Lecture 3: Crossed Products by Finite Groups; the Rokhlin Property

N. Christopher Phillips

University of Oregon

15 July 2016
The Second Summer School on Operator Algebras and Noncommutative Geometry 2016

East China Normal University, Shanghai

11–29 July 2016

- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 6 (20 July 2016): Applications and Problems.
Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
Lecture 6 (20 July 2016): Applications and Problems.
Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
Lecture 6 (20 July 2016): Applications and Problems.
A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.
A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.
A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.
Recall: Group actions on C*-algebras

Definition
Let $G$ be a group and let $A$ be a C*-algebra.

Recall: Group actions on C*-algebras

**Definition**

Let $G$ be a group and let $A$ be a C*-algebra. An action of $G$ on $A$ is a homomorphism $g \mapsto \alpha_g$ from $G$ to $\text{Aut}(A)$. That is, for each $g \in G$, we have an automorphism $\alpha_g : A \to A$, and $\alpha_1 = \text{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

When $G$ is a topological group, we require that the action be continuous: $(g, a) \mapsto \alpha_g(a)$ is jointly continuous from $G \times A$ to $A$. 
Recall: Group actions on C*-algebras

Definition

Let $G$ be a group and let $A$ be a C*-algebra. An action of $G$ on $A$ is a homomorphism $g \mapsto \alpha_g$ from $G$ to Aut$(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g : A \to A$, and $\alpha_1 = \text{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$. 
Recall: Group actions on $\text{C}^*$-algebras

**Definition**

Let $G$ be a group and let $A$ be a $\text{C}^*$-algebra. An *action of $G$ on $A$* is a homomorphism $g \mapsto \alpha_g$ from $G$ to $\text{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g : A \to A$, and $\alpha_1 = \text{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

When $G$ is a topological group, we require that the action be continuous: $(g, a) \mapsto \alpha_g(a)$ is jointly continuous from $G \times A$ to $A$. 

N. C. Phillips (U of Oregon)
Definition

Let $G$ be a group and let $A$ be a C*-algebra. An action of $G$ on $A$ is a homomorphism $g \mapsto \alpha_g$ from $G$ to $\text{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g : A \to A$, and $\alpha_1 = id_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

When $G$ is a topological group, we require that the action be continuous: $(g, a) \mapsto \alpha_g(a)$ is jointly continuous from $G \times A$ to $A$. 
Recall: Group actions on C*-algebras

**Definition**

Let $G$ be a group and let $A$ be a C*-algebra. An *action of $G$ on $A$* is a homomorphism $g \mapsto \alpha_g$ from $G$ to $\text{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g : A \to A$, and $\alpha_1 = \text{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

When $G$ is a topological group, we require that the action be continuous: $(g, a) \mapsto \alpha_g(a)$ is jointly continuous from $G \times A$ to $A$. 
Recall: The action of $SL_2(\mathbb{Z})$ on the torus

Recall: Every action of a group $G$ on a compact space $X$ gives an action of $G$ on $C(X)$. 
Recall: The action of $\text{SL}_2(\mathbb{Z})$ on the torus

Recall: Every action of a group $G$ on a compact space $X$ gives an action of $G$ on $C(X)$.

The group $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. 
Recall: The action of $SL_2(\mathbb{Z})$ on the torus

Recall: Every action of a group $G$ on a compact space $X$ gives an action of $G$ on $C(X)$.

The group $SL_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. $SL_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix},
\begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix},
\text{ and }
\begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
$$
Recall: The action of $\text{SL}_2(\mathbb{Z})$ on the torus

Recall: Every action of a group $G$ on a compact space $X$ gives an action of $G$ on $C(X)$.

The group $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. $\text{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

Restriction gives actions of these on $S^1 \times S^1$. 
Recall: The action of $\text{SL}_2(\mathbb{Z})$ on the torus

Recall: Every action of a group $G$ on a compact space $X$ gives an action of $G$ on $C(X)$.

