# EVANS-KRYLOV ESTIMATES FOR A NONCONVEX MONGE AMPÈRE EQUATION

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Abstract. We establish Evans-Krylov estimates for certain nonconvex fully nonlinear elliptic and parabolic equations by exploiting partial Legendre transformations. The equations under consideration arise in part from the study of the "pluriclosed flow" introduced by the first author and Tian [28].

## 1. Introduction

1.1. **Statement of Main Estimate.** Consider $U \subset \mathbb{R}^k \times \mathbb{R}^l$ with coordinates $\{x_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^l$, and let $u \in C^\infty(U)$ be convex in the $x$ variables and concave in the $y$ variables. We will consider two equations in this setting. First, we have the *real twisted Monge-Ampère equation*

$$F(u) := \log\det u_{x_i x_j} - \log\det(-u_{y_i y_j}) = 0. \tag{1.1}$$

Next we consider the *parabolic real twisted Monge-Ampère equation*

$$H(u) := \frac{\partial}{\partial t} u - F(u) = 0. \tag{1.2}$$

We also consider similar equations using complex variables. In particular, consider $U \subset \mathbb{C}^k \times \mathbb{C}^l$ with coordinates $\{z_i\}_{i=1}^k$ and $\{w_i\}_{i=1}^l$, and let $u \in C^\infty(U)$ be plurisubharmonic in the $z$ variables and plurisuperharmonic in the $w$ variables. We have the *complex twisted Monge-Ampère equation*

$$F_{\mathbb{C}}(u) := \log\det u_{z_i \bar{z}_j} - \log\det(-u_{w_i \bar{w}_j}) = 0. \tag{1.3}$$

Lastly we have the *parabolic complex twisted Monge-Ampère equation*,

$$H_{\mathbb{C}}(u) := \frac{\partial}{\partial t} u - F_{\mathbb{C}}(u) = 0. \tag{1.4}$$

With uniform convexity assumptions made (see Definition 2.1), equations (1.1) and (1.3) are uniformly elliptic, whilst (1.2) and (1.4) are uniformly parabolic. However, these equations are *neither convex nor concave*, and as such the Evans-Krylov theory does not apply to these equations.

The celebrated result of Evans and Krylov states that for uniformly elliptic (or parabolic) equations, one can conclude interior Hölder estimates on the second derivatives from $C^{1,1}$ estimates, provided the equation is convex or concave. There are a few more general cases where such a statement can be made: Caffarelli and Cabre [3] showed results in the case where the functional $F$ is a minimum of convex and concave functions. Caffarelli and Yuan [5] demonstrate the estimates under a partial convexity condition. Yuan [31] has proved such estimates in the specific case of the 3-dimensional special Lagrangian equations. In full generality these estimates are known to fail for nonconvex equations : Nadirashvilli and Vlăduţ [22] exhibit a 2-homogeneous function satisfying a uniformly elliptic equation, which is $C^{1,1}$ and not $C^2$ at the origin.

The main purpose of this paper is to establish a $C^{2,\alpha}$ estimate for solutions of twisted Monge Ampère equations from uniform bounds on the Hessian.

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**Theorem 1.1.** *Given $k, l \in \mathbb{N}$ and $\lambda, \Lambda > 0$ there exists $C = C(k, l, \lambda, \Lambda)$ and $\alpha = \alpha(k, l, \lambda, \Lambda) > 0$ such that*

- *If $u \in \mathcal{E}^{k,l,\mathbb{R}}_{B_2,\lambda,\Lambda}$ is a solution to (1.1) then $||u||_{C^{2,\alpha}(B_1)} \leq C$.*
- *If $u \in \mathcal{E}^{k,l,\mathbb{R}}_{Q_2,\lambda,\Lambda}$ is a solution to (1.2) then $||\frac{\partial}{\partial t}u||_{C^{\alpha}(Q_1)} + ||u||_{C^{2,\alpha}(Q_1)} \leq C$.*
- *If $u \in \mathcal{E}^{k,l,\mathbb{C}}_{B_2,\lambda,\Lambda}$ is a solution to (1.3) then $||u||_{C^{2,\alpha}(B_1)} \leq C$.*
- *If $u \in \mathcal{E}^{k,l,\mathbb{C}}_{Q_2,\lambda,\Lambda}$ is a solution to (1.4) then $||\frac{\partial}{\partial t}u||_{C^{\alpha}(Q_1)} + ||u||_{C^{2,\alpha}(Q_1)} \leq C$.*

The key insight to establish these estimates comes from applying a partial Legendre transformation in the real case. More specifically, as the functions in question are mixed concave/convex, we can apply a Legendre transformation to the concave variables to obtain a strictly convex function. This transformation was described by Darboux [7], who observed how it has the effect of linearizing the two-dimensional real Monge-Ampère equation. This was exploited to establish regularity properties for the 2-dimensional Monge Ampère equation by many authors, and later in general dimension [24]. For the classical theory, see [12, 13, 14, 15, 25, 26] and for more recent applications see [11]. In Proposition 2.3 below we show that more generally one can transform solutions to (1.1) into the real elliptic Monge-Ampère equation. This observation alone suffices to establish an analogue of the rigidity result of Calabi-Jorgens-Pogorelov ([6, 17, 23]) for uniformly concave solutions of (1.1)

This result can be used in conjunction with a blowup argument to establish the first two claims of Theorem 1.1. However, even with the natural hypotheses for equations (1.3) and (1.4) of plurisub/superharmonicity, there is no well-defined version of a complex Legendre transformation. Thus this method alone cannot establish Theorem 1.1 in these cases. Nonetheless, we use the transformation law to take quantities which are subsolutions to parabolic equations associated to the complex parabolic Monge-Ampère equation, and express them in terms of the inverse Legendre-transformed coordinates, assuming this transformation were defined. As it turns out, these quantities in the original coordinates are *always* defined (irrespective of whether the Legendre transformation is defined), and still are subsolutions of certain parabolic equations. This key observation can be exploited to adapt the usual proof of the Evans-Krylov estimate for convex equations to our setting, establishing Theorem 1.1.

1.2. **Consequences for pluriclosed flow.** In [27, 28, 29, 30] the second author and Tian introduced and developed a geometric flow of pluriclosed metrics on complex manifolds. Briefly, given $(M^{2n}, g_0, J)$ a Hermitian manifold such that the associated Kähler form $\omega_0$ is pluriclosed, i.e. $\partial\overline{\partial}\omega_0 = 0$, we say that a one-parameter family $g_t$ of metrics metrics is a solution to *pluriclosed flow* if the associated Kähler forms satisfy

$$\frac{\partial}{\partial t}\omega = \partial\partial^*_\omega\omega + \overline{\partial\partial^*_\omega}\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g. \tag{1.5}$$

This evolution equation is strictly parabolic and preserves the pluriclosed condition. Moreover, in [30] it was discovered that pluriclosed flow preserves generalized Kähler geometry in an appropriate sense. Recall that a generalized Kähler manifold is a quadruple $(M^{2n}, g, J_A, J_B)$ consisting of a smooth manifold with two integrable complex structures $J_A, J_B$, a metric $g$ which is compatible with both, and moreover satisfies the conditions

$$d^c_A\omega_A = -d^c_B\omega_B, \qquad dd^c_A\omega_A = 0.$$

If we impose the further condition that $[J_A, J_B] = 0$, one obtains an integrable splitting of $T_{\mathbb{C}}M$, and moreover the pluriclosed flow in this setting reduces (up to background terms) to the parabolic complex twisted Monge Ampère equation ([27] Theorem 1.1). This is expounded upon in §5. Combining Theorem 1.1 with a blowup argument yields higher order regularity of the flow in the presence of uniform metric estimates. This result plays a key role in establishing new long time existence and convergence results for the pluriclosed flow, and these will be described in a future work.

**Theorem 1.2.** *Let $(M^{2n}, g_0, J_A, J_B)$ be a compact generalized Kähler manifold satisfying $[J_A, J_B] = 0$. Let $g_t$ be the solution to pluriclosed flow with initial condition $g_0$. Suppose the solution exists on $[0, \tau)$, $\tau < \tau^*(g)$ (see Definition 5.6), and there exists constants $\lambda, \Lambda$ such that*

$$\lambda g_0 \leq g_t \leq \Lambda g_0.$$

*Given $k \in \mathbb{N}$, $\alpha \in [0, 1)$ there exists a constant $C = C(k, \alpha, g_0, \lambda, \Lambda, \tau)$ such that*

$$||g_t||_{C^{k,\alpha}} \leq C.$$

1.3. **Outline.** In §2 we recall the partial Legendre transformation and determine transformation laws for the PDEs in question. Inspired by these transformation laws, in §3 we state monotonicity formulas for certain combinations of second derivatives along solutions to (1.2), (1.4). As the proofs consist of lengthy, tedious calculations we relegate them to an appendix, §6. Using this key input we establish Theorem 1.1 in §4. In §5 we recall how to reduce solutions to the pluriclosed flow on commuting generalized Kähler manifolds to solutions of a scalar PDE which reduces to (1.4) on flat space, and then use Theorem 1.1 to establish Theorem 1.2.

## 2. Background on Legendre Transformation

2.1. **Real Legendre Transformation.** We briefly recall the Legendre transformation and one of its key properties for us. For a smooth convex function $u : \mathbb{R}^n \to \mathbb{R}$, the *Legendre transformation* is defined first by the change of variables

$$y_i(x) = \frac{\partial u}{\partial x_i}(x),$$

then by declaring

$$w(y) = x_i \frac{\partial u}{\partial x_i}(x) - u(x).$$

Observe that the Jacobian of the coordinate change takes the form

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial^2 u(x)}{\partial x_i x_j}.$$

On the other hand, the Legendre transformation is involutive, so it follows that

$$\frac{\partial x_i}{\partial y_j} = \frac{\partial^2 w(y)}{\partial y_i \partial y_j}.$$

Thus we see that the Legendre transformation "inverts the Monge Ampère operator" in the sense that

$$\det \frac{\partial^2 w}{\partial y_i \partial y_j} = \left( \det \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^{-1}.$$

This basic fact lies at the heart of our constructions below.

2.2. **Transformation Laws for PDE and Rigidity Results.** In this subsection we build on the observation in §2.1 of the transformation law for the Monge Ampère operator under Legendre transformation to define a partial Legendre transformation which can convert the twisted Monge Ampère equation into the usual Monge Ampère equation. To begin we define a class of functions with certain convexity hypotheses.

