is less than $24.536$, the supplier will find the maximum price so that $D(p) \geq C$.

Now suppose there are two ASIC producers each with capacities $C_1$ and $C_2$. Suppose that $C_1 = C_2 = 20$. If each wishes to max out their capacity, the total demand should be

$$\frac{1000}{p+8} - 50 = 40$$

so one could imagine that each may charge price

$$p = \frac{1000}{90} - 8 \approx 3.1111$$

as this would be the largest price each could charge while saturating their demand.

At this price both sellers sell 20 units and make a profit of

$$20 * (3.1111 - 1) \approx 42.222.$$

Now what happens if one producer increases the price? Suppose that one firm decided to charge.

$$p = 3.33.$$

What we expect to happen is that the total demand will be determined by the higher of the two prices, namely

$$D = \frac{1000}{3.33 + 8} - 50 = 38.261$$

Buyers prefer the lower price and 20 units will be filled at the price 3.11, while the remaining 18.261 units will be purchased at price 3.33. So the profits will be

$$\begin{aligned} 18.261 * (3.33 - 1) &\approx 42.548 \\ 20 * (3.11 - 1) &\approx 42.222. \end{aligned}$$

So it follows that in this case, the value that saturates the capacity is not profit maximizing for both: either can raise prices, sell less and still increase profits. This leads to the next question - why doesn't the second firm also raise their price? Indeed they can. In fact their incentive is larger to do so - they will not lose sales until they meet or exceed the price of the firm with the higher price. It follows that $p_2 < p_1$ is not an equilibrium. So any Nash Equilibrium must have $p_1 = p_2$. However, this is also impossible. If $p_1 = p_2 > 3.111$ neither capacity is saturated and the demand is being split. Either firm can $\varepsilon$-undercut to get a significant jump in demand. We conclude there is no Nash equilibrium. This is the Edgeworth paradox.

One objection to this construction is that the "step-function" effect of $\varepsilon$-undercutting doesn't reflect a realistic model. Would this same phenomenon occur if the jump was smoothed out a bit? The answer is that the phenomenon is robust and may often persist even when the discontinuity is smoothed out.

3.4.1. *Hotelling differentiation and smoothed out transition function.* In reality, buyers are often slightly differentiated. Thus some consumers will have a larger threshold for price discrepancies and won't immediately switch suppliers. A classic example is described by the following: Suppose two suppliers of the same product are situated at the opposite sides of a small town. Consumers will take into consideration both price and transportation cost to travel to the supplier. If the transportation cost is small, most consumers will prefer the cheapest price, however, if consumers are modeled on a continuum, one would expect the demand transitioning from one to another when prices are changed continuously.

Suppose that there are two prices $p_1, p_2$ not far from one another. Instead of a hard cutoff function, one could imagine a differentiable function $\eta(p_1, p_2, t)$

with the property that

$$\begin{array}{c}\eta(p_1,p_2,t)=0 \mid p_1-p_2\geq t\\ \eta(p_1,p_2,t)=1 \mid p_2-p_1\geq t\\ \eta(p_1,p_2,t)=\frac{1}{2} \mid p_1=p_2\end{array}$$

where

$$\left|\frac{d\eta}{dp_1}\right|\approx\left|\frac{d\eta}{dp_2}\right|\approx\frac{1}{2t}$$

on the transition region where

$$|p_1-p_2|<t.$$

Then the amount of product sold by firm 1 when $p_1 \geq p_2$ would (provided we are away from constraints $C_1, C_2$) be given by

$$D(p_1)\eta(p_1,p_2,t)$$

while the amount sold by firm 2 would be

$$D(p_1)\left[1-D(p_1)\eta(p_1,p_2,t)\right].$$

This can provide a smooth transition from one to the other. The larger price $p_1$ at which marginal purchases are being made will be determined by the demand function, hence $D(p_1)$ is the total demand and hence the sum of the two expressions.

**Exercise 6.** *Construct a continuous function transitioning from supplier* 1 *and supplier* 2 *given prices* $p_1$ *and* $p_2$

**Exercise 7.** *With the function constructed in Exercise 6 , consider the above example in section 3.4. Show that for small enough values of* $t$*, there is no Nash equilibrium.*

3.5. **Four Regimes.** We find that there are essentially three regimes for modeling. A fourth classification would be an inhomogeneous mixture of the three.

(1) **Bertrand regime**. In this regime, the capacity for ASIC producers to produce is essentially unlimited or is at least larger than any reasonably expected demand. The economics behave like the Bertrand model predicts. This regime is where there is maximum competition among ASIC producers, and machines will be offered at near cost. This is probably not a realistic model in 2022.
(2) **Edgeworth regime.** When the supply is constrained, but not too tightly, we get a situation where market dynamics allow for no Nash equilibrium. This is characterized by moderate constraints on supply.

(3) **Strongly constrained regime**. When the supply is strongly constrained, Nash equilibriums will be predictable and often computable. The incentive for competing suppliers to raise prices so that their sales are below capacity which occurs in the Edgeworth paradox is not present: The best pricing options will always ensure that demand saturates supply.

(4) **Mixed**. When there are differentiated miners and suppliers, there can be an abundance of one type of ASIC and dearth of another. The market can be predictable in some segment of the curve and not in others. For example there may be an abundance of inefficient mining equipment and a lack of efficient mining equipment.

3.6. **Miner and supplier differentiation.** In practice, while there may be only a few suppliers of ASICs, they provide an array of efficiencies and prices, and once these have been sold and are distributed, there will be a large array of mining equipment for sale, not only as new equipment, market but also on the used market.

In the sequel we're going to assume that both the miners and the ASIC producers are competitive: Due to ample mining equipment already being in the possession of entities who may be motivated to sell for different reasons, sellers will behave somewhat competitively. However, competitive does not mean unconstrained. While there may be a number of sellers the number of units can be constrained, and this leads to interesting pricing dynamics. As we will see, with constraints, this typically does not mean that equipment will be sold at costs.

When considering the possibility that miners have an option to choose which supplier to buy from and that these suppliers might be offering different prices, the different costs and inefficiencies serve to differentiate.

In order to set this up, we consider a system with $n$ mining firms, and each mining firm has electricity costs, given by

$$c_i = \text{electricity price, given in units of } \$/\texttt{kW-h}.$$

Suppose that there are $m$ different producers of ASICs, and suppose that they each produce a machine that has electric inefficiency $e_j$, that is

$$e_j = \text{electricity required to produce hashes, given in units of } \texttt{kW-h/TH.}$$

(For simplicity, and without loss of generality, we treat a supplier producing separate units is two suppliers. This shouldn't change our computations.) Now the supplier offers their units at price $p_j$ where

$$p_j = \text{price of equipment that will produce a } \texttt{TH,} \text{ given in units of } \$/\ \texttt{TH.}$$

(Continuing with our assumptions,this will be over a fixed time epoch approximating the profitability like of the machine, say a year or two.)

Now for the miner with cost $c_i$, the cost to purchase from supplier $j$ and produce a unit of hash would be

$$e_j c_i + p_j.$$

Naturally, given a distribution of options, the miner with electrical costs $c$ is going to buy from the producer that results in the lowest per hash cost to the miner. So given a set of prices $\{p_j\}$, we define a function

$$u(c) = \min_j (e_j c + p_j). \tag{3.6}$$

Notice that $u$ is a function taking as argument any $c$ (which can be extended to a continuum) and is given by the minimum of a family of linear functions, hence is a piecewise linear, continuous, and concave function. Given a set of inefficiencies $\{e_j\}$ which are fixed, the values $\{p_j\}$ which are to be determined by the sellers will define this function $u$. A miner with cost $c$ may purchase from firm $j$ whenever

$$u(c) = e_j c + p_j$$

is satisfied.

