Lecture 2: Introduction to Crossed Products and More Examples of Actions

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University of Oregon

13 July 2016
Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
Lecture 6 (20 July 2016): Applications and Problems.
The Second Summer School on Operator Algebras
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Lecture 6 (20 July 2016): Applications and Problems.
A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
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- Applications of the tracial Rokhlin property.
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Recall: Group actions on C*-algebras

Definition
Let $G$ be a group and let $A$ be a C*-algebra.

We saw some examples coming from actions on compact Hausdorff spaces.

We also saw the inner action: if $g \mapsto z_g$ is a (continuous) homomorphism from $G$ to the unitary group $U(A)$ of a unital C*-algebra $A$, then $\alpha_g(a) = z_g a z_g^*$ defines an action of $G$ on $A$. (We write $\alpha_g = \text{Ad}(z_g)$.)

Exercise
Prove that $g \mapsto \text{Ad}(z_g)$ really is a continuous action of $G$ on $A$.

Finally, we looked at one example of an infinite tensor product action (next slide).
**Recall: Group actions on C*-algebras**

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Let $G$ be a group and let $A$ be a C*-algebra. An *action of $G$ on $A$* is a homomorphism $g \mapsto \alpha_g$ from $G$ to $\text{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g : A \rightarrow A$, and $\alpha_1 = \text{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

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Let $A_n = (M_2)^\otimes n$, the tensor product of $n$ copies of the algebra $M_2$ of $2 \times 2$ matrices. Thus $A_n \cong M_{2^n}$.
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\varphi_n : A_n \rightarrow A_{n+1} = A_n \otimes M_2
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by \( \varphi_n(a) = a \otimes 1 \). Let \( A \) be the (completed) direct limit \( \lim_{n \rightarrow} A_n \). (This is just the \( 2^\infty \) UHF algebra.)
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$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

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Crossed Products

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Infinite tensor product action example (continued)

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Commutative diagram to define the order 2 automorphism $\alpha_n \in \text{Aut}(A_n)$ is $\alpha_n = \text{Ad}(z_n)$. 

The action of $Z_2$ is not inner (see later), although it is "approximately inner" (that is, a pointwise limit of inner actions).
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Infinite tensor product action example (continued)

Recall: \( A_n = (M_2)^\otimes n \cong M_{2^n} \).

\( \varphi_n: A_n \to A_{n+1} = A_n \otimes M_2 \) is \( \varphi_n(a) = a \otimes 1 \), and \( A = \lim_{\longrightarrow} n A_n \).

\( \nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in U(M_2) \), and \( z_n = \nu^\otimes n \in U(A_n) \).

\( \alpha_n \in \text{Aut}(A_n) \) is \( \alpha_n = \text{Ad}(z_n) \).

Commutative diagram to define the order 2 automorphism \( \alpha \in \text{Aut}(A) \):

\[
\begin{array}{cccccccc}
\mathbb{C} & \xrightarrow{\varphi^0} & M_2 & \xrightarrow{\varphi^1} & M_4 & \xrightarrow{\varphi^2} & M_8 & \xrightarrow{\varphi^3} & \cdots & \xrightarrow{} & A \\
\downarrow{\alpha^0} & & \downarrow{\alpha^1} & & \downarrow{\alpha^2} & & \downarrow{\alpha^3} & & & & \downarrow{\alpha} \\
\mathbb{C} & \xrightarrow{\varphi^0} & M_2 & \xrightarrow{\varphi^1} & M_4 & \xrightarrow{\varphi^2} & M_8 & \xrightarrow{\varphi^3} & \cdots & \xrightarrow{} & A
\end{array}
\]

The action of \( \mathbb{Z}_2 \) is not inner (see later), although it is “approximately inner” (that is, a pointwise limit of inner actions).
Recall: $A_n = (M_2)^{\otimes n} \cong M_{2^n}$.

$\varphi_n: A_n \to A_{n+1} = A_n \otimes M_2$ is $\varphi_n(a) = a \otimes 1$, and $A = \lim_{\to} A_n$.

$\nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in U(M_2)$, and $z_n = \nu^{\otimes n} \in U(A_n)$.