**Definition 2.1.** Given $U \subset \mathbb{R}^k \times \mathbb{R}^l$ and constants $\lambda, \Lambda > 0$ we set

$$\mathcal{E}_U^{k,l,\mathbb{R}} := \{u \in C^\infty(U) \mid \ D^2u\big|_{\mathbb{R}^k} > 0, \quad D^2u\big|_{\mathbb{R}^l} < 0\},$$
$$\mathcal{E}_{U,\lambda,\Lambda}^{k,l,\mathbb{R}} := \{u \in \mathcal{E}_U^{k,l,\mathbb{R}} \mid \lambda I_k \leq D^2u\big|_{\mathbb{R}^k} \leq \Lambda I_k, \quad \lambda I_l \leq (-\, D^2u\big|_{\mathbb{R}^l}) \leq \Lambda I_l\}.$$

Similarly, given $U \subset \mathbb{C}^k \times \mathbb{C}^l$

$$\mathcal{E}_U^{k,l,\mathbb{C}} := \{u \in C^\infty(U) \mid \ \sqrt{-1}\partial\overline{\partial}u\big|_{\mathbb{C}^k} > 0, \quad \sqrt{-1}\partial\overline{\partial}u\big|_{\mathbb{C}^l} < 0\},$$
$$\mathcal{E}_{U,\lambda,\Lambda}^{k,l,\mathbb{C}} := \{u \in \mathcal{E}_U^{k,l,\mathbb{C}} \mid \lambda I_k \leq \sqrt{-1}\partial\overline{\partial}u\big|_{\mathbb{C}^k} \leq \Lambda I_k, \quad \lambda I_l \leq (-\sqrt{-1}\partial\overline{\partial}u\big|_{\mathbb{C}^l}) \leq \Lambda I_l\}.$$

**Definition 2.2.** Given $u \in \mathcal{E}_\Omega^{k,l,\mathbb{R}}$, the *partial Legendre transformation* is the map

$$\mathcal{PL}_{k,l} : \mathcal{E}_\Omega^{k,l,\mathbb{R}} \to \mathcal{E}_{\hat{\Omega}}^{n,0,\mathbb{R}},$$

defined as follows. Let $(x, y)$ denote coordinates on the domain $\Omega \subset \mathbb{R}^k \times \mathbb{R}^l$ where the function is originally defined. Let $(x, z)$ denote coordinates of the set $\hat{\Omega}$ that is the image of $\Omega$ under the map

$$(x, y) \longmapsto \left(x, \frac{\partial u}{\partial y}(x, y)\right). \tag{2.1}$$

Now on $\hat{\Omega}$, define

$$y(x, z) = \left\{y \mid \frac{\partial u}{\partial y}(x, y) = z\right\}. \tag{2.2}$$

Then we have $\mathcal{PL}_{k,l}(u) = w(x, z)$, where

$$w(x, z) = u(x, y(x, z)) - \langle y(x, z), z\rangle\,.$$

**Proposition 2.3.** *(Transformation law for twisted Monge Ampère) Given $u \in \mathcal{E}^{k,l,\mathbb{R}}$, one has*

$$D^2\mathcal{PL}_{k,l}u = \begin{pmatrix} u_{xx} - u_{xy}u_{yy}^{-1}u_{yx} & u_{yx}u_{yy}^{-1} \\ u_{yy}^{-1}u_{yx} & -u_{yy}^{-1} \end{pmatrix} \geq 0. \tag{2.3}$$

*Moreover,*

$$\det D^2\mathcal{PL}_{k,l}u = \frac{\det D^2u\big|_{\mathbb{R}^k}}{\det\left(-D^2u|_{\mathbb{R}^l}\right)}. \tag{2.4}$$

*Proof.* Let $\Phi(x, y) = (x, z)$ denote the coordinate transformation defined by (2.2). The defining equation (2.2) can be rewritten as $y = D_yu(x, z)$. Differentiating this yields

$$I = \frac{\partial y}{\partial y} = D_yD_yu(x, z)\frac{\partial z}{\partial y},$$

thus $\frac{\partial z}{\partial y} = (D_yD_yu)^{-1}$. Also

$$0 = \frac{\partial y}{\partial x} = D_xD_yu + D_yD_yu\frac{\partial z}{\partial x},$$

thus $\frac{\partial z}{\partial x} = -(D_yD_yu)^{-1}D_xD_yu$. Thus

$$D\Phi = \begin{pmatrix} I_k & 0 \\ -(D_yD_yu)^{-1}D_xD_yu & (D_yD_yu)^{-1} \end{pmatrix}. \tag{2.5}$$

Using this, a further direct calculation shows

$$\nabla w = \ (u_x, -z)\,.$$

Thus

$$D^2 w = \begin{pmatrix} u_{xx} + u_{yx} \cdot \frac{\partial z}{\partial x} & u_{xy} \frac{\partial z}{\partial y} \\ -\frac{\partial z}{\partial x} & -\frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} u_{xx} - u_{xy} u_{yy}^{-1} u_{yx} & u_{yx} u_{yy}^{-1} \\ u_{yy}^{-1} u_{yx} & -u_{yy}^{-1} \end{pmatrix}, \tag{2.6}$$

establishing (2.3). The formula (2.4) follows from the block determinant formula

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det \left( A - BD^{-1} C \right).$$

To conclude the matrix is nonnegative, write it symbolically as

$$\begin{pmatrix} A + BCB^T & -BC \\ -C^T B^T & C \end{pmatrix}.$$

Then one can easily check that

$$\begin{aligned} &\begin{pmatrix} A + BCB^T & -BC \\ -C^T B^T & C \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \cdot \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \\ &= A\vec{v} \cdot \vec{v} + C \left( \vec{w} - B^T \vec{v} \right) \cdot \left( \vec{w} - B^T \vec{v} \right) \geq 0. \end{aligned}$$

□

**Corollary 2.4.** *Given $\lambda, \Lambda > 0$, let $u \in \mathcal{E}^{k,l}_{\mathbb{R}^n,\lambda,\Lambda}$ be a solution to (1.1). Then $u$ is a quadratic polynomial.*

*Proof.* Given a lower bound on the concavity in the second variable, we conclude the coordinate $z$ is global, for each $x$. Directly using (2.3) we see that $\mathcal{PL}_{k,l}$ is an entire convex function. Also, using (2.4) and (1.1) we conclude that

$$\det D^2 \mathcal{PL}_{k,l} u = 1.$$

It follows from ([6, 17, 23]) that $\mathcal{PL}_{k,l} u$ is a quadratic polynomial, and its Hessian is constant. Comparing against (2.3), we conclude that $u_{yy}^{-1}$ is constant, thus $u_{xy}$ is constant, and then finally $u_{xx}$ is constant. Thus $u$ is a quadratic polynomial. □

**Remark 2.5.** The direct Legendre transformation method of Corollary 2.4 gives a Liouville-Bernstein type property for solutions to (1.1) which are uniformly concave in the second variable. We will give a second proof of Evans-Krylov type regularity in §3-4 which does not actually require direct use of the Legendre transformation, but rather makes estimates directly on the matrix (2.3), which is always defined in the original coordinates, and has the crucial subsolution property necessary in the proof of the Evans-Krylov estimate. This has the key advantage of extending to the complex settings, where plurisubharmonicity does not suffice to define a genuine Legendre transform.

2.3. **Complex Legendre Transformation.** While convexity hypotheses are natural for understanding the real Monge Ampère equation, the natural hypothesis to impose for complex Monge-Ampère type equations is plurisubharmonicity. Many Legendre-type transformations have been proposed for functions on $\mathbb{C}^n$ satisfying conditions weaker than convexity, see for instance ([8, 16, 18, 20]). More recently there is a proposal ([1]) to define a Legendre transform for plurisubharmonic functions asymptotically, using Bergman kernels.

Now by direct analogy, if a complex Legendre transform where defined, the complex Hessian of the transform would be the inverse of the complex Hessian of the function, and we could argue as above to apply known results about the complex Monge-Ampère equation to the twisted equation. While we cannot find such a function, we operate directly on the partially transformed complex Hessian, proceeding obliviously as if the transform where defined. This yields a number of nontrivial maximum principles for (1.3) and (1.4), which would otherwise be difficult to discover or motivate.

## 3. Evolution Equations

3.1. **Real Case.** As explained in §2 we will now derive a priori estimates for (1.1)-(1.4) without the explicit use of the Legendre transformation. The basic idea is to think of the Legendre transformation "infinitesimally" and use it as a change of variables on the tangent space which helps us identify the right quantities/linearized operators which have favorable maximum principles. The first step is to recall that, in the context of the "pure" Monge Ampère equation for a function $w$, the linearized operator is

$$\mathcal{L} = w^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta}.$$

We now rewrite this operator using the partial Legendre transformation.

**Lemma 3.1.** *Let $u \in \mathcal{E}^{k,l}$, and $w = \mathcal{PL}_{k,l}u$. Furthermore let $T = D\Phi$ denote the change of basis matrix for the Legendre transformation as in (2.5).*

$$L^{ab} = w^{\alpha\beta} T^a_\alpha T^b_\beta.$$

*Then*

$$L = \begin{pmatrix} u_{xx}^{-1} & 0 \\ 0 & -u_{yy}^{-1} \end{pmatrix}.$$

*Proof.* We emphasize that this formula is describing the matrix $L$ in terms of the basis vectors $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\}$. First note that a direct calculation shows that

$$w^{-1} = \begin{pmatrix} u_{xx}^{-1} & u^{xx}u_{xy} \\ u_{yx}u^{xx} & -u_{yy}^{-1} + u_{yx}u^{xx}u_{xy} \end{pmatrix}.$$

We now directly compute $L$ block by block. First:

$$L^{x_i x_j} = w^{\alpha\beta} T^{x_i}_\alpha T^{x_j}_\beta = w^{x_k x_l} T^{x_i}_{x_k} T^{x_j}_{x_l} = u^{x_k x_l} \delta^i_k \delta^j_l = u^{x_i x_j}.$$

Next we have

$$\begin{aligned} L^{x_i z_j} &= w^{\alpha\beta} T^{x_i}_\alpha T^{z_j}_\beta \\ &= w^{x_k x_l} T^{x_i}_{x_k} T^{z_j}_{x_l} + w^{x_k y_l} T^{x_i}_{x_k} T^{z_j}_{y_l} + w^{y_k x_l} T^{x_i}_{y_k} T^{z_j}_{x_l} + w^{y_k y_l} T^{x_i}_{y_k} T^{z_j}_{y_l} \\ &= u^{x_k x_l} \delta^i_k \left(-u^{y_j y_p} u_{y_p x_l}\right) + u^{x_k x_p} u_{x_p y_l} \delta^i_k u^{y_j y_l} + 0 + 0 \\ &= 0. \end{aligned}$$

A similar calculation shows that $L^{z_i x_j} = 0$. Lastly we have

$$\begin{aligned} L^{z_i z_j} &= w^{\alpha\beta} T^{z_i}_\alpha T^{z_j}_\beta \\ &= w^{x_k x_l} T^{z_i}_{x_k} T^{z_j}_{x_l} + w^{x_k y_l} T^{z_i}_{x_k} T^{z_j}_{y_l} + w^{y_k x_l} T^{z_i}_{y_k} T^{z_j}_{x_l} + w^{y_k y_l} T^{z_i}_{y_k} T^{z_j}_{y_l} \\ &= u^{x_k x_l} (-u^{y_i y_p} u_{y_p x_k})(-u^{y_j y_q} u_{y_q x_l}) + u^{x_k x_p} u_{x_p y_l} (-u^{y_i y_q} u_{y_q x_k}) u^{y_j y_l} \\ &\quad + u_{y_k x_p} u^{x_p x_l} u^{y_i y_k} (-u^{y_j y_p} u_{y_p x_l}) + \left(-u_{y_k y_l} + u_{y_k x_p} u^{x_p x_q} u_{x_q y_l}\right) u^{y_i y_k} u^{y_j u_l} \\ &= -u^{y_i y_j}. \end{aligned}$$

□

**Proposition 3.2.** *Let $u \in \mathcal{E}^{k,l,\mathbb{R}}$ be a solution to (1.2). Let $W = D^2 \mathcal{PL}_{k,l} u$. Then*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) \frac{\partial u}{\partial t} = 0,$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) W \leq 0.$$

*Proof.* See Lemma 6.3. □

3.2. **Complex Case.** As discussed in §2, there is as yet not a clear Legendre transformation defined for plurisubharmonic functions. Nonetheless, we draw inspiration from §3.1 and obtain maximum principle estimates which correspond to usual maximum principle estimates for the complex Monge Ampère equation, assuming the Legendre transformation were defined. In particular, assume $u \in \mathcal{E}^{k,l}$ and set

$$\mathcal{L} := u^{\overline{z}_b z_a} \frac{\partial^2}{\partial z_a \partial \overline{z}_b} - u^{\overline{w}_b w_a} \frac{\partial^2}{\partial w_a \partial \overline{w}_b}. \tag{3.1}$$

Also, define

$$W = \begin{pmatrix} u_{z_i \overline{z}_j} - u_{z_i \overline{w}_k} u^{\overline{w}_k w_l} u_{w_l \overline{z}_j} & u_{z_i \overline{w}_k} u^{\overline{w}_k w_i} \\ u^{\overline{w}_j w_k} u_{w_k \overline{z}_i} & -u^{\overline{w}_j w_i} \end{pmatrix}. \tag{3.2}$$

The form of this matrix is of course derived from the corresponding quantity in (2.3). By a direct calculation one observes that equation (1.4) is equivalent to

$$\frac{\partial}{\partial t} u = \log \det W. \tag{3.3}$$

As in the real case, the crucial input in obtaining Evans-Krylov type estimates for (1.4) and (1.3) is a subsolution property for the matrix $W$. The proposition below is simply a long calculation which follows by elementary applications of the Cauchy-Schwarz inequality, and is carried out in §6.