3.7. **Unconstrained supplier markets.** When suppliers are able to increase their supply, as in the situation of the Bertrand model, the pricing will usually be close to cost, but with the exception that sometimes differentiation, as described in the previous section, will allow some suppliers to enjoy a monopoly over some segment of the market. If each supplier prices their equipment at price $p_j$ we can construct $u(c)$ as in (3.6.)

Again, this is a concave function, determined by the minimum of linear functions. One can split up the domain of costs $C$ as a union of closed intervals of the form

$$C_j = \{c \mid u(c) = e_j c + p_j\}$$

and think of this as the region over which supplier $j$ is the preferred supplier. Notice that by increasing the price $p_j$ the supplier will shrink this region. If the shrinking of this region (as a set of real numbers) does not result in a shrinking of the number of miners who prefer this supplier, the supplier has monopolistic control over this region.

**Example 3.1.** *Suppose there are three producers, with production costs* $(7, 8, 11)$ *respectively, and who produce machines with inefficiencies* $(5, 4, 3)$ *respectively.*

*If the producers sell at costs, the function $u$ will be defined by*

$$u(c) = \begin{array}{c} 5c+7 : c \leq 1 \\ 4c+8 : 1 \leq c \leq 3 \\ 3c+11 : 3 \leq c \end{array}$$

*and the domain is split up according to preferred producer. If the majority of the miners have electric costs within the region $1 \leq c \leq 3$ it may make sense for the 2nd supplier to price monopolistically (price somewhat above cost in order to maximize profit) because the real competition is only on the edge of the region. For example, by raising price from $8$ to $8.25$ the miner cost function becomes*

$$u(c) = \begin{array}{c} 5c+7 : c \leq 1.25 \\ 4c+8.25 : 1.25 \leq c \leq 2.75 \\ 3c+11 : 2.75 \leq c \end{array}$$

*showing that the supplier has gained a per unit profit of 0.25, and has lost only the miners with costs in the regions $1 \leq c \leq 1.25$ and $2.75 \leq c \leq 3$.*

A deeper look at this type of problem leads us into concepts from optimal transportation.

3.8. **The Legendre Transform and conventions from optimal transportation.** The form of the definition (3.6) is reminiscent of the Legendre transform. We will see that price and miner cost are related by transforms. Because the price determines a pairing from miners to suppliers, these type of functions and transforms arise naturally in the study of matching problems in economics, in particular the optimal transportation problems. We follow conventions used in [6, Chapter 5]

3.8.1. *The classical optimal transportation problem for finite source and target.* We introduce this topic by describing the classical optimal transportation problem for finite sets of point masses.

Suppose that $\alpha_1, ..., \alpha_m$ are positive values which represent masses at points $x_1, ..., x_m$ and $\mu_1, ..., \mu_n$ are positive values representing masses at points $y_1, .., y_n$ such that

$$\sum \alpha_j = \sum \mu_i,$$

and consider the cost function

$$\psi(x, y) = \frac{1}{2} |x - y|^2 .$$

An **optimal transportation plan** is a set of values

$$b_{ij} \geq 0$$

for

$$i \in \{1, .., n\}$$
$$j \in \{1, ..., m\}$$

satisfying

$$\sum_i b_{ij} = \alpha_j \text{ for each } j \in \{1, ..., m\} \tag{3.7}$$

$$\sum_j b_{ij} = \mu_i \text{ for each } i \in \{1, .., n\} \tag{3.8}$$

such that

$$\sum_{i,j} b_{ij}\psi(e_j, c_i) \leq \sum_{i,j} \tilde{b}_{ij}\psi(e_j, c_i)$$

for all other

$$\tilde{b}_{ij} \geq 0$$

that satisfy the constraints (3.7) (3.8).

This is often described as the earth mover's problem. Suppose you have $m$ piles of dirt at locations $x_1, ..., x_m$, with volumes $\alpha_1, ..., \alpha_m$, respectively, and $n$ holes at locations $y_1, .., y_n$ and the total volume of dirt is equal to the total amount of space in the holes. How do you fill in all the holes with the available dirt while minimizing the work done in transporting dirt?

This one-dimensional version of the problem (assume the points are all on the line) has an easy to describe solution: Starting to move the dirt from the left-most point to the left-most hole. If this dirt fills up the hole, proceed to the second left-most hole and begin filling this hole. Otherwise use the second left-most pile of dirt to continue to fill the hole. Continue to move monotonically to the right until the right-most dirt pile is used to fill the right-most hole.

The literature surrounding the optimal transportation problem in general dimension is complex and rich. Fortunately, for our needs we will only study one-dimensional problems. More general results have been developed which to apply to markets with more dimensions of differentiation, see [2] and references therein. For most nicely behaved cost functions $\psi$, the solutions will exhibit this monotonic behavior. The solutions may appear easy to describe in the one-dimensional case, provided we know the source and target masses $\{\alpha_j\}$ and $\{\mu_i\}$. However, in many cases we will be interested in solving for these masses as a key step in finding a Nash equilibrium.

In what follows, we may be able to solve the problems without using optimal transportation, however, because the language is well-developed and because optimal transportation is an important and worthwhile area of mathematics, we will go ahead and develop the presentation in terms of optimal transportation.

3.8.2. *The optimal transportation problem, in general, and for the ASIC market.* Let

$$C = \{c \in \mathbb{R}_+ \mid c \text{ is the electric costs of some miner}\}$$
$$E = \{e \in \mathbb{R}_+ \mid e \text{ is the electric inefficiency of an ASIC for sale}\}.$$

Then the operational cost is given by

$$\psi(c,e) = ce,$$

which describes the cost to create a unit of hash for miner with cost $c$ and ASIC with inefficiency $e$.

For a general $\psi$, we say that a function $p : E \to \mathbb{R}$ is $\psi$-convex, if there exists a function $\zeta(c)$ such that

$$p(e) = \sup_{c\in C} \left(\zeta(c) - \psi(c,e)\right).$$

We define the $\psi$-transform of $p$ via

$$p^{\psi}(c) = \inf_{e\in E}(p(e) + \psi(c,e))$$

and the $\psi$-subdifferential of $p$

$$\partial_{\psi} p = \left\{(e,c) \in E \times C : p^{\psi}(c) - p(e) = \psi(c,e)\right\}.$$

For a function $u : C \to \mathbb{R}$, we say that $u$ is $\psi$-concave if

$$u(c) = v^{\psi}(c)$$

for some function $v : E \to \mathbb{R}$.

