On the category of von Neumann algebras

Andre Kornell

a graduate student under Marc Rieffel at UC Berkeley

October 20, 2012

Definitions

Definition

Let $\mathbf{W}^*$ denote the category of von Neumann algebras and unital normal $*$-homomorphisms.

Definition

Let $^\circ\mathbf{W}^*$ denote the opposite of $\mathbf{W}^*$:

- The objects of $^\circ\mathbf{W}^*$ are von Neumann algebras, denoted $^\circ\mathcal{M}$ instead of $\mathcal{M}$.
- The morphisms of $^\circ\mathbf{W}^*$ from $^\circ\mathcal{M}$ to $^\circ\mathcal{N}$ are unital normal $*$-homomorphisms from $\mathcal{N}$ to $\mathcal{M}$, denoted $^\circ\pi : {^\circ\mathcal{M}} \to {^\circ\mathcal{N}}$ instead of $\pi : \mathcal{N} \to \mathcal{M}$.

Viewpoint

We'll view each object of $^\circ\mathbf{W}^*$ as something like a set, and each morphism of $^\circ\mathbf{W}^*$ as something like a function.

Terminology

If $\mathcal{M}$ is a von Neumann algebra, then $^\circ\mathcal{M}$ is a (quantum) collection.

Terminology

If $\pi : \mathcal{N} \to \mathcal{M}$ is a unital normal $*$-homomorphism, then $^\circ\pi : {^\circ\mathcal{M}} \to {^\circ\mathcal{N}}$ is a (quantum) function.

If $\mathcal{M}$ is the algebra of observables of some quantum system, then $^\circ\mathcal{M}$ is the collection of its configurations.

Sets are collections

Definition

Let **Set** denote the category of sets and functions.

Proposition

There's a full and faithful functor ${}^\circ\ell^\infty : \mathbf{Set} \to {}^\circ\mathbf{W}^*$ that

- takes each set to its von Neumann algebra of bounded functions, and
- takes each function $f : X \to Y$ to the quantum function ${}^\circ\ell^\infty(X) \to {}^\circ\ell^\infty(Y)$ given by precomposition with f.

Thus, **Set** is a subcategory of ${}^\circ\mathbf{W}^*$, and a quantum function between two sets is just a function in the ordinary sense.

Set is bicartesian closed

The category **Set** is bicartesian closed:

- **Set** has an initial object. $(\emptyset)$
- **Set** has a terminal object. $(\{*\})$
- **Set** has all binary coproducts. $(X \sqcup Y)$
- **Set** has all binary products. $(X \times Y)$
- **Set** has all exponentials. $(X^Y = \{f : Y \to X\})$

A category has all exponentials in case for every object Y, the functor $(-) \times Y$ has a right adjoint, denoted $(-)^Y$.

$$Hom(X \times Y, Z) \cong Hom(X, Z^Y)$$

${}^\circ\mathbf{W}^*$ is not bicartesian closed

- ${}^\circ\mathbf{W}^*$ has an initial object. (${}^\circ 0$)
- ${}^\circ\mathbf{W}^*$ has a terminal object. (${}^\circ\mathbb{C}$)
- ${}^\circ\mathbf{W}^*$ has all binary coproducts. (${}^\circ(\mathcal{M} \oplus \mathcal{N})$)
- ${}^\circ\mathbf{W}^*$ has all binary products. (${}^\circ(\mathcal{M} * \mathcal{N})$)

I call $\mathcal{M} * \mathcal{N}$ the *free product* of $\mathcal{M}$ and $\mathcal{N}$ because it's analogous to the free product of C^*-algebras. It's different than Voiculescu's free product. For example, $\mathbb{C}^2 * \mathbb{C}^2$ isn't σ-finite.

Theorem

The category ${}^\circ\mathbf{W}^*$ fails to have all exponentials.

Proof.

$$\mathbb{C}^2 * (\mathbb{C} \oplus \mathbb{C}) \not\cong (\mathbb{C}^2 * \mathbb{C}) \oplus (\mathbb{C}^2 * \mathbb{C})$$

Apply category theory. □

${}^\circ\mathbf{W}^*$ is a closed monoidal category

The usual tensor product construction yields a functor $-\overline{\otimes}- : \mathbf{W}^* \times \mathbf{W}^* \to \mathbf{W}^*$.