**Proposition 3.3.** *Let $u \in \mathcal{E}^{k,l,\mathbb{C}}$ be a solution to (1.4). Then*

$$\left( \frac{\partial}{\partial t} - \mathcal{L} \right) \frac{\partial u}{\partial t} = 0,$$
$$\left( \frac{\partial}{\partial t} - \mathcal{L} \right) W \leq 0.$$

*Proof.* This follows directly from Lemmas 6.1 and 6.2. □

## 4. Evans-Krylov Type Estimates

In this section we establish a general oscillation estimate (Theorem 4.3 below) of Evans-Krylov type which replaces the convexity hypothesis with its main implication in the original proofs of Evans-Krylov: that the elliptic operator acts on a matrix of second partials that is itself a subsolution of a uniformly elliptic equation in the matrix sense. In conjunction with the subsolution properties arising from the partial Legendre transformation (Propositions 3.2 and 3.3), we obtain Theorem 1.1 as an immediate corollary. The proof is closely modeled after ([21] Lemma 14.6). We also refer the reader to the original works ([9], [19]) as well as more recent versions of the proof ([2], [4]). To begin we recall some standard notation and results.

**Definition 4.1.** Given $(w, s) \in \mathbb{C}^n \times \mathbb{R}$, let

$$Q((w,s),R) := \{(z,t) \in \mathbb{C}^n \times \mathbb{R} |\ t \leq s, \quad \max\{|z-w|, |t-s|^{\frac{1}{2}}\} < R\},$$
$$\Theta(R) := Q((w, s-4R^2), R).$$

**Theorem 4.2.** *([21] Theorem 7.37) Let $u$ be a nonnegative function on $Q(4R)$ such that*

$$-u_t + a^{ij} u_{ij} \leq 0,$$

*where*

$$\lambda \delta_i^j \leq a^{ij} \leq \Lambda \delta_i^j. \tag{4.1}$$

*There are positive constants $C, p > 1$ depending only on $n, \lambda, \Lambda$ such that*

$$\left(R^{-n-2}\int_{\Theta(R)} u^p\right)^{\frac{1}{p}} \leq C \inf_{Q(R)} u. \tag{4.2}$$

**Theorem 4.3.** *Suppose that $u \in \mathcal{E}^{k,l}_{Q(R),\lambda,\Lambda} \cap C^{2,1}(Q(R))$ satisfies*

$$\frac{\partial}{\partial t}u = F(W(D^2 u))$$

*for $F$ a $(\lambda, \Lambda)$-elliptic functional and suppose that the Hermitian matrix $W$ satisfies*

$$\left(\frac{\partial}{\partial t} - L\right) W \leq 0,$$

*for some $(\lambda, \Lambda)$-elliptic operator $L$. Then there are positive constants $\alpha, C$ depending only on $n, \lambda, \Lambda$ such that for all $\rho < R$,*

$$\operatorname{osc}_{Q(\rho)} u_t + \operatorname{osc}_{Q(\rho)} W \leq C(n, \lambda, \Lambda)\left(\frac{\rho}{R}\right)^{\alpha}\left(\operatorname{osc}_{Q(R)} u_t + \operatorname{osc}_{Q(R)} W\right).$$

*Proof.* We are assuming that $F$ is $(\lambda, \Lambda)$-elliptic on a convex set containing the range of $W$. Thus for any two points $(x, t_1), (y, t_2) \in Q(4R)$ there exists a matrix $a^{ij}$ satisfying (4.1) such that

$$a^{ij}\left((x, t_1), (y, t_2)\right)\left(W_{ij}(x, t_1) - W_{ij}(y, t_2)\right) = \frac{\partial u}{\partial t}(x, t_1) - \frac{\partial u}{\partial t}(y, t_2). \tag{4.3}$$

Now by ([21] Lemma 14.5) we can choose a finite set of unit vectors $v_\alpha$ such that

$$a^{ij} = \sum_{\alpha=1}^{N} f_\alpha v^i_\alpha \overline{v^j_\alpha}$$

for $f_\alpha$ depending on $(x, t_1), (y, t_2)$ yet always satisfying

$$f_\alpha \in [\lambda_*, \Lambda_*],$$

for some constants $\lambda_*, \Lambda_*$ depending only on $\lambda, \Lambda$. Now let $f_0 = 1$, and let

$$w_0 := -\frac{\partial u}{\partial t}$$
$$w_\alpha := W_{v_\alpha \overline{v_\alpha}}.$$

In this notation (4.3) reads

$$\sum_{\alpha=0}^{N} f_\alpha\left(w_\alpha(x, t_1) - w_\alpha(y, t_2)\right) = 0.$$

It follows that for every pair $(x, t_1), (y, t_2) \in Q(4R)$, we have

$$f_\alpha\left(w_\alpha(y, t_2) - w_\alpha(x, t_1)\right) = \sum_{\beta \neq \alpha} f_\beta\left(w_\beta(x, t_1) - w_\beta(y, t_2)\right).$$

Now let

$$M_{s\alpha} = \sup_{Q(sR)} w_\alpha, \qquad m_{s\alpha} = \inf_{Q(sR)} w_\alpha, \qquad P(sR) = \sum_\alpha M_{s\alpha} - m_{s\alpha}.$$

As each quantity $w_\alpha$ is a subsolution to a uniformly parabolic equation, it follows that $M_{4\alpha} - w_\alpha$ is a supersolution to a uniformly parabolic equation, and hence by Theorem 4.2 we obtain

$$\left(R^{-n-2}\int_{\Theta(R)} (M_{4\alpha} - w_\alpha)^p\right)^{\frac{1}{p}} \leq C_1 \inf_{Q(R)} (M_{4\alpha} - w_\alpha) = C_1(M_{4\alpha} - M_{1\alpha}).$$

Summing these inequalities and applying Minkowski's inequality yields, for any fixed $\beta$,

$$\begin{aligned}\left(R^{-n-2}\int_{\Theta(R)}\sum_{\alpha\neq\beta}(M_{4\alpha}-w_\alpha)^p\right)^{\frac{1}{p}} &\leq \sum_{\alpha\neq\beta}\left(R^{-n-2}\int_{\Theta(R)}(M_{4\alpha}-w_\alpha)^p\right)^{\frac{1}{p}} \\ &\leq C_1\sum_{\alpha\neq\beta}M_{4\alpha}-M_{1\alpha} \\ &\leq C_1\sum_{\alpha\neq\beta}M_{4\alpha}-m_{4\alpha}-(M_{1\alpha}-m_{1\alpha}) \\ &\leq C_1\left[P(4R)-P(R)\right].\end{aligned} \tag{4.4}$$

Now choose $(x,t)\in Q(4R)$ such that

$$W_\beta\left(x,t_1\right)=m_{4\beta.}$$

We have

$$\lambda_*\left(W_\beta\left(y,t\right)-m_{4\beta}\right)\leq f_\beta\left(y,t\right)\left(W_\beta\left(y,t\right)-m_{4\beta}\right)=\sum_{\alpha\neq\beta}f_\alpha\left(y,t\right)\left(w_\alpha(x,t_1)-w_\alpha(y,t)\right).$$

Thus for all $(y,t)\in Q(4R)$

$$\begin{aligned}\left(W_\beta\left(y,t\right)-m_{4\beta}\right) &\leq \frac{1}{\lambda_*}\sum_{\alpha\neq\beta}f_\alpha\left(y,t\right)\left(w_\alpha(x,t_1)-w_\alpha(y,t)\right) \\ &\leq \frac{1}{\lambda_*}\sum_{\alpha\neq\beta}f_\alpha\left(y,t\right)\left(M_{4\alpha}-w_\alpha(y,t)\right) \\ &\leq \frac{\Lambda_*}{\lambda_*}\sum_{\alpha\neq\beta}\left(M_{4\alpha}-w_\alpha(y,t)\right).\end{aligned}$$

Hence, using convexity of $s\to s^p$ we have

$$\begin{aligned}\left(W_\beta\left(y,t\right)-m_{4\beta}\right)^p &\leq \left[\frac{\Lambda_*}{\lambda_*}\sum_{\alpha\neq\beta}\left(M_{4\alpha}-w_\alpha(y,t)\right)\right]^p \\ &\leq \left(\frac{\Lambda_*}{\lambda_*}\right)^p N^{p-1}\sum_{\alpha\neq\beta}\left(M_{4\alpha}-w_\alpha(y,t)\right)^p.\end{aligned}$$

Integrating in the $(y,t)$ variables and applying (4.4) we have

$$\begin{aligned}\left(R^{-n-2}\int_{\Theta(R)}\left(W_\beta\left(y,t\right)-m_{4\beta}\right)^p\right)^{\frac{1}{p}} &\leq \frac{\Lambda_*}{\lambda_*}N^{1-\frac{1}{p}}\left(R^{-n-2}\int_{\Theta(R)}\sum_{\alpha\neq\beta}\left(M_{4\alpha}-w_\alpha(y,t)\right)^p\right)^{\frac{1}{p}} \\ &\leq \frac{\Lambda_*}{\lambda_*}NC_1\left[P(4R)-P(R)\right].\end{aligned}$$

Summing over $\beta$ we have

$$\sum_\beta\left(R^{-n-2}\int_{\Theta(R)}\left(W_\beta\left(y,t\right)-m_{4\beta}\right)^p\right)^{\frac{1}{p}}\leq(N+1)\,\frac{\Lambda_*}{\lambda_*}NC_1\left[P(4R)-P(R)\right]. \tag{4.5}$$

Also, a direct application of Theorem 4.2 yields

$$\sum_\beta \left(R^{-n-2}\int_{\Theta(R)} \left(M_{4\beta}-W_\beta\left(y,t\right)\right)^p\right)^{\frac{1}{p}} \leq \sum_\beta C_1\left(M_{4\beta}-M_{1\beta}\right) \tag{4.6}$$
$$\leq C_1\left[P(4R)-P(R)\right]$$

Applying Minkowski's inequality, followed by (4.5), (4.6) gives

$$\begin{aligned}
P(4R) &= \sum_\beta M_{4\beta}-m_{4\beta}\\
&= \sum_\beta \left(R^{-n-2}\int_{\Theta(R)} \left(M_{4\beta}-m_{4\beta}\right)^p\right)^{1/p}\\
&= \sum_\beta \left(R^{-n-2}\int \left(M_{4\beta}-W_\beta\left(y,t\right)+W_\beta\left(y,t\right)-m_{4\beta}\right)^p\right)^{1/p}\\
&\leq \sum_\beta \left(R^{-n-2}\int_{\Theta(R)} \left(M_{4\beta}-W_\beta\left(y,t\right)\right)^p\right)^{\frac{1}{p}}+\sum_\beta \left(R^{-n-2}\int_{\Theta(R)} \left(W_\beta\left(y,t\right)-m_{4\beta}\right)^p\right)^{\frac{1}{p}}\\
&\leq \left(\left(N+1\right)\frac{\Lambda_*}{\lambda_*}N+1\right)C_1\left[P(4R)-P(R)\right].
\end{aligned}$$

Rearranging this yields

$$\left(\left(N+1\right)\frac{\Lambda_*}{\lambda_*}N+1\right)C_1P(R) \leq \left[\left(\left(N+1\right)\frac{\Lambda_*}{\lambda_*}N+1\right)C_1-1\right]P(4R),$$

or in other words

$$P(R) \leq \mu P(4R)$$

for appropriately chosen $\mu$. A standard iteration argument ([10, Theorem 8.23]) now yields

$$P(\rho) \leq \left(\frac{\rho}{R}\right)^\alpha P(R), \tag{4.7}$$

for

$$\alpha = \frac{1}{2}\frac{\log\frac{1}{\mu}}{\log(4)}.$$

Now observe that in our application of ([21], Lemma 14.5) we can choose the set of vectors to contain all vectors of the form $e_j$, $\left(e_j \pm e_k\right)/\sqrt{2}$ and $\left(e_j \pm \sqrt{-1}e_k\right)/\sqrt{2}$ for any particular coordinate basis. Trivially we observe

$$\sum_j \operatorname{osc}_{Q(\rho)} W_{j\bar{j}} \leq P(\rho),$$

and then using polarization and the triangle inequality we may similarly bound the oscillation of any of the components of $W$.