We also define the $\psi$-transform

$$u^{\psi}(e) = \sup_{c\in C} \left(u(c) - \psi(c,e)\right)$$

and the $\psi$-superdifferential

$$\partial^{\psi} u = \left\{(e,c) \in E \times C : u(c) - u^{\psi}(e) = \psi(c,e)\right\}$$

It is a well known fact (Legendre Duality, cf. [6, Proposition 5.8]) that $p$ is $\psi$-convex if and only if

$$\left(p^{\psi}\right)^{\psi} = p.$$

**Exercise 8.** *Show that a function $f(x)$ is convex if and only if it is $\psi$-convex for $\psi(x,y) = -xy$*

We are going to look for pair of function $(p, e)$ where $p$ will describe a price in terms of efficiency, and $u$ will describe the cost to the miners. These will be $\psi$-transforms of one another, and will describe the pairing support for an optimal transportation problem. We define the **pairing support** to be the set

$$\Gamma_{(p,u)} = \{(e, c) \in E \times C : u(c) - p(e) = \psi(c, e)\}.$$

In general, if

$$\{(j, i) \mid b_{ij} > 0\} \subset \left\{(e_j, c_j) \in \Gamma_{(p,u)}\right\}$$

then the $\{b_{ij}\}$ will define an optimal transporation plan between mass distributions $\alpha$ and $\mu$ if $\{b_{ij}\}$ satisfies (3.7) and (3.8).

Notice that if we start we a price function $p$, the compute $u = p^{\psi}$; we've created a pairing support $\Gamma_{(p,u)}$. This may or not be the correct pairing support for the problem we want to solve: The solutions requires constraints (3.7) and (3.8). Now by raising or lower different $p_j$ values, we change the regions

$$C_j = \{c \mid u(c) = e_j c + p_j\}.$$

One can change these so that each $C_j$ contains the appropriate amount of mass $\alpha_j$. That is

$$\sum_{c_i \in C_j} \mu_i \geq \alpha_j.$$

Note that the possible splitting of mass $\mu_i$s may be necessary to satisfy the problem. This corresponds to endpoints of the domains $C_j$ landing precisely on a values $c_i$. This is why the above is an inequality, instead of an equality.

The optimal transportation problem stated for miners and suppliers can be described as follows. Suppose that we are given $n$ miners with various electric costs $\{c_i\}$ looking to purchase equipment in order to produce $\mu_i$ units of hash (assume for now these are predetermined.) Suppose there $m$ suppliers of ASICs each with electric inefficiencies $\{e_j\}$, and each produces $\alpha_j$ units of hash. Suppose also that

$$\sum \mu_i = \sum \alpha_j.$$

How do we match them so that the total amount spent on electricity is minimized? In this case we are simply fixing the weights and seeking a "general welfare" solution, we aren't allowing the parties to optimize their own objectives. This a "central planner's problem:" A socialist authority could declare what units are to be sold to which miners. Of course, we aren't interested in this sort of solution - we would prefer to study solutions that are not imposed by central authorities. However, the structure of the solutions are the same. We will develop solutions to the "free market" problem in the next section.

Here and in the sequel, we will order the miners by cost efficiency and the suppliers in order by decreasing inefficiency. For the cost function $\psi(e, c) = ec$, solutions will pair the least efficient with the most efficient, so we can move from left to right monotonically with this indexing. This can be seen by noting that the function $u$ is concave, so the slopes $e_j$ will be decreasing as one moves from the left to the right along the $c$-axis. For $\psi(e,c) = ec$, the function $u$ has a predictable shape: It's a concave function, determined by the minimum of a set of lines. Each of these lines have positive slope, and the slope decreases while $c$ moves to the right. This means as $c$ becomes larger, the value(s) $e^*$ for which

$$e^*(c) = \{e \mid u(c) = p(e^*) + e^* c\}$$

must be monotone decreasing with $c$.

**Exercise 9.** *Let $a_1 < a_2$ and $b_1 < b_2$. Show that $a_1 b_2 + a_2 b_1 < a_1 b_1 + a_2 b_2$. Explain why this is relevant to the preceding paragraph.*

**Example 3.2.** *For a simple example, suppose there are 3 miners and 2 suppliers, and suppose the miners each want to produce 1/3 units, and the suppliers have 1/2 units each for sale. Suppose mining costs are $(3,4,5)$ and electric inefficiencies are $(7,6)$.*

We can think of a pairing as $\{b_{ij} : i \in (1,2,3), j \in (1,2)\}$ where

$$b_{ij} = \text{how many units are purchased by miner } i \text{ from supplier } j$$

Summing over the miners, we need

$$\sum_{i=1}^{3} b_{ij} = \frac{1}{2} \text{ for } j = 1 \text{ or } 2 \tag{3.9}$$

and summing over the suppliers

$$\sum_{j=1}^{2} b_{ij} = \frac{1}{3} \text{ for } i = 1, 2 \text{ or } 3. \tag{3.10}$$

Then the goal is the minimize

$$\sum b_{ij} c_i e_j.$$

Among the set of positive $\{b_{ij}\}$ with constraints (3.9)(3.10).

We may begin to solve the problem by pairing the most efficient electricity with the least efficient ASIC. In general, we should always expect that

$$(e_1, c_1) \in \Gamma_{(p,u)}.$$

In this case, the miner with electric costs $c = 3$ will buy as many units as possible from the supplier with inefficiency 7. This can only be 1/3 so we conclude that

$$b_{11} = 1/3$$
$$b_{12} = 0$$

and that

$$u(3) - p(7) = 3 * 7$$

which is equivalent to

$$(7,3) \in \Gamma_{(p,u)}.$$

We next look at the second miner. There is still $\frac{1}{6}$ available from supplier 2. So we will have

$$b_{21} = \frac{1}{6}$$

and

$$u(4) - p(7) = 4 * 7$$

hence $(7,4) \in \Gamma_{(p,u)}$. This miner still needs to purchase 1/6 more which they will do from the other supplier, hence

$$b_{22} = \frac{1}{6}$$

and $(6,4) \in \Gamma_{(p,u)}$ :

$$u(4) - p(6) = 4 * 6.$$

Finally, the miner with electric costs 5 will purchase from the second supplier the remaining $\frac{1}{3}$ that is

$$b_{32} = \frac{1}{3}$$

and $(6,5) \in \Gamma_{(p,u)}$

$$u(5) - p(6) = 5 * 6.$$

This information allows us to piece together the functions $p$ and $u$ provided we fix a particular value. For example, if we let $p(6) = p_0$, we conclude

$$u(5) = 5 * 6 + p(6) = p_0 + 30$$
$$u(4) = 4 * 6 + p(6) = p_0 + 24$$
$$p(7) = u(4) - 4 * 7 = p_0 - 4$$
$$u(3) = 3 * 7 + p(7) = p_0 + 17.$$

The optimal transportation problem we solved above is for a fixed allocation to each miner and a fixed allocation from each supplier. It doesn't specify the price, only up to a constant, but it does specify the differences in prices. Again, this is a socialized optimization problem. If the surplus is socialized, raising or lowering prices by a constant is essentially robbing Peter to pay Paul. We see in the next section that supply constrained markets give suppliers an upper hand in determining prices to suit their profit objectives. In fact, suppliers can choose the price function $p$ in order to guarantee that buyers choose optimal allocations. This is related to the notion of a Cournot-Nash equilibrium, see [1].

3.9. **Constrained markets and the main Theorem.** A relevant and interesting setting is when the ASIC producers are constrained, that is, each is bound by some capacity, and the mining firms are not: they can expand provided the equipment is available and it is profitable to do so. Before proving a theorem, we encode this condition in a definition that although may be hard to verify, defines a notion of supply constrained in a very practical way.

Suppose that there are $m$ suppliers, each with capacity $\alpha_j$, and production costs $\tilde{c}_j$. Suppose we know the demand functions

$$D_j\left(p_1, ..., p_m\right) = \text{demand for ASIC } j \text{ at prices } \vec{p}$$

which determine demand for any set of prices.