The group $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. $\text{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$ 

Restriction gives actions of these on $S^1 \times S^1$. 
Recall: The action of $SL_2(\mathbb{Z})$ on the torus

Recall: Every action of a group $G$ on a compact space $X$ gives an action of $G$ on $C(X)$.

The group $SL_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. $SL_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

Restriction gives actions of these on $S^1 \times S^1$. 
Reminder: The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_\theta$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. 

Some standard facts, presented without proof:

- If $\theta \not\in \mathbb{Q}$, then $A_\theta$ is simple. In particular, any two unitaries $u$ and $v$ in any C*-algebra satisfying $vu = e^{2\pi i \theta} uv$ generate a copy of $A_\theta$.
- If $\theta = m/n$ in lowest terms, with $n > 0$, then $A_\theta$ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber $M_n$. In particular, if $\theta = 0$, or if $\theta \in \mathbb{Z}$, then $A_\theta \cong C(S^1 \times S^1)$.

The algebra $A_\theta$ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).
Reminder: The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_\theta$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. Some standard facts, presented without proof:
Reminder: The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_\theta$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. Some standard facts, presented without proof:

- If $\theta \not\in \mathbb{Q}$, then $A_\theta$ is simple. In particular, any two unitaries $u$ and $v$ in any C*-algebra satisfying $vu = e^{2\pi i \theta} uv$ generate a copy of $A_\theta$. 

The algebra $A_\theta$ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).
Reminder: The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_\theta$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. Some standard facts, presented without proof:

- If $\theta \notin \mathbb{Q}$, then $A_\theta$ is simple. In particular, any two unitaries $u$ and $v$ in any C*-algebra satisfying $vu = e^{2\pi i \theta} uv$ generate a copy of $A_\theta$.
- If $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then $A_\theta$ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber $M_n$. 

N. C. Phillips (U of Oregon)
Reminder: The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_\theta$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. Some standard facts, presented without proof:

- If $\theta \notin \mathbb{Q}$, then $A_\theta$ is simple. In particular, any two unitaries $u$ and $v$ in any C*-algebra satisfying $vu = e^{2\pi i \theta} uv$ generate a copy of $A_\theta$.
- If $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then $A_\theta$ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber $M_n$.
- In particular, if $\theta = 0$, or if $\theta \in \mathbb{Z}$, then $A_\theta \cong C(S^1 \times S^1)$.
Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_\theta$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. Some standard facts, presented without proof:

- If $\theta \notin \mathbb{Q}$, then $A_\theta$ is simple. In particular, any two unitaries $u$ and $v$ in any C*-algebra satisfying $vu = e^{2\pi i \theta} uv$ generate a copy of $A_\theta$.
- If $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then $A_\theta$ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber $M_n$.
- In particular, if $\theta = 0$, or if $\theta \in \mathbb{Z}$, then $A_\theta \cong C(S^1 \times S^1)$.

The algebra $A_\theta$ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).
Reminder: The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_\theta$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. Some standard facts, presented without proof:

- If $\theta \notin \mathbb{Q}$, then $A_\theta$ is simple. In particular, any two unitaries $u$ and $v$ in any C*-algebra satisfying $vu = e^{2\pi i \theta} uv$ generate a copy of $A_\theta$.
- If $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then $A_\theta$ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber $M_n$.
  
- In particular, if $\theta = 0$, or if $\theta \in \mathbb{Z}$, then $A_\theta \cong C(S^1 \times S^1)$.

The algebra $A_\theta$ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).
Reminder: The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra $A_\theta$, the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. Some standard facts, presented without proof:

- If $\theta \notin \mathbb{Q}$, then $A_\theta$ is simple. In particular, any two unitaries $u$ and $v$ in any C*-algebra satisfying $vu = e^{2\pi i \theta} uv$ generate a copy of $A_\theta$.
- If $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then $A_\theta$ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber $M_n$.
- In particular, if $\theta = 0$, or if $\theta \in \mathbb{Z}$, then $A_\theta \cong C(S^1 \times S^1)$.