We also note that clearly

$$P(R) \leq N \operatorname{osc}_{Q(\rho)} W. \tag{4.8}$$

Thus

$$\operatorname{osc}_{Q(\rho)} W \leq C_2(N)\left(\frac{\rho}{R}\right)^\alpha \operatorname{osc}_{Q(R)} W,$$

which is the conclusion of the theorem. $\square$

*Proof of Theorem 1.1.* Let us give the argument for (1.4), the other cases being similar. Let $u \in \mathcal{E}^{k,l}_{Q_2,\lambda,\Lambda}$ be a solution to (1.4). Observe that uniform estimates on the complex Hessian of $u$ imply uniform upper and lower bounds on the corresponding matrix $W$. By (3.3) and Proposition 3.3, we see that $u$ satisfies the hypotheses of Theorem 4.3, and therefore we conclude a $C^\alpha$ estimate for $\frac{\partial}{\partial t}u$ and $W$. Examining (3.2), we see that the lower right block $-u^{\overline{w}_j w_i}$ has a $C^\alpha$ estimate, which implies that $u_{w_i\overline{w}_j}$ does as well, using the uniform lower bounds on the matrix. Combining this estimate with the estimate on the upper right and lower left blocks we see that the derivatives $u_{z_i\overline{w}_j}$ and $u_{w_i\overline{z}_j}$ also enjoy $C^\alpha$ estimates. Finally, these estimates together with the estimates for the upper left block of $W$ imply a $C^\alpha$ estimate for $u_{z_i\overline{z}_j}$. □

**Corollary 4.4.** *Let $u_t$ be a solution to (1.4) on $(-\infty,0] \times \mathbb{C}^n$ such that $u_t \in \mathcal{E}^{k,l}_{\mathbb{C}^n,\lambda,\Lambda}$ for all $t \in (-\infty,0]$. Then $\nabla\partial\overline{\partial}u_t = \frac{\partial^2 u}{\partial t^2} = 0$ for all $t$.*

*Proof.* Suppose there exists a point such that $\left|\nabla\partial\overline{\partial}u\right| \neq 0$. By translating in space and time we can assume without loss of generality this point is $(0,0)$. Fix some $\mu > 0$ and consider

$$v(x,t) := \mu^{-2}u(\mu x, \mu^2 t)$$

By direct calculation one verifies that $v$ is a solution to (1.4) on $(-\infty,0] \times \mathbb{C}^n$ and moreover $v_t \in \mathcal{E}^{k,l}_{\mathbb{C}^n,\lambda,\Lambda}$ for all $t \in (-\infty,0]$. Also observe that $\left|\nabla\partial\overline{\partial}v\right|(0,0) = \mu\left|\nabla\partial\overline{\partial}u\right|(0,0)$. However, we know by Theorem 1.1 that there is a uniform $C,\alpha > 0$ so that $\left|\partial\overline{\partial}v\right|_{C^\alpha(Q((0,0),1)} \leq C$. By modifying $v$ by a time-independent affine function we can ensure a uniform $C^0$ estimate for $v$ on $Q((0,0),1)$ as well. At this point we may apply Schauder estimates on $Q((0,0),1)$ to obtain an a priori bound for $\left|\nabla\partial\overline{\partial}v\right|(0,0)$. For $\mu$ chosen sufficiently large this is a contradiction, finishing the proof. □

## 5. Evans-Krylov Regularity for Pluriclosed Flow

5.1. **Commuting Generalized Kähler manifolds.** In this section we exploit the estimates of §4 to establish a priori regularity results for the pluriclosed flow. We briefly recall here the discussion in ([27]) wherein the pluriclosed flow in the setting of generalized Kähler geometry with commuting complex structures is reduced to a parabolic flow of the kind (1.4).

Let $(M^{2n}, g, J_A, J_B)$ be a generalized Kähler manifold satisfying $[J_A, J_B] = 0$. Define

$$\Pi := J_A J_B \in \mathrm{End}(TM).$$

It follows that $\Pi^2 = \mathrm{Id}$, and $\Pi$ is $g$-orthogonal, hence $\Pi$ defines a $g$-orthogonal decomposition into its $\pm 1$ eigenspaces, which we denote

$$TM = T_+M \oplus T_-M.$$

Moreover, on the complex manifold $(M^{2n}, J_A)$ we can similarly decompose the complexified tangent bundle $T^{1,0}_{\mathbb{C}}$. For notational simplicity we will denote

$$T^{1,0}_{\pm} := \ker(\Pi \mp I) : T^{1,0}_{\mathbb{C}}(M, J_A) \to T^{1,0}_{\mathbb{C}}(M, J_A).$$

We use similar notation to denote the pieces of the complex cotangent bundle. Other tensor bundles inherit similar decompositions. The one of most importance to us is

$$\begin{aligned}\Lambda^{1,1}_{\mathbb{C}}(M, J_A) = &\left(\Lambda^{1,0}_+ \oplus \Lambda^{1,0}_-\right) \wedge \left(\Lambda^{0,1}_+ \oplus \Lambda^{0,1}_-\right)\\ = &\left[\Lambda^{1,0}_+ \wedge \Lambda^{0,1}_+\right] \oplus \left[\Lambda^{1,0}_+ \wedge \Lambda^{0,1}_-\right] \oplus \left[\Lambda^{1,0}_- \wedge \Lambda^{0,1}_+\right] \oplus \left[\Lambda^{1,0}_- \wedge \Lambda^{0,1}_-\right].\end{aligned}$$

Given $\mu \in \Lambda^{1,1}_{\mathbb{C}}(M, J_A)$ we will denote this decomposition as

$$\mu := \mu^+ + \mu^{\pm} + \mu^{\mp} + \mu^- \tag{5.1}$$

**Definition 5.1.** Let $(M^{2n}, J_A, J_B)$ be a bicomplex manifold such that $[J_A, J_B] = 0$. Let

$$\chi(J_A, J_B) = c_1^+(T_+^{1,0}) - c_1^-(T_+^{1,0}) + c_1^-(T_-^{1,0}) - c_1^+(T_-^{1,0}).$$

The meaning of this formula is the following: fix Hermitian metrics $h_\pm$ on the holomorphic line bundles $\det T_\pm^{1,0}$, and use these to define elements of $c_1(T_\pm^{1,0})$, and then project according to the decomposition (5.1). In particular, given such metrics $h_\pm$ we let $\rho(h_\pm)$ denote the associated representatives of $c_1(T_\pm^{1,0})$, and then let

$$\chi(h_\pm) = \rho^+(h_+) - \rho^-(h_+) + \rho^-(h_-) - \rho^+(h_-).$$

This definition yields a well-defined class in a certain cohomology group, defined in [27], which we now describe.

**Definition 5.2.** Let $(M^{2n}, J_A, J_B)$ be a bihermitian manifold with $[J_A, J_B] = 0$. Given $\phi_A \in \Lambda^{1,1}_{J_A,\mathbb{R}}$, let $\phi_B = -\phi_A(\Pi \cdot, \cdot) \in \Lambda^{1,1}_{J_B,\mathbb{R}}$. We say that $\phi_A$ is *formally generalized Kähler* if

$$\begin{aligned} d^c_{J_A}\phi_A =&\ - d^c_{J_B}\phi_B \\ dd^c_{J_A}\phi_A =&\ 0. \end{aligned} \tag{5.2}$$

**Definition 5.3.** Let $(M^{2n}, g, J_A, J_B)$ denote a generalized Kähler manifold such that $[J_A, J_B] = 0$. Let

$$\mathcal{H} := \frac{\left\{ \phi_A \in \Lambda^{1,1}_{J_A,\mathbb{R}} \mid \phi_A \text{ satisfies } (5.2) \right\}}{\left\{ \delta_+\delta_+^c f - \delta_-\delta_-^c f \right\}}.$$

With this kind of cohomology space, we can define the analogous notion to the "Kähler cone," which we refer to as $\mathcal{P}$, the "positive cone."

**Definition 5.4.** Let $(M^{2n}, g, J_A, J_B)$ denote a generalized Kähler manifold such that $[J_A, J_B] = 0$. Let

$$\mathcal{P} := \left\{ [\phi] \in \mathcal{H} \mid \exists \omega \in [\phi], \omega > 0 \right\}.$$

**Definition 5.5.** Let $(M^{2n}, g, J_A, J_B)$ be a generalized Kähler manifold such that $[J_A, J_B] = 0$. We say that $\chi = \chi(J_A, J_B) > 0$, (resp. $(\chi < 0,\ \chi = 0)$ if $\chi \in \mathcal{P}$, (resp. $-\chi \in \mathcal{P}, \chi = 0$).

5.2. **Scalar reduction and Evans-Krylov estimate.** In this subsection we describe how to reduce (1.5) to a scalar PDE in the setting of commuting generalized Kähler manifolds. This is then used in conjunction with Theorem 1.1 to establish Corollary 1.2.

First we recall that it follows from ([27] Proposition 3.2, Lemma 3.4) that the pluriclosed flow in this setting reduces to

$$\frac{\partial}{\partial t}\omega = \ - \chi(g_\pm). \tag{5.3}$$

From the discussion in §5.1 we see that a solution to (5.3) induces a solution to an ODE in $\mathcal{P}$, namely

$$[\omega_t] = [\omega_0] - t\chi.$$

This suggests the following definition.

**Definition 5.6.** Given $(M^{2n}, g, J_A, J_B)$ a generalized Kähler manifold, let

$$\tau^*(g) := \sup \left\{ t \geq 0 | [\omega] - t\chi \in \mathcal{P} \right\}.$$

Now fix $\tau < \tau^*$, so that by hypothesis if we fix arbitrary metrics $\widetilde{h}_\pm$ on $T^{1,0}_\pm$, there exists $a \in C^\infty(M)$ such that

$$\omega_0 - \tau\chi(\widetilde{h}_\pm) + \left(\delta_+\delta_+^c - \delta_-\delta_-^c\right) a > 0.$$

Now set $h_\pm = e^{\pm\frac{a}{2\tau}}\widetilde{h}_\pm$. Thus $\omega_0 - \tau\chi(h_\pm) > 0$, and by convexity it follows that

$$\hat{\omega}_t := \omega_0 - t\chi(h_\pm)$$

is a smooth one-parameter family of metrics. Furthermore, given a function $f \in C^\infty(M)$, let

$$\omega_f := \hat{\omega} + \left(\delta_+\delta_+^c - \delta_-\delta_-^c\right) f,$$

with $g^f$ the associated Hermitian metric. Now suppose that $u_t$ satisfies

$$\frac{\partial}{\partial t}u = \log\frac{\det g_+^u \det h_-}{\det h_+ \det g_-^u}. \tag{5.4}$$

An elementary calculation using the transgression formula for the first Chern class ([27] Lemma 3.4) yields that $\omega_u$ solves (5.3).