**Definition 1.** A market is **strongly supply constrained** if the following condition holds. For any $j$, if

$$D_j\left(p_1, ..., p_m\right) = \alpha_j$$

then

$$\left(\tilde{p}_j - \tilde{c}_j\right) D_j\left(p_1, ., \tilde{p}_j, p_m\right) \leq \left(p - \tilde{c}_j\right) D_j\left(p_1, ., p_j, p_m\right) \tag{3.11}$$
$$\text{for all } \tilde{p}_j \geq p_j.$$

Essentially, the definition says that if prices are chosen precisely to saturate the capacity and no lower, increasing prices will yield a decrease in profit. This is possible to check in some explicit examples. It's unclear whether the definition should be more restrictive by requiring the hypthoses to hold for all $j$ simultaneously. The definition may be hard to check in practice, but it's really more an heuristic characterzing the strongly constrained regime, allowing us to make a clear statement of a theorem involving optimal transportation.

**Theorem 3.1.** *Suppose that a market is strongly supply constrained. Let $\mu_i$ be a set of buyer demands with*

$$\sum_i \mu_i = \sum_j \alpha_j$$

*and let $\left(p, p^{\psi}\right)$ be a pair solving the optimal transportation problem between $\mu$ and $\alpha$, and suppose also that for each $j$*

$$D_j(p) = \alpha_j. \tag{3.12}$$

*Then the price function $p$ represents a Nash equilibrium from the perspective of both suppliers and buyers.*

*Proof.* First, we consider suppliers. Since each supplier is at capacity, decreasing the price cannot increase their sales, so will only result in lower profits. Increasing prices, on the other hand will not increase profits, by (3.11) which follows from the hypothesis and (3.12). Implicitly, when we computed the demand functions, we are assuming that buyers are already buying from suppliers based on a Nash equilibrium from the perspective of the buyers, given the fixed prices $p$. □

Before stating our most general theorem, we continue the set up from section 3.2. Let

$$\begin{aligned}
P_B &= \text{Price of Bitcoin in dollar}\\
\chi &= \text{Block reward in Bitcoins}\\
q &= \text{total hashes that will be produced by all miners with machines already acquired,}\\
&\quad \text{over the upcoming epoch}\\
q_i &= \text{total hashes produced by miner firm } i \text{ over the upcoming epoch,}\\
&\quad \text{with equipment already owned}\\
q_0 &= \text{total hashes that will be produced by miners over the epoch, by miners}\\
&\quad \text{not on the market for new ASICs}\\
\Delta q_i &= \text{total hashes purchased by miner } i\\
c_i &= \text{ expected electrical cost over the expected lifetime for miner } i\\
p_j &= \text{price of the equipment required to produce one unit of hash from producer } j\\
e_j &= \text{electrical inefficiency of equipment sold by producer } j\\
\tilde{e}_i &= \text{average electrical inefficiency of equipment already owned by mining firm } i\\
\alpha_j &= \text{hashrate available for sale by producer } j\\
Q &= \text{total hashrate during epoch } = q + \sum_{i=1}^{n} \Delta q_i.
\end{aligned}$$

The problem: Determine the prices $p_j$, the quantities $\Delta q_i$ and pairings $b_{ij}$ representing purchases from miner $i$ from producer $j$ so that the demand for each ASIC at this price function is equal to the supply.

We assume that the costs are ordered increasingly, that is

$$c_1 \leq c_2 \leq ... \leq c_n$$

while the reverse is true for the inefficiencies

$$e_1 \geq e_2 \geq ... \geq e_m.$$

**Theorem 3.2.** *Suppose the supply market is strongly supply constrained. There exists a unique set of prices $\{p(e_j)\}$ and quantities $\Delta q_i$ demanded by the buyers at costs $u = p^\psi$, such that $\Delta q_i$ will be paired optimally to capacities $\{\alpha_j\}$ according to $(p, u)$. This pairing represents a Nash equilibrium for all participants.*

*Proof.* **Existence**. Our goal is to show that there is a set of prices for which the demand function

$$D_j\left(p_1, ..., p_m\right) = \alpha_j \text{ for all } j.$$

We define the function

$$u(c_i) = \inf_{e_j}\left(e_j c_i + p(e_j)\right).$$

The profit function for each miner is given by

$$\pi_i = \frac{\chi P}{q + \sum_j \Delta q_j}\left(q_i + \Delta q_i\right) - c_i \tilde{e}_i q_i - u(c_i)\Delta q_i.$$

Implicit in this statement are two key assumptions: 1) either the miner is purchasing from a single supplier, or the miner is indifferent to the suppliers that they are purchasing from, and 2) The miner is able to purchase the amount they want from the supplier(s) that they prefer to purchase from.

Our strategy is to use these assumptions, and derive the conditions for the appropriate optimal transportation problem. Then we will show that the optimal transportation problem has a solution. Necessarily, these two assumptions will be true for a solution of the optimal transportation problem. Thus we will have a solution to the problem.

The miner will purchase hash rate until doing so becomes no longer marginally profitable, that is when

$$\begin{aligned}\frac{\partial \pi_i}{\partial \Delta q_i} &= \frac{\chi P_B\left(Q - q_i - \Delta q_i\right)}{Q^2} - u(c_i) \\ &= 0.\end{aligned} \tag{3.13}$$

As before, we sum this relationship over $i$ :

$$\sum_{i=1}^{n} \frac{\chi P_B\left(Q - q_i - \Delta q_i\right)}{Q^2} = \sum_{i=1}^{n} u(c_i) \tag{3.14}$$

and get

$$\begin{aligned}&= \frac{\chi P_B}{Q^2}\left(nQ - \sum_{i=1}^{n} q_i - \sum_{i=1}^{n} \Delta q_i\right) \\ &= \frac{\chi P_B}{Q^2}\left(\left(n-1\right)Q + q_0\right)\end{aligned}$$

that is

$$\sum_{i=1}^{n} u(c_i) = \chi P_B \left( \frac{q_0}{Q^2} + \frac{(n-1)}{Q} \right) = \Theta. \tag{3.15}$$

In this constrained situation we assume that $\Delta q$ is precisely the available capacity to be purchased, that is

$$\sum_{i=1}^{n} \Delta q_i = \sum_{j=1}^{m} \alpha_j$$

and $q$ is known, so this quantity $\Theta$ is determined at the outset of the problem. Using (3.13), we can also solve for the amount of hash purchased by each miner in this plan

$$q - q_i + \sum_k \Delta q_k - \Delta q_i = \frac{u(c_i)}{\chi P_B} \left( q + \sum_k \Delta q_k \right)^2$$

$$\Delta q_i = Q - q_i - \frac{u(c_i)}{\chi P_B} Q^2.$$

Now switching to the optimal transport framework, let

$$\mu_i = \Delta q_i.$$

Thus the problem reduces to the following: Find a pair of function $(p, u)$ such that:

(1)
$$u = p^{\psi},$$

(2) the optimal transportation problem pairing the weights $\{\alpha_j\}$ to $\{\mu_i\}$ given by
$$\mu_i = Q - q_i - \frac{u(c_i)}{\chi P_B} Q^2 \tag{3.16}$$
has pairing support in the set
$$\{(e_j, c_i) : p(e_j) - u(c_i) = \psi(e_j, c_i)\}$$

(3) and finally
$$\sum u(c_i) = \Theta.$$

If we can show such a pair $(p, u)$ exists, it follows that the demand equations will be satisfied and the pairing will represent a solution.