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The action of $\mathbb{Z}_2$ is not inner (see later), although it is “approximately inner” (that is, a pointwise limit of inner actions).
General infinite tensor product actions

We had $A = \lim_{\to n} (M_2)^{\otimes n}$ with the action of $\mathbb{Z}_2$ generated by the direct limit automorphism

$$\lim_{\to n} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes n}$$
General infinite tensor product actions

We had $A = \lim_{n \to \infty} (M_2)^{\otimes n}$ with the action of $\mathbb{Z}_2$ generated by the direct limit automorphism

$$\lim_{n \to \infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes n}$$

We write this automorphism as

$$\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ on } A = \bigotimes_{n=1}^{\infty} M_2.$$
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In general, one can use an arbitrary group,
General infinite tensor product actions

We had \( A = \lim_{n \to \infty} (M_2) \otimes^n \) with the action of \( \mathbb{Z}_2 \) generated by the direct limit automorphism

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In general, one can use an arbitrary group, one need not choose the same unitary representation in each tensor factor.
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If all the factors are finite dimensional matrix algebras (not necessarily of the same size) and the action in each factor is inner, the action is frequently called a “product type action”.

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More examples of product type actions

We will later use the following two additional examples:

$$\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ on } A = \bigotimes_{n=1}^{\infty} M_3,$$

and

$$\bigotimes_{n=1}^{\infty} \text{Ad} \left( \text{diag}(-1, 1, 1, \ldots, 1) \right) \text{ on } A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$
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\[ \bigotimes_{n=1}^\infty \text{Ad} \left( \text{diag}(-1, 1, 1, \ldots, 1) \right) \text{ on } A = \bigotimes_{n=1}^\infty M_{2n+1}. \]

In the second one, there are supposed to be \(2^n\) ones on the diagonal, giving a \((2^n + 1) \times (2^n + 1)\) matrix.
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\]

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\[
\prod_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ on } A = \prod_{n=1}^{\infty} M_3,
\]

and

\[
\prod_{n=1}^{\infty} \text{Ad}(\text{diag}(-1, 1, 1, \ldots, 1)) \text{ on } A = \prod_{n=1}^{\infty} M_{2^n+1}.
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In the second one, there are supposed to be \(2^n\) ones on the diagonal, giving a \((2^n + 1) \times (2^n + 1)\) matrix.
The tensor product of copies of conjugation by the regular representation

Let $G$ be a finite group. Set $m = \text{card}(G)$. 

\[ \text{Let } G \text{ act on } l^2(G) \sim \mathbb{C}^m \text{ via the left regular representation. That is, if } g \in G, \text{ then } g \text{ acts on } l^2(G) \text{ by the unitary operator } (z \xi)(h) = \xi(g^{-1}h) \text{ for } \xi \in l^2(G) \text{ and } h \in G. \]

Now let $G$ act on $M_m \sim L(l^2(G))$ by conjugation by the left regular representation: $g \mapsto \text{Ad}(zg)$. Then take $A = \lim_{\to} \bigotimes_{n=1}^\infty (M_m)$ (which is the $\infty UHF$ algebra), with the action of $G$ given by $g \mapsto \bigotimes_{n=1}^\infty \text{Ad}(zg)$.

The first example we gave of a product type action is the case $G = \mathbb{Z}_2$. The left regular representation of $\mathbb{Z}_2$ is generated by \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] rather than \[
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N. C. Phillips (U of Oregon)

Crossed Products

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The tensor product of copies of conjugation by the regular representation

Let $G$ be a finite group. Set $m = \text{card}(G)$. Let $G$ act on the Hilbert space $l^2(G) \cong \mathbb{C}^m$ via the left regular representation. That is,

$$g \mapsto \text{Ad}(z_g).$$

Now let $G$ act on $M_m \cong L(l^2(G))$ by conjugation by the left regular representation:

$$g \mapsto \infty \bigotimes_{n=1}^\infty \text{Ad}(z_g).$$

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Let $G$ be a finite group. Set $m = \text{card}(G)$. Let $G$ act on the Hilbert space $l^2(G) \cong \mathbb{C}^m$ via the left regular representation. That is, if $g \in G$, then $g$ acts on $l^2(G)$ by the unitary operator $(z_g \xi)(h) = \xi(g^{-1}h)$ for $\xi \in l^2(G)$ and $h \in G$. 