The algebra $A_\theta$ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).
The action of $\text{SL}_2(\mathbb{Z})$ on the rotation algebra

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.
The action of $SL_2(\mathbb{Z})$ on the rotation algebra

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

The group $SL_2(\mathbb{Z})$ acts on $A_\theta$ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$
The action of $SL_2(\mathbb{Z})$ on the rotation algebra
Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vv = e^{2\pi i \theta} uv$.

The group $SL_2(\mathbb{Z})$ acts on $A_\theta$ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi in_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi in_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$
The action of $SL_2(\mathbb{Z})$ on the rotation algebra

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta}uv$.

The group $SL_2(\mathbb{Z})$ acts on $A_\theta$ by sending the matrix $n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$ to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

Exercise: Check that $\alpha_n$ is an automorphism, and that $n \mapsto \alpha_n$ is a group homomorphism.
The action of $SL_2(\mathbb{Z})$ on the rotation algebra

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

The group $SL_2(\mathbb{Z})$ acts on $A_\theta$ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi in_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi in_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$ 

Exercise: Check that $\alpha_n$ is an automorphism, and that $n \mapsto \alpha_n$ is a group homomorphism.

This action is the analog of the action of $SL_2(\mathbb{Z})$ on $S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2$. It reduces to that action when $\theta = 0$. 
The action of $SL_2(\mathbb{Z})$ on the rotation algebra

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

The group $SL_2(\mathbb{Z})$ acts on $A_\theta$ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi in_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi in_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

Exercise: Check that $\alpha_n$ is an automorphism, and that $n \mapsto \alpha_n$ is a group homomorphism.

This action is the analog of the action of $SL_2(\mathbb{Z})$ on $S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2$. It reduces to that action when $\theta = 0$. 
The action of $SL_2(\mathbb{Z})$ on the rotation algebra

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

The group $SL_2(\mathbb{Z})$ acts on $A_\theta$ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi in_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi in_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$ 

Exercise: Check that $\alpha_n$ is an automorphism, and that $n \mapsto \alpha_n$ is a group homomorphism.

This action is the analog of the action of $SL_2(\mathbb{Z})$ on $S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2$. It reduces to that action when $\theta = 0$. 

The action of $\text{SL}_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$. 

Recall that $\text{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

Restriction gives actions of these groups on the irrational rotation algebras. In terms of generators of $A_\theta$, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by $u \mapsto u^*$ and $v \mapsto v^*$, and the action of $\mathbb{Z}_4$ is generated by $u \mapsto v$ and $v \mapsto u^*$. 

Exercise: Find the analogous formulas for $\mathbb{Z}_3$ and $\mathbb{Z}_6$, and check that they give actions of these groups.
The action of $\text{SL}_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

Recall that $\text{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad 
\begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad 
\begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
$$
The action of $SL_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

Recall that $SL_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

Restriction gives actions of these groups on the irrational rotation algebras.

Exercise: Find the analogous formulas for $Z_3$ and $Z_6$, and check that they give actions of these groups.
The action of $SL_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

Recall that $SL_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

Restriction gives actions of these groups on the irrational rotation algebras.

In terms of generators of $A_\theta$, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by

\[
u \mapsto v^* \quad \text{and} \quad u \mapsto u^*,
\]
The action of \( \text{SL}_2(\mathbb{Z}) \) on the rotation algebra (continued)

Recall: \( A_{\theta} \) is the universal \( \text{C}^* \)-algebra generated by two unitaries \( u \) and \( v \) satisfying the commutation relation \( vu = e^{2\pi i \theta} uv \).

Recall that \( \text{SL}_2(\mathbb{Z}) \) has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

Restriction gives actions of these groups on the irrational rotation algebras.