With this in mind, we construct a set of admissible values for the function $u$. Let

$$\mathcal{A} = \left\{ \vec{u} = (u(c_1), .., u(c_n)) \in \mathbb{R}^n \mid \begin{array}{c} \sum u(c_i) = \Theta \\ (c_{i+1} - c_i)\, e_m \leq u\,(c_{i+1}) - u(c_i) \leq (c_{i+1} - c_i)\, e_1, \forall i \end{array} \right\}$$

One can check that set of values is a compact convex subset of Euclidean space. We are going to use the Brouwer fixed point theorem on this set.

Define cumulative measure functions:

$$\begin{aligned}
\rho_1(\vec{u}) &= Q - q_1 - \frac{u(c_1)}{\chi P_B} Q^2 \\
\rho_2(\vec{u}) &= \rho_1(\vec{u}) + Q - q_2 - \frac{u(c_2)}{\chi P_B} Q^2 \\
\rho_3(\vec{u}) &= \rho_2(\vec{u}) + Q - q_3 - \frac{u(c_3)}{\chi P_B} Q^2 \\
&\ldots \\
\rho_n(\vec{u}) &= \Delta q
\end{aligned}$$

which represent the cumulative hashrate purchased as we move from most efficient miner to least efficient.

Next construct a function $\eta_0$ on $[0, \Delta q]$ as follows

$$\eta_0(x) = \begin{pmatrix} e_1 \text{ if } x < \alpha_1 \\ e_2 \text{ if } \alpha_1 \leq x < \alpha_1 + \alpha_2 \\ \ldots \\ e_n \text{ if } \alpha_1 + \alpha_2 + \alpha_{n-1} \leq x \end{pmatrix}.$$

Notice that $\eta_0$ is not a continuous function, so it won't make a good ingredient in the fixed point theorem. Instead, we approximate $\eta_0$ with a function $\eta_\varepsilon(x)$ with the properties that

(1) $\eta_\varepsilon(x) = \eta_0(x)$ for all $x$ such that $|x - \alpha_j| > \varepsilon$ for all $\alpha_j$
(2) $\eta_\varepsilon(x)$ is continuous
(3) $\eta_\varepsilon(x)$ is monotone decreasing.

Next we define

$$V_\varepsilon(\vec{u}) = \begin{pmatrix} v_1, \\ v_2, \\ v_3 \\ \ldots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1, \\ v_1 + (c_2 - c_1)\, \eta_\varepsilon(\rho_1), \\ v_2 + (c_3 - c_2)\, \eta_\varepsilon(\rho_2) \\ \ldots \\ v_{n-1} + (c_n - c_{n-1})\, \eta_\varepsilon(\rho_{n-1}) \end{pmatrix} \tag{3.17}$$

where

$$v_1 = \frac{\Theta - \sum_{k=1}^{n-1}(n-k)\eta(\rho_k)(c_{k+1} - c_k)}{n}.$$

Observe that

$$\begin{aligned}
\sum v_i &= v_1 + (v_1 + (c_2 - c_1)\, \eta_\varepsilon(\rho_1)) + (v_1 + (c_2 - c_1)\, \eta_\varepsilon(\rho_1) + (c_3 - c_2)\, \eta_\varepsilon(\rho_2)) \ldots \\
&= n v_1 + (n-1)(c_2 - c_1)\, \eta_\varepsilon(\rho_1) + (n-2)(c_3 - c_2)\, \eta_\varepsilon(\rho_2) + \ldots + (c_n - c_{n-1})\, \eta_\varepsilon(\rho_{n-1}) \\
&= \Theta.
\end{aligned}$$

So the values $V(\vec{u})$ are in the set $\mathcal{A}$. Thus

$$V_\varepsilon : \mathcal{A} \to \mathcal{A}$$

is a continuous map, and has a fixed point, such that

$$V_\varepsilon(\vec{u}) = \vec{u}.$$

We want to show that this determines a solution to the problem, passing to the limit if necessary. First, we suppose that for some $\varepsilon > 0$ the values $\eta'_\varepsilon(\rho_i) = 0$ for all $i$. In this case, constructi the measure values

$$\begin{aligned} \mu_1 &= Q - q_1 - \frac{u(c_1)}{\chi P_B} Q^2 \\ \mu_2 &= Q - q_2 - \frac{u(c_2)}{\chi P_B} Q^2 \\ &\dots \\ \mu_n &= Q - q_n - \frac{u(c_n)}{\chi P_B} Q^2 \end{aligned}$$

and go about computing the optimal transport map from $\{\mu_i\}$ to $\{\alpha_j\}$. Suppose that this optimal transport plan is defined by $(\tilde{p}, \tilde{u})$. Note that due to the order-preserving property, and the fact that $\eta'_\varepsilon(\rho_i) = 0$ there will be a clean value $e_j(i)$ such that

$$\rho_i \in \left( \sum_{k=1}^{j-1} \alpha_k, \sum_{k=1}^{j} \alpha_k \right)$$

that is

$$\begin{aligned} (e_j, c_i) &\in \Gamma_{(\tilde{p},\tilde{u})} \\ (e_j, c_{i+1}) &\in \Gamma_{(\tilde{p},\tilde{u})}. \end{aligned}$$

thus

$$\begin{aligned} \tilde{u}(c_i) - p(e_j) &= c_i e_j \\ \tilde{u}(c_{i+1}) - p(e_j) &= c_{i+1} e_j. \end{aligned}$$

Thus

$$\begin{aligned} \tilde{u}(c_{i+1}) - \tilde{u}(c_i) &= (c_{i+1} - c_i)\, e_j \\ &= (c_{i+1} - c_i)\, \eta_0(\rho_i). \end{aligned}$$

But by the construction (3.17) these are exactly the same relations satisfied by the $\{u(c_i)\}$. It follows that $\tilde{u} = u$ and $u$ describes the optimal pairing between the measure $\{\mu_i\}$ to $\{\alpha_j\}$. This is what we wanted to show.

Now we suppose that for each $\varepsilon > 0$, the solution to $V_\varepsilon(\vec{u}) = \vec{u}$ does not satisfy $\eta'_\varepsilon(\rho_i) = 0$, for some $i$. Take a sequence of $\varepsilon_k \to 0$, and for each $\varepsilon_k$ let $\vec{u}_{(k)}$ be a fixed point solution, and let

$$\Lambda_k = \left(\vec{u}_{(k)}, \rho\left(\vec{u}_{(k)}\right), \eta_{\varepsilon_k}\left(\rho\left(\vec{u}_{(k)}\right)\right)\right) \in \mathcal{A}\times [0,1]^n \times [e_n, e_1]^n.$$

This gives a sequence of points $\{\Lambda_k\}$ inside a closed and bounded set in Euclidean space. By the Heine-Borel Theorem, this must have a limit inside the same set. Namely, there will be a value

$$\Lambda_\infty = \left(\vec{u}_\infty, \rho\left(\vec{u}_\infty\right), \eta^*\right) \in \mathcal{A}\times [0,1]^n \times [e_n, e_1]^n.$$

such that (possibly choosing a subsequence of $\{k\}$)

$$\begin{aligned} \vec{u}_k &\to \vec{u}_\infty \\ \rho\left(\vec{u}_k\right) &\to \rho\left(\vec{u}_\infty\right) \\ \eta_{\varepsilon_k}\left(\rho\left(\vec{u}_{(k)}\right)\right) &\to \eta^*. \end{aligned}$$

One can check that if $\rho_i \to \rho_i^*$ where

$$\rho_i^* \in \left(\sum_{k=1}^{j-1} \alpha_k, \sum_{k=1}^{j} \alpha_k\right)$$

then necessarily

$$\eta_i^* = e_j$$

However, if

$$\rho_i^* = \sum_{i}^{j-1} \alpha_j$$

for some $k$, then

$$\eta_i^* \in [e_j, e_{j-1}]$$

This means that the differences in the values $\vec{u}_\infty$ are determined by these $\eta_i^*$.