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Crossed products

Let $G$ be a locally compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on a C*-algebra $A$. 

There is a crossed product C*-algebra $C^*(G, A, \alpha)$, which is a kind of generalization of the group C*-algebra $C^*(G)$. Crossed products are quite important in the theory of C*-algebras. One motivation (mentioned already): Suppose $G$ is a semidirect product $N \rtimes H$. The action of $H$ on $N$ gives an action $\alpha: H \to \text{Aut}(C^*(N))$, and one has $C^*(G) \sim C^*(H, C^*(N), \alpha)$. (Exercise: Prove this when $H$ and $N$ are finite.) Thus, crossed products appear even if one is only interested in group C*-algebras and unitary representations of groups.

Another motivation (not applicable to finite groups acting on spaces): The noncommutative version of $X/G$ is the fixed point algebra $A_G$. In particular, for compact $G$, one can check that $C(X/G) \sim C(X)_G$. For noncompact groups, often $X/G$ is very far from Hausdorff and $A_G$ is far too small. The crossed product provides a much more generally useful algebra, which is the “right” substitute for the fixed point algebra when the action is free.
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Another motivation (not applicable to finite groups acting on spaces): The noncommutative version of $X/G$ is the fixed point algebra $A^G$. In particular, for compact $G$, one can check that $C(X/G) \cong C(X)^G$. For noncompact groups, often $X/G$ is very far from Hausdorff and $A^G$ is far too small. The crossed product provides a much more generally useful algebra, which is the “right” substitute for the fixed point algebra when the action is free.
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One motivation (mentioned already): Suppose $G$ is a semidirect product $N \rtimes H$. The action of $H$ on $N$ gives an action $\alpha: H \to \text{Aut}(C^*(N))$, and one has $C^*(G) \cong C^*(H, C^*(N), \alpha)$. (Exercise: Prove this when $H$ and $N$ are finite.)
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Recall the group case. If $G$ is a discrete group, then multiplication in $\mathbb{C}[G]$ is $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$, and adjoint is $(a \cdot u_g)^* = \overline{a}u_{g^{-1}}$. If $w$ is a unitary representation of $G$ on $H$, the unital $*$-homomorphism $\pi_w : \mathbb{C}[G] \to L(H)$ is $\pi_w \left( \sum_{g \in G} a_g \cdot u_g \right) = \sum_{g \in G} a_g \cdot w_g$. 
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The assignment $w \mapsto \pi_w$
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Let $\alpha : G \to \text{Aut}(A)$ be an action of a topological group $G$ on a C*-algebra $A$. A **covariant representation** of $(G, A, \alpha)$ on a Hilbert space $H$ is a pair $(\nu, \sigma)$ consisting of a unitary representation $\nu : G \to U(H)$ and a representation $\sigma : A \to L(H)$.
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Assume $G$ is finite. Recall: $\mathbb{C}[G]$ is all formal linear combinations $\sum_{g \in G} a_g \cdot u_g$ with $a_g \in \mathbb{C}$ for $g \in G$. Multiplication in $\mathbb{C}[G]$ is $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$, and adjoint is $(a \cdot u_g)^* = \overline{a} u_{g^{-1}}$. 

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\[ A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}. \]
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N. C. Phillips (U of Oregon)  Crossed Products  13 July 2016  14 / 30
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\( \alpha: G \to \text{Aut}(A) \) is an action of a discrete group \( G \) on a C*-algebra \( A \). (Don’t assume \( G \) is finite, and don’t assume \( A \) is unital.)
Defining the crossed product: the general discrete case

\( \alpha: G \to \text{Aut}(A) \) is an action of a discrete group \( G \) on a C*-algebra \( A \). (Don’t assume \( G \) is finite, and don’t assume \( A \) is unital.) The skew group ring is