- $(\mathbf{W}^*, \overline{\otimes})$ is a symmetric monoidal category.
- $({}^\circ\mathbf{W}^*, \overline{\otimes})$ is a symmetric monoidal category.

Theorem

The symmetric monoidal category $({}^\circ\mathbf{W}^*, \overline{\otimes})$ *is closed:*

For every collection ${}^\circ\mathcal{N}$, the functor ${}^\circ(-\overline{\otimes}\mathcal{N})$ has right adjoint, denoted by ${}^\circ(-^{*\mathcal{N}})$.

$$Hom({}^\circ(\mathcal{R}\overline{\otimes}\mathcal{N}), {}^\circ\mathcal{M}) \cong Hom({}^\circ\mathcal{R}, {}^\circ(\mathcal{M}^{*\mathcal{N}}))$$

The story of $^\circ\mathbf{W}^*$

If $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, and $\pi : \mathcal{M} \to \mathcal{N}$ is a unital normal $*$-homomorphism, then we say:

1. $^\circ\mathcal{M}$ and $^\circ\mathcal{N}$ are collections.
2. $^\circ\pi : {}^\circ\mathcal{N} \to {}^\circ\mathcal{M}$ is a function.
3. $^\circ(\mathcal{M} \oplus \mathcal{N})$ is the disjoint union of $^\circ\mathcal{M}$ and $^\circ\mathcal{N}$.
4. $^\circ(\mathcal{M} \overline{\otimes} \mathcal{N})$ is the Cartesian product of $^\circ\mathcal{M}$ and $^\circ\mathcal{N}$.
5. $^\circ(\mathcal{M}^{*\mathcal{N}})$ is the collection of functions from $^\circ\mathcal{N}$ to $^\circ\mathcal{M}$.
6. $^\circ(\mathcal{M} * \mathcal{N})$ is the collection of functions from $\{0, 1\}$ to the disjoint union of $^\circ\mathcal{M}$ and $^\circ\mathcal{N}$, whose value on 0 is in $^\circ\mathcal{M}$ and whose value on 1 is in $^\circ\mathcal{N}$.

Free exponential von Neumann algebra

I call the von Neumann algebra $\mathcal{M}^{*\mathcal{N}}$ the free exponential of $\mathcal{M}$ and $\mathcal{N}$. It comes with a coevaluation morphism:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varepsilon} & \mathcal{M}^{*\mathcal{N}}\overline{\otimes}\mathcal{N} \\ & \searrow & \big\downarrow {\scriptstyle !\overline{\otimes}1} \\ & & \mathcal{R}\overline{\otimes}\mathcal{N} \end{array}$$

The free exponential is a quantum monoid with counit γ, and cocomposition κ:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varepsilon} & \mathcal{M}^{*\mathcal{M}}\overline{\otimes}\mathcal{M} \\ & \searrow^{1} & \big\downarrow {\scriptstyle \gamma\overline{\otimes}1} \\ & & \mathbb{C}\overline{\otimes}\mathcal{M} \end{array} \qquad \begin{array}{ccc} \mathcal{M} & \xrightarrow{\varepsilon} & \mathcal{M}^{*\mathcal{M}}\overline{\otimes}\mathcal{M} \\ \big\downarrow {\scriptstyle \varepsilon} & & \big\downarrow {\scriptstyle \kappa\overline{\otimes}1} \\ \mathcal{M}^{*\mathcal{M}}\overline{\otimes}\mathcal{M} & \xrightarrow{1\overline{\otimes}\varepsilon} & \mathcal{M}^{*\mathcal{M}}\overline{\otimes}\mathcal{M}^{*\mathcal{M}}\overline{\otimes}\mathcal{M} \end{array}$$

Unital normal completely positive maps

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be von Neumann algebras.

Definition

A linear map $\psi : \mathcal{M} \to \mathcal{N}$ is said to be *completely positive* in case for each square matrix $[m_{ij}] \in M_n(\mathcal{M})$,

$$[m_{ij}] \geq 0 \quad \Longrightarrow \quad [\psi(m_{ij})] \geq 0.$$

Proposition

A function $\psi : \mathcal{M} \to \mathcal{N}$ is a unital normal completely positive linear map iff there is a Hilbert space $\mathcal{L}$ and an isometry $v : \mathcal{K} \to \mathcal{H} \otimes \mathcal{L}$ such that

$$\psi(m) = v^*(m \otimes 1)v.$$

Probabilistic functions

Terminology

If $\psi : \mathcal{M} \to \mathcal{N}$ is a unital normal completely positive map, then ${}^\circ\psi : {}^\circ\mathcal{N} \to {}^\circ\mathcal{M}$ is a probabilistic (quantum) function.