At this point, we may check that this set of $\vec{u}_\infty$ values will define an optimal transport plan from the measures $\mu_i$ constructed from $\vec{u}_\infty$ to $\{a_j\}$. We are done with existence.

**Uniqueness**. We show that the function $u$ solving the problem is unique.

Suppose there are two solutions, $u$ and $u', u$ with $u \neq u'$. We assume without loss of generality that

$$u(c_1) < u'(c_1).$$

or if, $u(c_1) = u'(c_1)$, then

$$u(c_2) < u'(c_2)$$

and so forth until a value $c_{k_0}$ is found where the functions are not equal. Beyond that, as the sums must be the same, there will be a first value $k > k_0$ such that

$$u(c_k) \geq u'(c_k).$$

It follows that

$$u(c_k) - u\left(c_{k-1}\right) > u'(c_k) - u'\left(c_{k-1}\right). \tag{3.18}$$

Now because

$$u(c_i) \leq u'(c_i) \text{ for } i = 1, ..., k-1$$

with at least one of these inequalities strict, we have that

$$\mu_i \geq \mu_i' \text{ for } i = 1, ..., k-1$$

and

$$\sum_{i=1}^{k-1} \mu_i > \sum_{i=1}^{k-1} \mu_i'. \tag{3.19}$$

Now let

$$r = \sup\left\{m \mid u(c_{k-1}) - p(e_m) = c_{k-1}e_m\right\}$$
$$r' = \sup\left\{m \mid u'(c_{k-1}) - p'(e_m) = c_{k-1}e_m\right\}.$$

The remainder of the proof (left as an exercise) follows from the following claim, which contradicts (3.18)

**Claim 1.** *For $r$ and $k$ defined above,*

$$r \geq r'$$

*and*

$$u(c_k) - u(c_{k-1}) \leq e_r\left(c_k - c_{k-1}\right)$$
$$u'(c_k) - u'(c_{k-1}) \leq e_r\left(c_k - c_{k-1}\right).$$

□

**Exercise 10.** *Prove Claim 1. Hint: First prove the following Lemma.*

**Lemma 3.1.** *Suppose that $(e_r, c_k) \in \Gamma$. Then for any $r \neq s$*

$$p(e_r) - p(e_s) + c_k\left(e_r - e_s\right) \leq 0.$$

*Suppose that $(e_s, c_{k-1}) \in \Gamma$. Then for any $r \neq s$*

$$p(e_r) - p(e_s) + c_{k-1}\left(e_r - e_s\right) \geq 0.$$

**Exercise 11.** *Argue that in the strongly constrained regime, every Nash equilibrium should arise as described in Theorem 3.1. Hint: First show that if the function $p$ is not a $\psi$-convex function, then there will be some supplier $j$ such that no miner prefers supplier $j$ as their most cost-efficient choice. This means that either 1)the demand for ASIC $j$ is not saturated; or, 2) There must be room for at least one other suppliers to increase price without losing any demand. Hence a non-convex price function $p(e)$ cannot describe equilibrium prices.*

3.10. **Supply constrained markets, examples.**

**Example 3.3.** *Suppose we have four mining firms each with electricity costs 10,11,12, and 15 respectively, and assume each has current capacity to produce 3 units of hash in the upcoming time epoch. Suppose there are two foundries producing ASICs for purchase with electric inefficiencies 7 and 5, and with output capacities 4 and 6 respectively. Suppose the price is 5000 and assume block reward is 1. (At this point we are more or less making up units so we can ignore them, like physicists, in order to keep the computations simple.) Find the allocation and the prices charged by the producers.*

Solution: First identify

$$q = 4 * 3 = 12$$
$$\alpha_1, \alpha_2 = 4, 6$$
$$\Delta q = 10$$
$$Q = 22$$
$$q_0 = 0$$

so by (3.15)

$$\sum_{i=1}^{4} u(c_i) = 5000 * \frac{3}{22} = \frac{7500}{11}. \tag{3.20}$$

So we want to find $u(10), u(11), u(12), u(15)$ such that (3.20) holds and so that the weights

$$\mu_i = 22 - 3 - \frac{u(c_i)}{5000} 22^2$$

are paired with $\alpha_1 = 4, \alpha_2 = 6$ in an optimal transportation plan involving $u$.

Now our proof of Theorem 3.2 involved the Brouwer fixed point theorem, which doesn't describe how to construct the solution. Constructing the solution may require some basic trial and error. We may be able to take an educated guess at the pairing support, check to see if it this pairing support will support the solution, and modify if necessary. That is, considering the monotonicity in the pairing, there are only so many possibilities for the matching: Certainly $\mu_1$ and $\alpha_1$ are paired, as are $\mu_4$ and $\alpha_2$ but because we aren't told precisely how big $\mu_1$ is we aren't sure what happens in the middle.

Perhaps the easiest way is to pay attention to where the break between suppliers happens: Either $\mu_1$ is split between $\alpha_1$ and $\alpha_2$ or there is a clean break between $\mu_1$ and $\mu_2$ or $\mu_2$ is split, and so forth.

To begin we will try the situation where $\mu_1$ is split. This necessarily means that $\mu_1 > \alpha_1$, and

$$\Gamma = \{(7, 10), (5, 10), (5, 11), (5, 12), (5, 15)\}$$

as only the price 7 is paired with the electrical cost 10. In this case,

$$\begin{aligned}
u(10) - p(7) &= 7 * 10 \text{ from } (7, 10) \in \Gamma \\
u(10) - p(5) &= 5 * 10 \text{ from } (5, 10) \in \Gamma \\
u(11) - p(5) &= 5 * 11 \text{ from } (5, 11) \in \Gamma \\
u(12) - p(5) &= 5 * 12 \text{ from } (5, 12) \in \Gamma \\
u(15) - p(5) &= 5 * 15 \text{ from } (5, 15) \in \Gamma.
\end{aligned}$$

Taking the difference of the first two equations, we get

$$p(5) = p(7) + 20.$$

Now taking differences of the remaining equations

$$\begin{aligned}
u(11) - u(10) &= 5 \\
u(12) - u(11) &= 5 \\
u(15) - u(12) &= 15.
\end{aligned}$$

Without knowing $u(10)$ we may add up

$$\sum u(c_i) = 4u(10) + 5 + 10 + 25 = 4u(10) + 40$$

so

$$\begin{aligned}
4u(10) + 40 &= \frac{7500}{11}, \\
u(10) &= \frac{\frac{7500}{11} - 40}{4} = \frac{1765}{11}
\end{aligned}$$

giving

$$\mu_1 = 22 - 3 - \frac{\frac{1765}{11}}{5000} 22^2 = 3.468.$$

Now unfortunately, in the matching we chose, we needed $\mu_1 > \alpha_1 = 4$ so this matching failed. But we are not too far off. Next try a matching where the split happens for $\mu_2$ : that is $\mu_1 < \alpha_1$ and $\mu_1 + \mu_2 > \alpha_1$ and

$$\Gamma = \{(7, 10), (7, 11), (5, 11), (5, 12), (5, 15)\}$$

Similar computations

$$\begin{aligned} u(10) - p(7) &= 7 * 10 \\ u(11) - p(7) &= 7 * 11 \\ u(11) - p(5) &= 5 * 11 \\ u(12) - p(5) &= 5 * 12 \\ u(15) - p(5) &= 5 * 15. \end{aligned}$$