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A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A, c_g = 0 \text{ for all but finitely many } g \in G \right\}.
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Exercise: Prove that \( A[G] \) is a *-algebra over \( \mathbb{C} \).
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If $G$ is discrete but not finite, $C^*(G, A, \alpha)$ is the completion of $A[G]$ in a suitable norm. (In general, there are several choices, but only one gives the right universal property.)
Defining the crossed product: the general discrete case

\(\alpha: G \rightarrow \text{Aut}(A)\) is an action of a discrete group \(G\) on a C*-algebra \(A\). (Don’t assume \(G\) is finite, and don’t assume \(A\) is unital.) The skew group ring is

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If \(G\) is discrete but not finite, \(C^*(G, A, \alpha)\) is the completion of \(A[G]\) in a suitable norm. (In general, there are several choices, but only one gives the right universal property.)

General locally compact case: See the appendix.
Defining the crossed product: the general discrete case

$\alpha: G \rightarrow \text{Aut}(A)$ is an action of a discrete group $G$ on a $C^*$-algebra $A$. (Don’t assume $G$ is finite, and don’t assume $A$ is unital.) The skew group ring is

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A, c_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

Multiplication in $A[G]$ is

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g bu_g^{-1}]) \cdot u_g u_h = (a\alpha_g(b)) \cdot u_{gh}$$

and adjoint is

$$(a \cdot u_g)^\ast = \alpha_g^{-1}(a^\ast) \cdot u_{g^{-1}}.$$

Exercise: Prove that $A[G]$ is a $\ast$-algebra over $\mathbb{C}$.

If $G$ is discrete but not finite, $C^*(G, A, \alpha)$ is the completion of $A[G]$ in a suitable norm. (In general, there are several choices, but only one gives the right universal property.)

General locally compact case: See the appendix.
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Exercise: Prove that $A[G]$ is a *-algebra over $\mathbb{C}$. 

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General locally compact case: See the appendix.
The universal property of the crossed product

Recall: If $G$ is finite and $w : G \to U(H)$ is a unitary representation, then $\pi_w : \mathbb{C}[G] \to L(H)$ is $\pi_w \left( \sum_{g \in G} a_g \cdot u_g \right) = \sum_{g \in G} a_g \cdot w_g$. Moreover, $w \mapsto \pi_w$ is a bijection from unitary representations to unital $\ast$-homomorphisms.
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N. C. Phillips (U of Oregon)
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Exercise: Keep $G$ finite, but no longer assume that $A$ is unital. Assume that you know $A[G]$ is a $C^*$-algebra. Prove the theorem with "unital $*$-homomorphisms" replaced by "nondegenerate representations". ($\rho: B \to L(H)$ is nondegenerate if $\rho(B) H = H$.)

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N. C. Phillips  (U of Oregon) Crossed Products 13 July 2016 18 / 30
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The norm on $A[G]$

Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a C*-algebra $A$. We construct a C* norm on the skew group ring $A[G]$. 

Recall:

$$(aug)(buh) = a\alpha_g(b)uh$$

and

$$(aug)^* = \alpha^{-1}_g(a^*)ug^{-1}.$$ 

Fix a faithful representation $\sigma_0 : A \to \text{L}(H_0)$ of $A$ on a Hilbert space $H_0$.

Set $H = l^2(G, H_0)$, the set of all $\xi = (\xi_g)_{g \in G}$ in $\bigoplus_{g \in G} H_0$, with the scalar product $\langle (\xi_g)_{g \in G}, (\eta_g)_{g \in G} \rangle = \sum_{g \in G} \langle \xi_g, \eta_g \rangle$. 

(Exercise: Prove that $H$ is a Hilbert space.) Then define $\sigma : A[G] \to \text{L}(H)$ as follows.

For $c = \sum_{g \in G} c_g u_g$ and $h \in G$,

$$(\sigma(c) \xi_h) = \sum_{g \in G} \sigma_0(\alpha^{-1}_h(c_g))(\xi_g)^{-1}_h.$$ 

Exercise: If $A$ and $\sigma_0$ are unital, find a representation $v : G \to \text{U}(H)$ and a unital representation $\sigma : A \to \text{L}(H)$ such that $(v, \sigma)$ is covariant.