Proposition

If X and Y are sets, then a probabilistic function $X \to Y$ is a function taking the elements of X to probability distributions on Y.

Example

- The space of probabilistic functions $\{*\} \to {}^\circ M_2(\mathbb{C})$ is the state space of $M_2(\mathbb{C})$.
- The set of functions $\{*\} \to {}^\circ M_2(\mathbb{C})$ is empty.

Probabilistic mixtures of functions

- Every probabilistic function $X \to Y$ is a mixture of the set *Hom*(X, Y) of functions from X to Y.
- Not every probabilistic function $^\circ\mathcal{N} \to {}^\circ\mathcal{M}$ is a mixture of the set *Hom*$(^\circ\mathcal{N}, {}^\circ\mathcal{M})$ of functions from $^\circ\mathcal{N}$ to $^\circ\mathcal{M}$.
- Every probabilistic function $^\circ\mathcal{N} \to {}^\circ\mathcal{M}$ is a mixture of the *collection* $^\circ\mathcal{M}^{*\mathcal{N}}$ of functions from $^\circ\mathcal{N}$ to $^\circ\mathcal{M}$.

Theorem

Let $\psi : \mathcal{M} \to \mathcal{N}$ be a unital normal completely positive map. There exists a normal state μ on $\mathcal{M}^{*\mathcal{N}}$ such that

$$\mathcal{M} \xrightarrow{\varepsilon} \mathcal{M}^{*\mathcal{N}} \overline{\otimes} \mathcal{N} \xrightarrow{\mu \overline{\otimes} 1} \mathcal{N}, \qquad \psi = (\mu \overline{\otimes} 1) \circ \varepsilon.$$

Quantum operations

Suppose that $\mathcal{M}_0$ and $\mathcal{M}_1$ are the von Neumann algebras of observables of quantum systems $\mathcal{Q}_0$ and $\mathcal{Q}_1$, respectively.

Definition (Kraus, 1970)

A *quantum operation* from $\mathcal{Q}_0$ to $\mathcal{Q}_1$ is a probabilistic function from ${}^\circ\mathcal{M}_0$ to ${}^\circ\mathcal{M}_1$.

Thus, a quantum operation is a probabilistic function transforming the configurations of $\mathcal{Q}_0$ to configurations of $\mathcal{Q}_1$.
If the quantum operation is a function, i. e., if it is given by a homomorphism, then we might say that it is "deterministic".

Stern-Gerlach Experiment

$$\{*\} \longrightarrow {}^\circ M_2(\mathbb{C}) \longrightarrow \{\uparrow, \downarrow\}$$

References

[1] A. Guichardet, *Sur la Catégorie des Algèbres de Von Neumann*, Bulletin des Sciences Mathématiques **90** (1966), 41-64.

[2] C. Heunen, N. P. Landman, and B. Spitters, *A Topos for Algebraic Quantum Theory*, arXiv:0709.4364 (2007).

[3] A. Kornell, *Quantum Functions*, arXiv:1101.1694 (2011).

[4] K. Kraus, *General State Changes in Quantum Theory*, Annals of Physics **64** (1971), 311-335.

[5] K. G. Schlesinger, *Toward Quantum Mathematics. I. From Quantum Set Theory to Universal Quantum Mechanics*, Journal of Mathematical Physics **40** (1999), no. 3.

[6] I. E. Segal, *Equivalence of Measure Spaces*, American Journal of Mathematics **73** (1951), no. 2, 275-313.

[7] P. M. Sołtan, *Quantum Families of Maps and Quantum Semigroups of Finite Quantum Spaces*, arXiv:0610922 (2006).

[8] G. Takeuti, *Quantum Set Theory*, Current Issues in Quantum Logic (1981), 303-322.

[9] N. Weaver, *Quantum Relations*, arXiv:1005.0354 (2010).