From the difference of the second and third equations we get

$$p(5) = p(7) + 22.$$

Subtracting the first and second equations, and so forth

$$\begin{aligned} u(11) - u(10) &= 7 \\ u(12) - u(11) &= 5 \\ u(15) - u(12) &= 15. \end{aligned}$$

Similarly to before,

$$\sum u(c_i) = 4u(10) + 46$$

and

$$u(10) = \frac{\frac{7500}{11} - 46}{4} = \frac{3497}{22}$$

leading to

$$\begin{aligned} \mu_1 &= 22 - 3 - \frac{\frac{3497}{22}}{5000} 22^2 \approx 3.6132 \\ \mu_2 &= 22 - 3 - \frac{\frac{3497}{22} + 7}{5000} 22^2 \approx 2.9356 \\ \mu_3 &\approx 2.4516 \\ \mu_4 &\approx 0.9996. \end{aligned}$$

Notice that because

$$\mu_1 + \mu_2 > \alpha_1$$

this particular matching is compatible with our assumptions. We can then determine the price via

$$u(10) - p(7) = 7 * 10.$$

Thus

$$\begin{aligned} p(7) &= 119.95 \\ p(5) &= 141.95. \end{aligned}$$

Now one thing we didn't do before diving into the solution of the problem is to attempt to prove that the regime is strongly supply constrained. However, we can always a posteriori go ahead and look at how the quantities sold will respond to a change in prices and conclude that the given solution is a Nash equilibrium.

Suppose that $p(7)$ were to be increased. As the value $p(7)$ determines the costs $u(10)$ and $u(11)$, moving equally in the opposite direction, the equation (3.16) suggests that a small change $\varepsilon$ in $p(7)$ will result in a decrease in $\mu_1$ of

$$-\frac{\varepsilon}{\chi P_B} Q^2 \approx -0.096\,8\varepsilon.$$

So the change in revenue obtained from selling to $\mu_1$ will be approximately

$$\begin{aligned} &3.\,613\,2\varepsilon - 0.096\,8 * 119.\,95\varepsilon \\ &\approx -7.\,998\,0\varepsilon. \end{aligned}$$

This confirms that a price increase for this supplier would be a bad idea. Note that we haven't mentioned the production cost, which could influence the computation. Nor have we considered the effect on $u(11)$ which should be smaller, but in the same direction.

In general, observe that differentiating the profit obtained results in

$$\frac{d}{dp(e_j)} \left[\alpha_j(p) * (p(e_j) - \tilde{c}_j)\right] = \alpha_j - \frac{Q^2}{\chi P_B} \left(p(e_j) - \tilde{c}_j\right)$$

so if the profit margins are large enough so that

$$\alpha_j \leq \frac{Q^2}{\chi P_B} \left(p(e_j) - \tilde{c}_j\right)$$

it will generally be a bad decision to raise prices in a manner that will cause the demand to decrease below capacity.

**Example 3.4.** *Same as Example 3.3, but suppose instead each firm is already producing 20 units of hash per time epoch and suppose that price is 10000.*

First identify

$$\begin{aligned} q &= 4 * 20 = 80 \\ \alpha_1, \alpha_2 &= 4, 6 \\ \Delta q &= 10 \\ Q &= 90 \end{aligned}$$

so

$$\sum_{i=1}^{4} u(c_i) = 10000 * \frac{3}{90} = \frac{1000}{3}. \tag{3.21}$$

So we want to find $u(10), u(11), u(12), u(15)$ such that (3.20) holds and so that

$$\mu_i = 90 - 20 - \frac{u(c_i)}{10000} 90^2$$

are paired with $\alpha_1 = 4, \alpha_2 = 6$ in an optimal transportation plan involving $u$.

Again we look for where the break between suppliers happens: Either $\mu_1$ is split between $\alpha_1$ and $\alpha_2$ or there is a clean break between $\mu_1$ and $\mu_2$, or $\mu_2$ is split, and so forth.

To begin we will try a situation where $\mu_1$ is split. This necessarily means that $\mu_1 > 4$. In this case, as above we get

$$u(10) = \frac{\frac{1000}{3} - 40}{4} = \frac{220}{3}$$

and

$$\mu_1 = 70 - \frac{\frac{220}{3}}{10000} 90^2 \approx 10.6.$$

but

$$10.6 > 10$$

so this is most certainly wrong. If we were to continue, we would find that the values $\mu_i$ would still add to 10, so some must be negative. So the problem with our assumption is that we allowed for a situation that the quantity purchased by the least efficient miner is negative; this is not feasible. So instead, we can assume that it is impossible to have a Nash equilibrium involving the 4th miner, who would be losing money by joining the market.

The solution is to drop the 4th miner and try again.

Dropping the 4th from the ASIC market sets up a new game. This time we have $q_0 = 20$ : This represents that the 4th miners is not interested in purchasing, but is continuing to mine. We assume they continue to mine their 20. (We could definitely ask whether they should continue to mine - since we aren't given the efficiency of the equipment they own, we don't have the information to make this determination, and we leave the assumption alone and proceed.)

This time, we get

$$\sum_{i=1}^{3} u(c_i) = 10000 \left( \frac{20}{90^2} + \frac{2}{90} \right) = \frac{20\,000}{81}. \tag{3.22}$$

and

$$u(10) = \frac{\frac{20\,000}{81} - 15}{3} = \frac{18\,785}{243}$$
$$u(11) = \frac{18\,785}{243} + 5$$
$$u(12) = \frac{18\,785}{243} + 10$$

also

$$\mu_1 = 70 - \frac{\frac{18\,785}{243}}{10000}90^2 \approx 7.383\,3$$
$$\mu_2 = 70 - \frac{\frac{18\,785}{243} + 5}{10000}90^2 \approx 3.333\,3$$
$$\mu_3 = 70 - \frac{\frac{18\,785}{243} + 10}{10000}90^2 \approx -0.716\,67.$$

This is closer, but still not a feasible solution, as $\mu_3 < 0$.

Repeating the process again but removing the third miner, we will get:

$$\mu_1 = 70 - 77.747 * \frac{81}{100} \approx 7.024\,9.$$
$$\mu_2 = 70 - 82.747 * \frac{81}{100} \approx 2.974\,9$$

$$p(7) \approx 7.746\,9$$
$$p(5) \approx 27.746\,9.$$

3.10.1. *Remarks.* There are many variations on these types of problems, we will not work examples of each. But we can observe general principles.

The more hashrate that is already in place, the larger effect the electricity cost has on pricing out the mining firms with higher electricity cost. This isn't hard to argue mathematically: As $q$ becomes large relative to $\Delta q$ the more likely it is that all of the equipment will be going to the miners with cheapest electricity.

We are not exploring, as it can become involved, what happens when a mining firm has access a variety of electricity sources.