*Proof of Theorem 1.2.* Since by hypothesis $\tau < \tau^*$ we can adopt the discussion above and reduce our solution to the scalar flow (5.4). We recall the quantity $\Upsilon = \Upsilon(g,h)$ defined by the difference of the Chern connections associated to $g$ and $h$. In particular, we have

$$\Upsilon_{ij}^k = \left(\nabla^g - \nabla^h\right)_{ij}^k = g^{\bar{l}k}g_{i\bar{l},j} - h^{\bar{l}k}h_{i\bar{l},j}.$$

For a reduced solution to pluriclosed flow as above, there exists some background tensor $A = A(\hat{g}, h, \tau)$ such that

$$\Upsilon_{ij}^k = g^{\bar{l}k}\left[(\delta_+\delta_+^c - \delta_-\delta_-^c)u\right]_{i\bar{l},j} + A_{ij}^k.$$

By a standard argument using Schauder estimates, to prove the theorem it suffices to show that there exists a uniform constant $C$ such that

$$\sup_{M\times[0,\tau)} |\Upsilon(g,h)|^2 \leq C.$$

If this were not the case, choose a sequence $(x_i, t_i)$ such that $t_i \to \tau$ and

$$\lambda_i := |\Upsilon|^2(x_i,t_i) = \sup_{M\times[0,t_i]} |\Upsilon|^2.$$

We now construct a blowup sequence of solutions centered at these points. Furthermore we can choose a small constant $R > 0$ and a normal coordinate chart for $g_0$ centered at $x_i$, covering $B_R(x_i, g_0)$. Using such charts for each $i$ and translating in time we obtain a solution to (5.4) on $Q((0,0),\rho)$ for some small uniform constant $\rho > 0$ such that $|\Upsilon_i|^2(0,0) = \lambda_i$.

Now fix some constant $B > 0$, let $\alpha_i := B\lambda_i$ and let

$$\begin{aligned}
\widetilde{u}_i &= \alpha_i^2\left(u_i(\alpha_i^{-1}x, \alpha_i^{-2}t) - u_i(0,0)\right)\\
\widetilde{\omega}_i &= \hat{\omega}\left(\alpha_i^{-1}x, \alpha_i^{-2}t\right)\\
\widetilde{h}_i &= h\left(\alpha_i^{-1}x, \alpha_i^{-2}t\right).
\end{aligned}$$

A direct calculation shows that for each $i$ the function $\widetilde{u}_i$ is a solution of

$$\frac{\partial}{\partial t}\widetilde{u}_i = \log\frac{\det \widetilde{g}_+^{\widetilde{u}_i}\det\widetilde{h}-}{\det\widetilde{h}_+\det\widetilde{g}_+^{\widetilde{u}_i}}.$$

For sufficiently large $i$ we may assume this solution exists on $Q((0,0),3)$. Moreover, we claim that for every $j \in \mathbb{N}$ we have $||\widetilde{u}_i||_{C^j(Q((0,0),3))} \leq C(j,B)$. By the choice of scaling parameters we have that $\left|\widetilde{\Upsilon}\right|^2_{\widetilde{g}} \leq CB$. This corresponds to a uniform $C^3$ estimate for $\widetilde{u}_i$, which after the application of Schauder estimates implies uniform $C^j$ bounds for all $j$. Given this, using standard compactness theorems we obtain a subsequence of $(\widetilde{u}_i, \widetilde{\omega}_i, \widetilde{h}_i)$ converging to a limit $(u_\infty, \omega_\infty, h_\infty)$ on $Q((0,0),2)$. As the background metrics $\omega$, $h$ were uniformly controlled before the blowup it follows that $\omega_\infty, h_\infty$ are flat, and moreover uniformly equivalent to the standard flat metric with a bound depending only on the background data $\omega, h$. By a rescaling in space and time and adding a function of time only, we can assume that $u_\infty$ is a solution to (1.4), which moreover by construction satisfies

$$u_\infty \in \mathcal{E}^{k,l}_{\lambda,\Lambda,Q((0,0),2)}, \qquad \left|\nabla^3 u_\infty\right|(0,0) = c(\lambda,\Lambda,\omega,h)B^{-1}.$$

However, as a solution to (1.4), by Theorem 1.1 and Schauder estimates there is an a priori interior $C^3$ estimate for $u_\infty$. For $B$ chosen sufficiently small this is a contradiction, finishing the proof. $\square$

## 6. Appendix: Evolution Equations

In this section we prove the crucial subsolution properties for the matrix $W$ along the real and complex twisted Monge-Ampère equations. The results are contained in Lemmas 6.3 and 6.1. We directly prove the case of complex variables first, which consists of lengthy calculations and applications of the Cauchy-Schwarz inequality. Again we note that these monotonicity properties are suggested by the discussion of Legendre transformations in §2. A similar direct calculation can yield the case of real variables, but we suppress this as it is lengthy and nearly identical to the complex case. Instead we show that the real case follows by formally extending variables and appealing to the complex setting.

**Lemma 6.1.** *Let $u_t$ be a solution to (1.4) such that $u_t \in \mathcal{E}^{k,l}_U$ for all $t$. Then*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\frac{\partial u}{\partial t} = 0. \tag{6.1}$$

*Also,*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) W = Q,$$

*where*

(6.2)

$$\begin{aligned}
Q_{\alpha_z\overline{\alpha}_z} = & - u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p\overline{z}_q\alpha_z} u_{z_r\overline{z}_s\overline{\alpha}_z} + u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p\overline{z}_q\alpha_z} u_{z_r\overline{z}_s\overline{w}_k} u^{\overline{w}_k w_l} u_{w_l\overline{\alpha}_z} \\
& - u_{\alpha_z\overline{w}_k} u^{\overline{w}_k w_p} u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p\overline{z}_q w_p} u_{z_r\overline{z}_s\overline{w}_q} u^{\overline{w}_q w_l} u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k} u^{\overline{w}_k w_l} u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p\overline{z}_q w_l} u_{z_r\overline{z}_s\overline{\alpha}_z} \\
& + u^{\overline{z}_b z_a} \left[ -u_{\alpha_z\overline{w}_k z_a} u^{\overline{w}_k w_p} u_{w_p\overline{w}_q\overline{z}_b} u^{\overline{w}_q w_l} u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k z_a} u^{\overline{w}_k w_l} u_{w_l\overline{\alpha}_z\overline{z}_b} \right. \\
& - u_{\alpha_z\overline{w}_k\overline{z}_b} u^{\overline{w}_k w_p} u_{w_p\overline{w}_q z_a} u^{\overline{w}_q w_l} u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k} u^{\overline{w}_k w_r} u_{w_r\overline{w}_s\overline{z}_b} u^{\overline{w}_s w_p} u_{w_p\overline{w}_q z_a} u^{\overline{w}_q w_l} u_{w_l\overline{\alpha}_z} \\
& + u_{\alpha_z\overline{w}_k} u^{\overline{w}_k w_p} u_{w_p\overline{w}_q z_a} u^{\overline{w}_q w_r} u_{w_r\overline{w}_s\overline{z}_b} u^{\overline{w}_s w_l} u_{w_l\overline{\alpha}_z} - u_{\alpha_z\overline{w}_k} u^{\overline{w}_k w_p} u_{w_p\overline{w}_q z_a} u^{\overline{w}_q w_l} u_{w_l\overline{\alpha}_z\overline{z}_b} \\
& \left. + u_{\alpha_z\overline{w}_k\overline{z}_b} u^{\overline{w}_k w_l} u_{w_l\overline{\alpha}_z z_a} - u_{\alpha_z\overline{w}_k} u^{\overline{w}_k w_r} u_{w_r\overline{w}_s\overline{z}_b} u^{\overline{w}_s w_l} u_{w_l\overline{\alpha}_z z_a} \right] \\
& - u^{\overline{w}_b w_a} \left[ -u_{\alpha_z\overline{w}_k\overline{w}_b} u^{\overline{w}_k w_p} u_{w_p\overline{w}_q w_a} u^{\overline{w}_q w_l} u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k} u^{\overline{w}_k w_r} u_{w_r\overline{w}_s\overline{w}_b} u^{\overline{w}_s w_p} u_{w_p\overline{w}_q w_a} u^{\overline{w}_q w_l} u_{w_l\overline{\alpha}_z} \right. \\
& \left. + u_{\alpha_z\overline{w}_k\overline{w}_b} u^{\overline{w}_k w_l} u_{w_l\overline{\alpha}_z w_a} - u_{\alpha_z\overline{w}_k} u^{\overline{w}_k w_r} u_{w_r\overline{w}_s\overline{w}_b} u^{\overline{w}_s w_l} u_{w_l\overline{\alpha}_z w_a} \right]
\end{aligned}$$

(6.3)

$$\begin{aligned}
Q_{\alpha_w\overline{\alpha}_w} = & - u^{\overline{\alpha}_w w_k} u^{\overline{w}_l \alpha_w} u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p\overline{z}_q w_k} u_{z_r\overline{z}_s\overline{w}_l} + u^{\overline{z}_l z_k} u^{\overline{\alpha}_w w_p} u_{w_p\overline{w}_q\overline{z}_l} u^{\overline{w}_q w_j} u_{w_j\overline{w}_k z_k} u^{\overline{w}_k \alpha_w} \\
& + u^{\overline{z}_l z_k} u^{\overline{\alpha}_w w_j} u_{w_j\overline{w}_k z_k} u^{\overline{w}_k w_p} u_{w_p\overline{w}_q\overline{z}_l} u^{\overline{w}_q \alpha_w} - u^{\overline{w}_l w_k} u^{\overline{\alpha}_w w_p} u_{w_p\overline{w}_q\overline{w}_l} u^{\overline{w}_q w_j} u_{w_j\overline{w}_r w_k} u^{\overline{w}_r \alpha_w},
\end{aligned}$$

(6.4)
$$\begin{aligned}Q_{\alpha_z\alpha_w} = & -u^{\overline{z}_b z_a}u^{\overline{z}_d z_c}u_{z_a\overline{z}_d\alpha_z}u_{z_c\overline{z}_b\overline{w}_k}u^{\overline{w}_k\alpha_w} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_l}u^{\overline{w}_p\alpha_w}u^{\overline{z}_b z_a}u^{\overline{z}_d z_c}u_{z_a\overline{z}_d w_l}u_{z_c\overline{z}_b\overline{w}_p}\\ & - u^{\overline{z}_b z_a}\left[-u_{\alpha_z\overline{w}_k z_a}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\overline{z}_b}u^{\overline{w}_q\alpha_w} - u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q z_a}u^{\overline{w}_q\alpha_w}\right.\\ & \left.+u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_s w_p}u_{w_p\overline{w}_q z_a}u^{\overline{w}_q\alpha_w} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q z_a}u^{\overline{w}_q w_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_s\alpha_w}\right]\\ & + u^{\overline{w}_b w_a}\left[-u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q w_a}u^{\overline{w}_q\alpha_w} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_s w_p}u_{w_p\overline{w}_q w_a}u^{\overline{w}_q\alpha_w}\right],\end{aligned}$$

$$Q_{\overline{\alpha}_z\overline{\alpha}_w} = \overline{Q}_{\alpha_z\alpha_w} \tag{6.5}$$

*Proof.* First we prove (6.1).