(This is one way one is really supposed to construct a C* norm.)
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Exercise: Prove these two inequalities. (The second requires looking at what $\pi(c)$ does to suitable elements in $H$. It is related to $\|a\| \geq \max_{j,k} |a_{j,k}|$ for a matrix $a = (a_{j,k})_{j,k=1,2,...,n}$.)
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The group $SL_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication.

This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$.

$SL_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

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- $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$,
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The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_{\theta}$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. 

If $\theta_1 - \theta_2 \in \mathbb{Z}$, then $A_{\theta_1} = A_{\theta_2}$. (The commutation relation is the same.)

Some standard facts, presented without proof:

- If $\theta \not\in \mathbb{Q}$, then $A_{\theta}$ is simple. In particular, any two unitaries $u$ and $v$ in any C*-algebra satisfying $vu = e^{2\pi i \theta} uv$ generate a copy of $A_{\theta}$.
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The gauge action on the rotation algebra

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. 

There is a unique action $\gamma: S_1 \times S_1 \to \text{Aut}(A_\theta)$ such that $\gamma(\lambda, \zeta)(u) = \lambda u$ and $\gamma(\lambda, \zeta)(v) = \zeta v$ for $\lambda, \zeta \in S_1$.

This essentially follows from the fact that $\lambda u$ and $\zeta v$ satisfy the same commutation relation that $u$ and $v$ do. One must also check that $\gamma(\lambda, \zeta)\mapsto \gamma(\lambda, \zeta)$ is a group homomorphism. (A bit of work is required to show that $\gamma(\lambda, \zeta)\mapsto \gamma(\lambda, \zeta)(a)$ is continuous for all $a \in A_\theta$. Exercise: Do it. Hint: Show that it is true for $a$ in the linear span of all $u^m v^n$, and then use an $\varepsilon/3$ argument.)

In particular, there are actions of $\mathbb{Z}_n$ on $A_\theta$ generated by the automorphism $u \mapsto e^{2\pi i \theta/n} u$ and $v \mapsto v$ and by the automorphism $u \mapsto u$ and $v \mapsto e^{2\pi i \theta/n} v$. 

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There is a unique action $\gamma: S^1 \times S^1 \to \text{Aut}(A_\theta)$ such that

$$\gamma(\lambda, \zeta)(u) = \lambda u \quad \text{and} \quad \gamma(\lambda, \zeta)(v) = \zeta v$$

for $\lambda, \zeta \in S^1$. This essentially follows from the fact that $\lambda u$ and $\zeta v$ satisfy the same commutation relation that $u$ and $v$ do.
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Appendix: The C*-algebra of a locally compact group, etc.

Let $G$ be a locally compact group. We recall that nondegenerate representations of the group C*-algebra $\mathbb{C}^*(G)$ on a Hilbert space $H$ are in one to one correspondence with the unitary representations of $G$ on $H$.

To construct $\mathbb{C}^*(G)$, one starts with $L^1(G)$ (using left Haar measure $\mu$) with convolution multiplication:

$$(a \ast b)(g) = \int_G a(h) b(h^{-1}g) \, d\mu(h).$$

(We omit the formula for the adjoint.)

If $G$ is discrete and $\delta_g \in l^1(G)$ is the standard basis vector corresponding to $g \in G$, this amounts to declaring that $\delta_g \ast \delta_h = \delta_{gh}$ and $\delta_g^* = \delta_{g^{-1}}$.

A unitary representation $g \mapsto v_g$ of $G$ on a Hilbert space $H$ gives a nondegenerate *-representation $\pi$ of $L^1(G)$ on $H$ via the formula

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N. C. Phillips (U of Oregon)
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The crossed product $C^*(G, A, \alpha)$ (for $G$ locally compact) is defined in such a way as to have a universal property which generalizes the universal property of the group $C^*$-algebra $C^*(G)$. 
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The crossed product $C^*(G, A, \alpha)$ (for $G$ locally compact) is defined in such a way as to have a universal property which generalizes the universal property of the group C*-algebra $C^*(G)$. We give the statements for the general case.
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**Definition**

Let $\alpha : G \to \text{Aut}(A)$ be an action of a locally compact group $G$ on a C*-algebra $A$. 