The first term in (3.13) describes the marginal revenue to a miner producing a unit of hash:

$$\frac{\chi P_B \left(Q - q_i - \Delta q_i\right)}{Q^2}.$$

This is clearly linear in price, but is nonlinear in $Q$ and decreasing in $q_i$ and $\Delta q_i$. If two mining firms have access to the same electric costs, the firm with the lower current hashrate will have more incentive to increase hashrate, and the firm with higher hashrate may be priced out of new equipment. This is generally good for decentralization as it allocates hashrate towards firms with less current production.

## 4. Two easy applications

4.1. **Subsidies for green mining.** A concern with Bitcoin mining is that the energy consumption is massive and grows with proportional to the block reward, posing ecological risk. On the counterside of this claim is a claim that a consistent global market for renewable energy incentivizes the buildout of renewable energy sources that otherwise may have not been tapped. It is certainly not in the scope of this text to offer an opinion, certainly both arguments are founded on some valid assumptions. Instead we only wish to give some frameworks and tools which will allow the debate to be waged on solid footing.

For example, it has been proposed that the move towards green mining can be encouraged by subsidies for green miners, giving green miners a large share of the market, ultimately crowding out the unheathly coal-burning miners, see [3]

We show how our methods can be used to explore this idea quantitatively. We take the assumption that the subsidy is given to green miners in the form of a per-hash subsidy on equipment purchases. Notice that a lump sum doesn't change the profit motives, but a per unit price change does.

As a toy model, we consider a case where there is a base hashrate $q_0$ and only two mining firms are in the market for hashrate, which is supplied by a single supplier. One is mining renewable energy at cheaper electric cost, while the other is mining using energy obtained from expensive coal. We consider the situation when a third party agrees to subsidize any purchase by the first (green) miner by contributing $\varepsilon$ to the cost. Setting the profit derivatives to zero

$$\frac{\partial \pi_1}{\partial \Delta q_1} = \frac{P\left(Q - q_1 - \Delta q_1\right)}{Q^2} - \left(u(c_1) - \varepsilon\right) = 0$$
$$\frac{\partial \pi_2}{\partial \Delta q_2} = \frac{P\left(Q - q_2 - \Delta q_2\right)}{Q^2} - u(c_2) = 0$$

where we see the $\varepsilon$ has gone to slightly reduce the cost. Summing, as before we get

$$\sum_{i=1}^{2} u(c_i) = P\left(\frac{q_0}{Q^2} + \frac{1}{Q}\right) + \varepsilon$$

There's no price differentiation, so necessarily

$$u(c_1) - p(e_1) = c_1 e_1$$
$$u(c_2) - p(e_1) = c_2 e_1$$

and

$$u(c_1) = \frac{P\left(\frac{q_0}{Q^2} + \frac{1}{Q}\right) + \varepsilon - (c_2 - c_1)\, e_1}{2}$$
$$u(c_2) = \frac{P\left(\frac{q_0}{Q^2} + \frac{1}{Q}\right) + \varepsilon + (c_2 - c_1)\, e_1}{2}.$$

Solving for

$$\Delta q_1 = \frac{Q}{2} - q_1 - \frac{q_0}{2} + \frac{\varepsilon + (c_2 - c_1)\, e_1}{2P} Q^2$$
$$\Delta q_2 = \frac{Q}{2} - q_2 - \frac{q_0}{2} + \frac{-\varepsilon - (c_2 - c_1)\, e_1}{2P} Q^2.$$

From this we conclude the following: for each per unit price subsidization of $\varepsilon$, a quantity of

$$\delta = \frac{\varepsilon Q^2}{2P}$$

that would have been purchased by the miner with cost $c_2$ will be purchased by the miner with cost $c_1$.

The price

$$p(e_1) = \frac{P\left(\frac{q_0}{Q^2} + \frac{1}{Q}\right) + \varepsilon - (c_2 + c_1)\, e_1}{2}$$

will have increased by $\varepsilon/2$.

We leave it to the reader to find real world quantities that make computation meaningful.

4.2. **Intermittent mining.** Another debate, not unrelated, is whether a mining firm can sustain profitability on cheaper but intermittent energy sources. Again, we only offer a framework to settle toy versions of this question, and leave the details of the debate to the more worldly. We consider a similar situation to the one above - there are only two miners in the market for new equipment, but suppose that the one with cheapest electricity is only able mine a fraction of the time $\eta \in (0,1)$. For simplicity in our toy model, assume that neither mining firm has current hashrate.

The profit function in this case

$$\pi_1 = \frac{P}{Q}\eta\Delta q_1 - (\eta c_1 e_1 + p(e_1))\,\Delta q_1$$
$$\pi_2 = \frac{P}{Q}\Delta q_2 - (c_2 e_1 + p(e_1))\,\Delta q_1$$

noticing that the $\eta$ shows up in the revenue term, the electricity term, but not the flat purchase price term. Differentiating,

$$\frac{\partial \pi_1}{\partial \Delta q_1} = \eta\frac{P\,(Q-\Delta q_1)}{Q^2} - (\eta c_1 e_1 + p(e_1)) = 0$$
$$\frac{\partial \pi_2}{\partial \Delta q_2} = \frac{P\,(Q-\Delta q_2)}{Q^2} - (c_2 e_1 + p(e_1)) = 0.$$

These equations can be written

$$\frac{P\,(Q-\Delta q_1)}{Q^2}\eta - ((\eta-1)c_1 e_1 + u(c_1)) = 0$$
$$\frac{P\,(Q-\Delta q_2)}{Q^2} - u(c_2) = 0.$$

Now from

$$u(c_2) = u(c_1) + (c_2 - c_1)\,e_1$$

we have

$$\frac{P\,(Q-\Delta q_2)}{Q^2} = \frac{P\,(Q-\Delta q_1)}{Q^2}\eta - (\eta-1)\,c_1 e_1 + (c_2 - c_1)\,e_1$$
$$= \frac{P\,(Q-\Delta q_1)}{Q^2}\eta - \eta c_1 e_1 + c_2 e_1.$$

Combine this with

$$\Delta q_2 + \Delta q_1 = \Delta q :$$
$$\frac{P\,(Q-\Delta q+\Delta q_1)}{Q^2} = \frac{P\,(Q-\Delta q_1)}{Q^2}\eta - \eta c_1 e_1 + c_2 e_1$$

or

$$(1+\eta)\,\Delta q_1 = (\eta-1)\,Q + \Delta q + \frac{(c_2 - \eta c_1)\,e_1 Q^2}{P}.$$

So one question we could ask, is, how low can $\eta$ fall before it becomes a bad idea to purchase new equipment at market prices?

The answer:

$$\eta = \frac{Q - \Delta q - \frac{c_2 e_1 Q^2}{P}}{Q - \frac{c_1 e_1 Q^2}{P}} < 1$$

describes the profitability threshold.

For example, suppose the current hashrate is $q_0 = 100$, there are 10 units for sale, the electric costs are 10 and 12 and the ASIC inefficiency is 5

$$\eta = \frac{100 - \frac{12*5*110^2}{10000}}{110 - \frac{10*5*110^2}{10000}} \approx 0.553\,54$$

meaning that a firm with lower electricity costs but wishing to obtain a unit that was meant to be run with 50% uptime would be priced out of the market.

**Exercise 12.** *Consider the example above. Is there a cost* $c_1 < 10$ *so that the first miner will obtain all the hash rate, even with* $\eta = 1/2$*?*

## References

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