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\log\det u_{\alpha_z\overline{\alpha}_z} - \log\det(-u_{\alpha_w\overline{\alpha}_w})\right)\\ &= u^{\overline{z}_b z_a}\left(\frac{\partial u}{\partial t}\right)_{z_a\overline{z}_b} - u^{\overline{w}_b w_a}\left(\frac{\partial u}{\partial t}\right)_{w_a\overline{w}_b}\\ &= \mathcal{L}\frac{\partial u}{\partial t}.\end{aligned}$$

Next we establish (6.3). We start by computing partial derivatives

$$\begin{aligned}(\log\det u_{z\overline{z}})_{,\alpha\beta} = & \left(u^{\overline{z}_q z_p}u_{z_p\overline{z}_q\alpha}\right)_{,\beta} = u^{\overline{z}_q z_p}u_{z_p\overline{z}_q\alpha\beta} - u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q\alpha}u_{z_r\overline{z}_s\beta},\\ (\log\det(-u_{yy}))_{,\alpha\beta} = & \left(u^{\overline{w}_q w_p}u_{w_p\overline{w}_q\alpha}\right)_{,\beta} = u^{\overline{w}_q w_p}u_{w_p\overline{w}_q\alpha\beta} - u^{\overline{w}_q w_r}u^{\overline{w}_s w_p}u_{w_p\overline{w}_q\alpha}u_{w_r\overline{w}_s\beta},\end{aligned}\tag{6.6}$$

Using this we compute

$$\begin{aligned}\frac{\partial}{\partial t}u^{\overline{\alpha}_w\alpha_w} = & -u^{\overline{\alpha}_w w_k}\left(\frac{\partial}{\partial t}u\right)_{w_k\overline{w}_l}u^{\overline{w}_l\alpha_w}\\ = & -u^{\overline{\alpha}_w w_k}\left(\log\det u_{z\overline{z}} - \log\det(-u_{w\overline{w}})\right)_{w_k\overline{w}_l}u^{\overline{w}_l\alpha_w}\\ = & \, u^{\overline{\alpha}_w w_k}u^{\overline{w}_l\alpha_w}\left(u^{\overline{w}_q w_p}u_{w_p\overline{w}_q w_k\overline{w}_l} - u^{\overline{w}_q w_r}u^{\overline{w}_s w_p}u_{w_p\overline{w}_q w_k}u_{w_r\overline{w}_s\overline{w}_l}\right.\\ & \left.-u^{\overline{z}_q z_p}u_{z_p\overline{z}_q w_k\overline{w}_l} + u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q w_k}u_{z_r\overline{z}_s\overline{w}_l}\right).\end{aligned}\tag{6.7}$$

Also we compute the partial derivatives

$$\begin{aligned}u^{\overline{\alpha}_w\alpha_w}_{\alpha\beta} = & -\left(u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_k\alpha}u^{\overline{w}_k\alpha_w}\right)_{,\beta}\\ = & \, u^{\overline{\alpha}_w w_p}u_{w_p\overline{w}_q\beta}u^{\overline{w}_q w_j}u_{w_j\overline{w}_k\alpha}u^{\overline{w}_k\alpha_w} - u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_k\alpha\beta}u^{\overline{w}_k\alpha_w}\\ & + u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_k\alpha}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\beta}u^{\overline{w}_q\alpha_w}.\end{aligned}\tag{6.8}$$

Thus we have

$$\begin{aligned}&\mathcal{L}(u^{\overline{\alpha}_w\alpha_w})\\ &= u^{\overline{z}_l z_k}(u^{\overline{\alpha}_w\alpha_w})_{,z_k\overline{z}_l} - u^{\overline{w}_k w_l}(u^{\overline{\alpha}_w\alpha_w})_{,w_k\overline{w}_l}\\ &= u^{\overline{z}_l z_k}\left(u^{\overline{\alpha}_w w_p}u_{w_p\overline{w}_q\overline{z}_l}u^{\overline{w}_q w_j}u_{w_j\overline{w}_k z_k}u^{\overline{w}_k\alpha_w} -\right.\\ &\qquad \left.u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_k z_k\overline{z}_l}u^{\overline{w}_k\alpha_w} + u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_k z_k}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\overline{z}_l}u^{\overline{w}_q\alpha_w}\right)\\ &\quad - u^{\overline{w}_l w_k}\left(u^{\overline{\alpha}_w w_p}u_{w_p\overline{w}_q\overline{w}_l}u^{\overline{w}_q w_j}u_{w_j\overline{w}_r w_k}u^{\overline{w}_r\alpha_w} -\right.\\ &\qquad \left.u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_r w_k\overline{w}_l}u^{\overline{w}_r\alpha_w} + u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_r w_k}u^{\overline{w}_r w_p}u_{w_p\overline{w}_q\overline{w}_l}u^{\overline{w}_q\alpha_w}\right).\end{aligned}\tag{6.9}$$

Putting together (6.7) and (6.9) yields

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\mathcal{L}\right)W_{\alpha_w\overline{\alpha}_w}\\
&\qquad = -u^{\overline{\alpha}_w w_k}u^{\overline{w}_l\alpha_w}\left(-u^{\overline{w}_q w_r}u^{\overline{w}_s w_p}u_{w_p\overline{w}_q w_k}u_{w_r\overline{w}_s\overline{w}_l}+u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q w_k}u_{z_r\overline{z}_s\overline{w}_l}\right)\\
&\qquad\quad + u^{\overline{z}_l z_k}\left(u^{\overline{\alpha}_w w_p}u_{w_p\overline{w}_q\overline{z}_l}u^{\overline{w}_q w_j}u_{w_j\overline{w}_k z_k}u^{\overline{w}_k\alpha_w}+u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_k z_k}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\overline{z}_l}u^{\overline{w}_q\alpha_w}\right)\\
&\qquad\quad - u^{\overline{w}_l w_k}\left(u^{\overline{\alpha}_w w_p}u_{w_p\overline{w}_q\overline{w}_l}u^{\overline{w}_q w_j}u_{w_j\overline{w}_r w_k}u^{\overline{w}_r\alpha_w}+u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_r w_k}u^{\overline{w}_r w_p}u_{w_p\overline{w}_q\overline{w}_l}u^{\overline{w}_q\alpha_w}\right)\\
&\qquad = -u^{\overline{\alpha}_w w_k}u^{\overline{w}_l\alpha_w}u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q w_k}u_{z_r\overline{z}_s\overline{w}_l}+u^{\overline{z}_l z_k}u^{\overline{\alpha}_w w_p}u_{w_p\overline{w}_q\overline{z}_l}u^{\overline{w}_q w_j}u_{w_j\overline{w}_k z_k}u^{\overline{w}_k\alpha_w}\\
&\qquad\quad + u^{\overline{z}_l z_k}u^{\overline{\alpha}_w w_j}u_{w_j\overline{w}_k z_k}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\overline{z}_l}u^{\overline{w}_q\alpha_w}-u^{\overline{w}_l w_k}u^{\overline{\alpha}_w w_p}u_{w_p\overline{w}_q\overline{w}_l}u^{\overline{w}_q w_j}u_{w_j\overline{w}_r w_k}u^{\overline{w}_r\alpha_w},
\end{aligned}
$$

finishing the proof of (6.3). Next we establish (6.2). First we compute using (6.6)

$$
\begin{aligned}
\frac{\partial}{\partial t}u_{\alpha_z\overline{\alpha}_z} &= \left(\log\det u_{\alpha_z\overline{\alpha}_z}-\log\det(-u_{\alpha_w\overline{\alpha}_w})\right)_{\alpha_z\overline{\alpha}_z}\\
&= u^{\overline{z}_q z_p}u_{z_p\overline{z}_q\alpha_z\overline{\alpha}_z}-u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q\alpha_z}u_{z_r\overline{z}_s\overline{\alpha}_z}-u^{\overline{w}_q w_p}u_{w_p\overline{w}_q\alpha_z\overline{\alpha}_z}+u^{\overline{w}_q w_r}u^{\overline{w}_s w_p}u_{w_p\overline{w}_q\alpha_z}u_{w_r\overline{w}_s\overline{\alpha}_z}.
\end{aligned}
$$

Also

$$
\mathcal{L}u_{\alpha_z\overline{\alpha}_z}=u^{\overline{z}_q z_p}u_{\alpha_z\overline{\alpha}_z z_p\overline{z}_q}-u^{\overline{w}_q w_p}u_{\alpha_z\overline{\alpha}_z w_p\overline{w}_q}.
$$

Thus

$$
\left(\frac{\partial}{\partial t}-\mathcal{L}\right)u_{\alpha_z\overline{\alpha}_z} = -u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q\alpha_z}u_{z_r\overline{z}_s\overline{\alpha}_z}+u^{\overline{w}_q w_r}u^{\overline{w}_s w_p}u_{w_p\overline{w}_q\alpha_z}u_{w_r\overline{w}_s\overline{\alpha}_z}. \tag{6.10}
$$

To compute the next term we first differentiate using (6.6)

(6.11)

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z}\right)\\
&= \left(\frac{\partial}{\partial t}u\right)_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z}-u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_p}\left(\frac{\partial}{\partial t}u\right)_{w_p\overline{w}_q}u^{\overline{w}_q w_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_l}\left(\frac{\partial}{\partial t}u_{w_l\overline{\alpha}_z}\right)\\
&= \left(u^{\overline{z}_q z_p}u_{z_p\overline{z}_q\alpha_z\overline{w}_k}-u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q\alpha_z}u_{z_r\overline{z}_s\overline{w}_k}-u^{\overline{w}_b w_a}u_{w_a\overline{w}_b\alpha_z\overline{w}_k}+u^{\overline{w}_d w_c}u^{\overline{w}_b w_a}u_{w_a\overline{w}_d\alpha_z}u_{w_c\overline{w}_b\overline{w}_k}\right)u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z}\\
&\quad - u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_p}\left(u^{\overline{z}_q z_p}u_{z_p\overline{z}_q w_p\overline{w}_q}-u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q w_p}u_{z_r\overline{z}_s\overline{w}_q}\right.\\
&\qquad\qquad \left.-u^{\overline{w}_b w_a}u_{w_a\overline{w}_b w_p\overline{w}_q}+u^{\overline{w}_d w_c}u^{\overline{w}_b w_a}u_{w_a\overline{w}_d w_p}u_{w_c\overline{w}_b\overline{w}_q}\right)u^{\overline{w}_q w_l}u_{w_l\overline{\alpha}_z}\\
&\quad + u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_l}\left(u^{\overline{z}_q z_p}u_{z_p\overline{z}_q w_l\overline{\alpha}_z}-u^{\overline{z}_q z_r}u^{\overline{z}_s z_p}u_{z_p\overline{z}_q w_l}u_{z_r\overline{z}_s\overline{\alpha}_z}-u^{\overline{w}_b w_a}u_{w_a\overline{w}_b w_l\overline{\alpha}_z}+u^{\overline{w}_d w_c}u^{\overline{w}_b w_a}u_{w_a\overline{w}_d w_l}u_{w_c\overline{w}_b\overline{\alpha}_z}\right).
\end{aligned}
$$

Next we compute the partial derivatives

$$
\begin{aligned}
\left(u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z}\right)_{,\alpha\beta} &= \left(u_{\alpha_z\overline{w}_k\alpha}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z}-u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\alpha}u^{\overline{w}_q w_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z\alpha}\right)_{,\beta}\\
&= u_{\alpha_z\overline{w}_k\alpha\beta}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z}-u_{\alpha_z\overline{w}_k\alpha}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\beta}u^{\overline{w}_q w_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_k\alpha}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z\beta}\\
&\quad - u_{\alpha_z\overline{w}_k\beta}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\alpha}u^{\overline{w}_q w_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_r}u_{w_r\overline{w}_s\beta}u^{\overline{w}_s w_p}u_{w_p\overline{w}_q\alpha}u^{\overline{w}_q w_l}u_{w_l\overline{\alpha}_z}\\
&\quad - u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\alpha\beta}u^{\overline{w}_q w_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\alpha}u^{\overline{w}_q w_r}u_{w_r\overline{w}_s\beta}u^{\overline{w}_s w_l}u_{w_l\overline{\alpha}_z}\\
&\quad - u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_p}u_{w_p\overline{w}_q\alpha}u^{\overline{w}_q w_l}u_{w_l\overline{\alpha}_z\beta}\\
&\quad + u_{\alpha_z\overline{w}_k\beta}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z\alpha}-u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_r}u_{w_r\overline{w}_s\beta}u^{\overline{w}_s w_l}u_{w_l\overline{\alpha}_z\alpha}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_k w_l}u_{w_l\overline{\alpha}_z\alpha\beta}.
\end{aligned}
$$