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The crossed product $\mathbb{C}^*(G, A, \alpha)$ (for $G$ locally compact) is defined in such a way as to have a universal property which generalizes the universal property of the group C*-algebra $\mathbb{C}^*(G)$. We give the statements for the general case.

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Let $\alpha : G \to \text{Aut}(A)$ be an action of a locally compact group $G$ on a C*-algebra $A$. A **covariant representation** of $(G, A, \alpha)$ on a Hilbert space $H$ is a pair $(\nu, \sigma)$ consisting of a unitary representation $\nu : G \to U(H)$ (the unitary group of $H$) and a representation $\sigma : A \to L(H)$ (the algebra of all bounded operators on $H$), satisfying the covariance condition

$$\nu(g) \sigma(a) \nu(g)^* = \sigma(\alpha(g)(a))$$

for all $g \in G$ and $a \in A$. It is called **nondegenerate** if $\sigma$ is nondegenerate.
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Let $\alpha: G \to \text{Aut}(A)$ be an action of a locally compact group $G$ on a $C^*$-algebra $A$. A covariant representation of $(G, A, \alpha)$ on a Hilbert space $H$ is a pair $(\nu, \sigma)$ consisting of a unitary representation $\nu: G \to U(H)$ (the unitary group of $H$) and a representation $\sigma: A \to L(H)$ (the algebra of all bounded operators on $H$), satisfying the covariance condition

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A dense subalgebra of the crossed product

The skew group ring $A[G]$, used when $G$ is discrete,
A dense subalgebra of the crossed product

The skew group ring $A[G]$, used when $G$ is discrete, is replaced by $C_c(G, A, \alpha)$,

\[
(ab)(g) = \int_G a(h) b(h^{-1}g) \, d\mu(h)
\]

and (using the modular function $\Delta$ of $G$)

\[
a^*(g) = \Delta(g)^{-1} \alpha(g)(a(g^{-1}a^*))
\]

One can define a (non C*) norm by

\[
\|a\|_1 = \int_G \|a(g)\| \, d\mu(g)
\]

The completion is called $L_1(G, A, \alpha)$.

(One can also define $L_1(G, A, \alpha)$ directly, using a more general version of Banach space valued integration.)

Exercise: Prove that $C_c(G, A, \alpha)$ is a normed $*$-algebra.

$C^*(G, A, \alpha)$ is the completion of $C_c(G, A, \alpha)$ (or $L_1(G, A, \alpha)$) in a suitable $C^*$ norm.

(It is the universal enveloping $C^*$-algebra of $L_1(G, A, \alpha)$.)
A dense subalgebra of the crossed product

The skew group ring $A[G]$, used when $G$ is discrete, is replaced by $C_c(G, A, \alpha)$, with product (using a left Haar measure $\mu$ on $G$ and Banach space valued integration)
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One can define a (non \( \mathbb{C}^* \)) norm by

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The universal property of the crossed product (continued)

Definition
Let $\alpha: G \to \text{Aut}(A)$ be an action of a locally compact group $G$ on a C*-algebra $A$, and let $(\nu, \sigma)$ be a covariant representation of $(G, A, \alpha)$ on a Hilbert space $H$. Then the integrated form of $(\nu, \sigma)$ is the representation $\pi: C^c_c(G, A, \alpha) \to \mathcal{L}(H)$ given by

$$\pi(a)\xi = \int_G \sigma(a(g))\nu_g\xi \, d\mu(g).$$

$C^c_c(G, A, \alpha)$ is then a completion of $C^c_c(G, A, \alpha)$, chosen to give:

Theorem
Let $\alpha: G \to \text{Aut}(A)$ be an action of a locally compact group $G$ on a C*-algebra $A$. Then the integrated form construction defines a bijection from the set of nondegenerate covariant representations of $(G, A, \alpha)$ on a Hilbert space $H$ to the set of nondegenerate representations of $C^c_c(G, A, \alpha)$ on the same Hilbert space.
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