Thus we have

(6.12)

$$
\begin{aligned}
&\mathcal{L}(u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z})\\
&= u^{\overline{z}_bz_a}\left(u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z}\right)_{,z_a\overline{z}_b} - u^{\overline{w}_bw_a}\left(u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z}\right)_{,w_a\overline{w}_b}\\
&= u^{\overline{z}_bz_a}\big[u_{\alpha_z\overline{w}_kz_a\overline{z}_b}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z} - u_{\alpha_z\overline{w}_kz_a}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{z}_b}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_kz_a}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z\overline{z}_b}\\
&\quad - u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}\\
&\quad - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a\overline{z}_b}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_z}\\
&\quad - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z\overline{z}_b}\\
&\quad +u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_zz_a} - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_zz_a} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_zz_a\overline{z}_b}\big]\\
&\quad - u^{\overline{w}_bw_a}\big[u_{\alpha_z\overline{w}_kw_a\overline{w}_b}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z} - u_{\alpha_z\overline{w}_kw_a}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{w}_b}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_kw_a}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z\overline{w}_b}\\
&\quad - u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}\\
&\quad - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a\overline{w}_b}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_z}\\
&\quad - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z\overline{w}_b}\\
&\quad +u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_zw_a} - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_zw_a} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_zw_a\overline{w}_b}\big].
\end{aligned}
$$

Putting together (6.10), (6.11) and (6.12) yields

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \mathcal{L}\right)W_{\alpha_z\overline{\alpha}_z}\\
&\qquad = -u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_q\alpha_z}u_{z_r\overline{z}_s\overline{\alpha}_z} + u^{\overline{w}_qw_r}u^{\overline{w}_sw_p}u_{w_p\overline{w}_q\alpha_z}u_{w_r\overline{w}_s\overline{\alpha}_z}\\
&\qquad\quad - \left(-u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_q\alpha_z}u_{z_r\overline{z}_s\overline{w}_k} + u^{\overline{w}_dw_c}u^{\overline{w}_bw_a}u_{w_a\overline{w}_d\alpha_z}u_{w_c\overline{w}_b\overline{w}_k}\right)u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z}\\
&\qquad\quad + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}\left(-u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_qw_p}u_{z_r\overline{z}_s\overline{w}_q} + u^{\overline{w}_dw_c}u^{\overline{w}_bw_a}u_{w_a\overline{w}_dw_p}u_{w_c\overline{w}_b\overline{w}_q}\right)u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}\\
&\qquad\quad - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}\left(-u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_qw_l}u_{z_r\overline{z}_s\overline{\alpha}_z} + u^{\overline{w}_dw_c}u^{\overline{w}_bw_a}u_{w_a\overline{w}_dw_l}u_{w_c\overline{w}_b\overline{\alpha}_z}\right)\\
&\qquad\quad + u^{\overline{z}_bz_a}\big[-u_{\alpha_z\overline{w}_kz_a}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{z}_b}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_kz_a}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z\overline{z}_b}\\
&\qquad\quad - u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}\\
&\qquad\quad + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_z} - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z\overline{z}_b}\\
&\qquad\quad +u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_zz_a} - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_zz_a}\big]\\
&\qquad\quad - u^{\overline{w}_bw_a}\big[-u_{\alpha_z\overline{w}_kw_a}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{w}_b}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_kw_a}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z\overline{w}_b}\\
&\qquad\quad - u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z} + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}\\
&\qquad\quad + u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_z} - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z\overline{w}_b}\\
&\qquad\quad +u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_zw_a} - u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_zw_a}\big]\\
&\qquad =: \sum_{i=1}^{24} A_i.
\end{aligned}
$$

We observe that $A_2 + A_{18} = A_4 + A_{17} = A_6 + A_{21} = A_8 + A_{22} = 0$, and hence

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \mathcal{L}\right) W_{\alpha_z \overline{\alpha}_z} \\
&\quad = - u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p \overline{z}_q \alpha_z} u_{z_r \overline{z}_s \overline{\alpha}_z} + u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p \overline{z}_q \alpha_z} u_{z_r \overline{z}_s \overline{w}_k} u^{\overline{w}_k w_l} u_{w_l \overline{\alpha}_z} \\
&\qquad - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p \overline{z}_q w_p} u_{z_r \overline{z}_s \overline{w}_q} u^{\overline{w}_q w_l} u_{w_l \overline{\alpha}_z} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_l} u^{\overline{z}_q z_r} u^{\overline{z}_s z_p} u_{z_p \overline{z}_q w_l} u_{z_r \overline{z}_s \overline{\alpha}_z} \\
&\qquad + u^{\overline{z}_b z_a} \left[ - u_{\alpha_z \overline{w}_k z_a} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q \overline{z}_b} u^{\overline{w}_q w_l} u_{w_l \overline{\alpha}_z} + u_{\alpha_z \overline{w}_k z_a} u^{\overline{w}_k w_l} u_{w_l \overline{\alpha}_z \overline{z}_b} \right. \\
&\qquad - u_{\alpha_z \overline{w}_k \overline{z}_b} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q z_a} u^{\overline{w}_q w_l} u_{w_l \overline{\alpha}_z} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_r} u_{w_r \overline{w}_s \overline{z}_b} u^{\overline{w}_s w_p} u_{w_p \overline{w}_q z_a} u^{\overline{w}_q w_l} u_{w_l \overline{\alpha}_z} \\
&\qquad + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q z_a} u^{\overline{w}_q w_r} u_{w_r \overline{w}_s \overline{z}_b} u^{\overline{w}_s w_l} u_{w_l \overline{\alpha}_z} - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q z_a} u^{\overline{w}_q w_l} u_{w_l \overline{\alpha}_z \overline{z}_b} \\
&\qquad \left. + u_{\alpha_z \overline{w}_k \overline{z}_b} u^{\overline{w}_k w_l} u_{w_l \overline{\alpha}_z z_a} - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_r} u_{w_r \overline{w}_s \overline{z}_b} u^{\overline{w}_s w_l} u_{w_l \overline{\alpha}_z z_a} \right] \\
&\qquad - u^{\overline{w}_b w_a} \left[ - u_{\alpha_z \overline{w}_k \overline{w}_b} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q w_a} u^{\overline{w}_q w_l} u_{w_l \overline{\alpha}_z} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_r} u_{w_r \overline{w}_s \overline{w}_b} u^{\overline{w}_s w_p} u_{w_p \overline{w}_q w_a} u^{\overline{w}_q w_l} u_{w_l \overline{\alpha}_z} \right. \\
&\qquad \left. + u_{\alpha_z \overline{w}_k \overline{w}_b} u^{\overline{w}_k w_l} u_{w_l \overline{\alpha}_z w_a} - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_r} u_{w_r \overline{w}_s \overline{w}_b} u^{\overline{w}_s w_l} u_{w_l \overline{\alpha}_z w_a} \right],
\end{aligned}
$$

completing the proof of (6.2). Next we establish (6.4). Using (6.6) we compute

$$
\begin{aligned}
&\frac{\partial}{\partial t} u_{\alpha_z \overline{w}_k} u^{\overline{w}_k \alpha_w} \\
&= \left(\frac{\partial}{\partial t} u\right)_{\alpha_z \overline{w}_k} u^{\overline{w}_k \alpha_w} - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_l} \left(\frac{\partial}{\partial t} u\right)_{w_l \overline{w}_p} u^{\overline{w}_p \alpha_w} \\
&= \left( u^{\overline{z}_b z_a} u_{z_a \overline{z}_b \alpha_z \overline{w}_k} - u^{\overline{w}_b w_a} u_{w_a \overline{w}_b \alpha_z \overline{w}_k} \right. \\
&\quad \left. - u^{\overline{z}_b z_a} u^{\overline{z}_d z_c} u_{z_a \overline{z}_d \alpha_z} u_{z_c \overline{z}_b \overline{w}_k} + u^{\overline{w}_b w_a} u^{\overline{w}_d w_c} u_{w_a \overline{w}_d \alpha_z} u_{w_c \overline{w}_b \overline{w}_k} \right) u^{\overline{w}_k \alpha_w} \\
&\quad - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_l} u^{\overline{w}_p \alpha_w} \left( u^{\overline{z}_b z_a} u_{z_a \overline{z}_b w_l \overline{w}_p} - u^{\overline{w}_b w_a} u_{w_a \overline{w}_b w_l \overline{w}_p} \right. \\
&\quad \left. - u^{\overline{z}_b z_a} u^{\overline{z}_d z_c} u_{z_a \overline{z}_d w_l} u_{z_c \overline{z}_b \overline{w}_p} + u^{\overline{w}_b w_a} u^{\overline{w}_d w_c} u_{w_a \overline{w}_d w_l} u_{w_c \overline{w}_b \overline{w}_p} \right).
\end{aligned} \tag{6.13}
$$

Next we compute partial derivatives

$$
\begin{aligned}
\left( u_{\alpha_z \overline{w}_k} u^{\overline{w}_k \alpha_w} \right)_{,\mu\rho} &= \left( u_{\alpha_z \overline{w}_k \mu} u^{\overline{w}_k \alpha_w} - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q \mu} u^{\overline{w}_q \alpha_w} \right)_{,\rho} \\
&= u_{\alpha_z \overline{w}_k \mu\rho} u^{\overline{w}_k \alpha_w} - u_{\alpha_z \overline{w}_k \mu} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q \rho} u^{\overline{w}_q \alpha_w} \\
&\quad - u_{\alpha_z \overline{w}_k \rho} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q \mu} u^{\overline{w}_q \alpha_w} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_r} u_{w_r \overline{w}_s \rho} u^{\overline{w}_s w_p} u_{w_p \overline{w}_q \mu} u^{\overline{w}_q \alpha_w} \\
&\quad - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q \mu\rho} u^{\overline{w}_q \alpha_w} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q \mu} u^{\overline{w}_q w_r} u_{w_r \overline{w}_s \rho} u^{\overline{w}_s \alpha_w}.
\end{aligned} \tag{6.14}
$$

Using this we compute

$$
\begin{aligned}
&\mathcal{L} \left( u_{\alpha_z \overline{w}_k} u^{\overline{w}_k \alpha_w} \right) \\
&= u^{\overline{z}_b z_a} \left( u_{\alpha_z \overline{w}_k} u^{\overline{w}_k \alpha_w} \right)_{z_a \overline{z}_b} - u^{\overline{w}_b w_a} \left( u_{\alpha_z \overline{w}_k} u^{\overline{w}_k \alpha_w} \right)_{w_a \overline{w}_b} \\
&= u^{\overline{z}_b z_a} \left[ u_{\alpha_z \overline{w}_k z_a \overline{z}_b} u^{\overline{w}_k \alpha_w} - u_{\alpha_z \overline{w}_k z_a} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q \overline{z}_b} u^{\overline{w}_q \alpha_w} \right. \\
&\quad - u_{\alpha_z \overline{w}_k \overline{z}_b} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q z_a} u^{\overline{w}_q \alpha_w} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_r} u_{w_r \overline{w}_s \overline{z}_b} u^{\overline{w}_s w_p} u_{w_p \overline{w}_q z_a} u^{\overline{w}_q \alpha_w} \\
&\quad \left. - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q z_a \overline{z}_b} u^{\overline{w}_q \alpha_w} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q z_a} u^{\overline{w}_q w_r} u_{w_r \overline{w}_s \overline{z}_b} u^{\overline{w}_s \alpha_w} \right] \\
&\quad - u^{\overline{w}_b w_a} \left[ u_{\alpha_z \overline{w}_k w_a \overline{w}_b} u^{\overline{w}_k \alpha_w} - u_{\alpha_z \overline{w}_k w_a} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q \overline{w}_b} u^{\overline{w}_q \alpha_w} \right. \\
&\quad - u_{\alpha_z \overline{w}_k \overline{w}_b} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q w_a} u^{\overline{w}_q \alpha_w} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_r} u_{w_r \overline{w}_s \overline{w}_b} u^{\overline{w}_s w_p} u_{w_p \overline{w}_q w_a} u^{\overline{w}_q \alpha_w} \\
&\quad \left. - u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q w_a \overline{w}_b} u^{\overline{w}_q \alpha_w} + u_{\alpha_z \overline{w}_k} u^{\overline{w}_k w_p} u_{w_p \overline{w}_q w_a} u^{\overline{w}_q w_r} u_{w_r \overline{w}_s \overline{w}_b} u^{\overline{w}_s \alpha_w} \right].
\end{aligned} \tag{6.15}
$$

Combining (6.13) and (6.15) yields

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\mathcal{L}\right)W_{\alpha_z\alpha_w}\\
=\ &\left(-u^{\overline{z}_bz_a}u^{\overline{z}_dz_c}u_{z_a\overline{z}_d\alpha_z}u_{z_c\overline{z}_b\overline{w}_k}+u^{\overline{w}_bw_a}u^{\overline{w}_dw_c}u_{w_a\overline{w}_d\alpha_z}u_{w_c\overline{w}_b\overline{w}_k}\right)u^{\overline{w}_k\alpha_w}\\
&-u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}u^{\overline{w}_p\alpha_w}\left(-u^{\overline{z}_bz_a}u^{\overline{z}_dz_c}u_{z_a\overline{z}_dw_l}u_{z_c\overline{z}_b\overline{w}_p}+u^{\overline{w}_bw_a}u^{\overline{w}_dw_c}u_{w_a\overline{w}_dw_l}u_{w_c\overline{w}_b\overline{w}_p}\right)\\
&-u^{\overline{z}_bz_a}\Big[-u_{\alpha_z\overline{w}_kz_a}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{z}_b}u^{\overline{w}_q\alpha_w}-u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_q\alpha_w}\\
&+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_q\alpha_w}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_s\alpha_w}\Big]\\
&+u^{\overline{w}_bw_a}\Big[-u_{\alpha_z\overline{w}_kw_a}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{w}_b}u^{\overline{w}_q\alpha_w}-u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_q\alpha_w}\\
&+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_q\alpha_w}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_s\alpha_w}\Big]\\
=:\ &\sum_{i=1}^{12}A_i.
\end{aligned}
$$

Observing that $W$ is Hermitian, and that the operator $\frac{\partial}{\partial t}-\mathcal{L}$ is Hermitian we obtain (6.5). □

**Lemma 6.2.** *With the setup above,*

$$Q\leq 0.$$

*Proof.* Using Lemma 6.1 we compute

$$
\begin{aligned}
Q(\alpha,\overline{\alpha})&=Q_{\alpha_z\overline{\alpha}_z}+Q_{\alpha_z\overline{\alpha}_w}+Q_{\alpha_w\overline{\alpha}_z}+Q_{\alpha_w\overline{\alpha}_w}\\
&=-u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_q\alpha_z}u_{z_r\overline{z}_s\overline{\alpha}_z}+u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_q\alpha_z}u_{z_r\overline{z}_s\overline{w}_k}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z}\\
&\quad-u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_qw_p}u_{z_r\overline{z}_s\overline{w}_q}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_qw_l}u_{z_r\overline{z}_s\overline{\alpha}_z}\\
&\quad+u^{\overline{z}_bz_a}\Big[-u_{\alpha_z\overline{w}_kz_a}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{z}_b}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_kz_a}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_z\overline{z}_b}\\
&\quad-u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}\\
&\quad+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_z}-u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z\overline{z}_b}\\
&\quad+u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_zz_a}-u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_zz_a}\Big]\\
&\quad-u^{\overline{w}_bw_a}\Big[-u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_l}u_{w_l\overline{\alpha}_z}\\
&\quad+u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_kw_l}u_{w_l\overline{\alpha}_zw_a}-u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_l}u_{w_l\overline{\alpha}_zw_a}\Big]\\
&\quad-u^{\overline{\alpha}_ww_k}u^{\overline{w}_l\alpha_w}u^{\overline{z}_qz_r}u^{\overline{z}_sz_p}u_{z_p\overline{z}_qw_k}u_{z_r\overline{z}_s\overline{w}_l}+u^{\overline{z}_lz_k}u^{\overline{\alpha}_ww_p}u_{w_p\overline{w}_q\overline{z}_l}u^{\overline{w}_qw_j}u_{w_j\overline{w}_kz_k}u^{\overline{w}_k\alpha_w}\\
&\quad+u^{\overline{z}_lz_k}u^{\overline{\alpha}_ww_j}u_{w_j\overline{w}_kz_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{z}_l}u^{\overline{w}_q\alpha_w}-u^{\overline{w}_lw_k}u^{\overline{\alpha}_ww_p}u_{w_p\overline{w}_q\overline{w}_l}u^{\overline{w}_qw_j}u_{w_j\overline{w}_rw_k}u^{\overline{w}_r\alpha_w}\\
&\quad-u^{\overline{z}_bz_a}u^{\overline{z}_dz_c}u_{z_a\overline{z}_d\alpha_z}u_{z_c\overline{z}_b\overline{w}_k}u^{\overline{w}_k\alpha_w}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_l}u^{\overline{w}_p\alpha_w}u^{\overline{z}_bz_a}u^{\overline{z}_dz_c}u_{z_a\overline{z}_dw_l}u_{z_c\overline{z}_b\overline{w}_p}\\
&\quad-u^{\overline{z}_bz_a}\Big[-u_{\alpha_z\overline{w}_kz_a}u^{\overline{w}_kw_p}u_{w_p\overline{w}_q\overline{z}_b}u^{\overline{w}_q\alpha_w}-u_{\alpha_z\overline{w}_k\overline{z}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_q\alpha_w}\\
&\quad+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_q\alpha_w}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_s\alpha_w}\Big]\\
&\quad+u^{\overline{w}_bw_a}\Big[-u_{\alpha_z\overline{w}_k\overline{w}_b}u^{\overline{w}_kw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_q\alpha_w}+u_{\alpha_z\overline{w}_k}u^{\overline{w}_kw_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_q\alpha_w}\Big]\\
&\quad+u^{\overline{\alpha}_ww_p}u^{\overline{w}_qw_k}u_{w_k\overline{\alpha}_z}u^{\overline{z}_bz_a}u^{\overline{z}_dz_c}u_{z_a\overline{z}_dw_p}u_{z_c\overline{z}_b\overline{w}_q}-u^{\overline{\alpha}_ww_k}u^{\overline{z}_bz_a}u^{\overline{z}_dz_c}u_{z_a\overline{z}_dw_k}u_{z_c\overline{z}_b\overline{\alpha}_z}\\
&\quad-u^{\overline{z}_bz_a}\Big[u^{\overline{\alpha}_ww_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_k}u_{w_k\overline{\alpha}_z}+u^{\overline{\alpha}_ww_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_r}u_{w_r\overline{w}_s\overline{z}_b}u^{\overline{w}_sw_k}u_{w_k\overline{\alpha}_z}\\
&\quad-u^{\overline{\alpha}_ww_p}u_{w_p\overline{w}_qz_a}u^{\overline{w}_qw_k}u_{w_k\overline{\alpha}_z\overline{z}_b}-u^{\overline{\alpha}_ww_p}u_{w_p\overline{w}_q\overline{z}_b}u^{\overline{w}_qw_k}u_{w_k\overline{\alpha}_zz_a}\Big]\\
&\quad+u^{\overline{w}_bw_a}\Big[u^{\overline{\alpha}_ww_r}u_{w_r\overline{w}_s\overline{w}_b}u^{\overline{w}_sw_p}u_{w_p\overline{w}_qw_a}u^{\overline{w}_qw_k}u_{w_k\overline{\alpha}_z}-u^{\overline{\alpha}_ww_p}u_{w_p\overline{w}_q\overline{w}_b}u^{\overline{w}_qw_k}u_{w_k\overline{\alpha}_zw_a}\Big]\\
&=:\sum_{i=1}^{36}A_i.
\end{aligned}
$$

We observe:

$$\begin{aligned} A_{21}+A_{22}+A_{29}+A_{30} =& \Re\left[\, u^{\overline{z}_b z_a} u^{\overline{z}_d z_c}\left(u_{z_c\overline{z}_b\overline{w}_p} u^{\overline{w}_p\alpha_w}\right)\left(u_{z_a\overline{z}_d w_l} u^{\overline{w}_k w_l} u_{\alpha_z\overline{w}_k} - u_{\alpha_z z_a\overline{z}_d}\right)\right] \\ \leq& \; -\left(A_1+A_2+A_3+A_4+A_{17}\right). \end{aligned}$$

using Cauchy-Schwarz. Next

$$\begin{aligned} A_{23}+A_{26}+A_{32}+A_{33} =& \Re\left[\, u^{\overline{z}_b z_a} u^{\overline{w}_k w_p}\left(u_{w_p\overline{w}_q\overline{z}_b} u^{\overline{w}_q\alpha_w}\right)\left(u_{\alpha_z z_a\overline{w}_k} - u_{z_a w_p\overline{w}_k} u^{\overline{w}_q w_p} u_{\alpha_z\overline{w}_q}\right)\right] \\ \leq& \; -\left(A_5+A_6+A_9+A_{10}+A_{18}\right). \end{aligned}$$

Next

$$\begin{aligned} A_{24}+A_{25}+A_{31}+A_{34} =& \Re\left[\, u^{\overline{z}_b z_a} u^{\overline{w}_k w_p}\left(u_{w_p\overline{w}_q z_a} u^{\overline{w}_q\alpha_w}\right)\left(u_{\alpha_z\overline{w}_k\overline{z}_b} - u_{\alpha_z\overline{w}_s} u^{\overline{w}_s w_r} u_{\overline{z}_b w_r\overline{w}_k}\right)\right] \\ \leq& \; -\left(A_7+A_8+A_{11}+A_{12}+A_{19}\right). \end{aligned}$$

Next

$$\begin{aligned} A_{27}+A_{28}+A_{35}+A_{36} =& \Re\left[\, u^{\overline{w}_b w_a} u^{\overline{w}_k w_p}\left(u_{w_p\overline{w}_q w_a} u^{\overline{w}_q\alpha_w}\right)\left(u_{\alpha_z\overline{w}_s} u^{\overline{w}_s w_r} u_{w_r\overline{w}_k\overline{w}_b} - u_{\alpha_z\overline{w}_k\overline{w}_b}\right)\right] \\ \leq& \; -\left(A_{13}+A_{14}+A_{15}+A_{16}+A_{20}\right). \end{aligned}$$

□

**Lemma 6.3.** *Let $u_t$ be a solution to (1.2) such that $u_t \in \mathcal{E}$ for all $t$. Then*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\frac{\partial u}{\partial t} = 0.$$

*Also,*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) W \leq 0.$$

*Proof.* Let $u_t$ be as in the statement. Define $v_t : \mathbb{C}^n \to \mathbb{R}$, by

$$v_t(z_1,\ldots,z_n) = u_t(\operatorname{Re} z_1,\ldots,\operatorname{Re} z_n).$$

Elementary calculations show that $v_t \in \mathcal{E}$ and that $v_t$ is a solution to (1.4). Moreover the matrix $W$ associated to $v_t$ via (3.2) agrees with the matrix $\nabla^2 w$ as in (2.6). The result follows from Lemmas 6.1 and 6.2. □

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