In terms of generators of \( A_{\theta} \), and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of \( \mathbb{Z}_2 \) is generated by

\[
u \mapsto v^* \quad \text{and} \quad v \mapsto v^*,
\]

and the action of \( \mathbb{Z}_4 \) is generated by

\[
u \mapsto v \quad \text{and} \quad v \mapsto u^*.
\]
The action of $SL_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

Recall that $SL_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
$$

Restriction gives actions of these groups on the irrational rotation algebras.

In terms of generators of $A_\theta$, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by

$$
u \mapsto \nu^* \quad \text{and} \quad v \mapsto v^*,$$

and the action of $\mathbb{Z}_4$ is generated by

$$
u \mapsto v \quad \text{and} \quad v \mapsto u^*.$$

Exercise: Find the analogous formulas for $\mathbb{Z}_3$ and $\mathbb{Z}_6$, and check that they give actions of these groups.
The action of $SL_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

Recall that $SL_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

Restriction gives actions of these groups on the irrational rotation algebras.

In terms of generators of $A_\theta$, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by

\[
uu \mapsto u^* \quad \text{and} \quad vv \mapsto v^*,
\]

and the action of $\mathbb{Z}_4$ is generated by

\[

uu \mapsto v \quad \text{and} \quad vv \mapsto u^*.
\]

Exercise: Find the analogous formulas for $\mathbb{Z}_3$ and $\mathbb{Z}_6$, and check that they give actions of these groups.
The action of $SL_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: $A_\theta$ is the universal C*-algebra generated by two unitaries $u$ and $v$ satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

Recall that $SL_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by
\[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.
\]
Restriction gives actions of these groups on the irrational rotation algebras.

In terms of generators of $A_\theta$, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by
\[ u \mapsto u^* \quad \text{and} \quad v \mapsto v^*, \]
and the action of $\mathbb{Z}_4$ is generated by
\[ u \mapsto v \quad \text{and} \quad v \mapsto u^*. \]

Exercise: Find the analogous formulas for $\mathbb{Z}_3$ and $\mathbb{Z}_6$, and check that they give actions of these groups.
Another example: The tensor flip

Assume (for convenience) that $A$ is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the “tensor flip” $a \otimes b \mapsto b \otimes a$. 

Similarly, the symmetric group $S_n$ acts on $A \otimes_n$. The tensor flip on the $2^\infty$ UHF algebra $A = \bigotimes_{n=1}^\infty M_2$ turns out to be essentially the product type action generated by $\bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ on $\bigotimes_{n=1}^\infty M_4$.

Exercise: Prove this. (Hint: Look at the tensor flip on $M_2 \otimes M_2$.

Another interesting example is gotten by taking $A$ to be the Jiang-Su algebra $Z$. It is simple, separable, unital, and nuclear. It has no nontrivial projections, its Elliott invariant is the same as for $C_\ast$, and $Z \otimes Z \cong Z$. 

N. C. Phillips (U of Oregon) 
Crossed Products; the Rokhlin Property 15 July 2016 9 / 41
Another example: The tensor flip

Assume (for convenience) that $A$ is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the “tensor flip” $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group $S_n$ acts on $A \otimes^n$.
Another example: The tensor flip

Assume (for convenience) that $A$ is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the “tensor flip” $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group $S_n$ acts on $A^\otimes n$.

The tensor flip on the $2^\infty$ UHF algebra $A = \bigotimes_{n=1}^{\infty} M_2$ turns out to be essentially the product type action generated by

$$\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ on } \bigotimes_{n=1}^{\infty} M_4.$$

Exercise: Prove this. (Hint: Look at the tensor flip on $M_2 \otimes M_2$.)
Another example: The tensor flip

Assume (for convenience) that $A$ is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the “tensor flip” $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group $S_n$ acts on $A \otimes^n$.

The tensor flip on the $2^\infty$ UHF algebra $A = \bigotimes_{n=1}^\infty M_2$ turns out to be essentially the product type action generated by

$$
\bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
$$
on $\bigotimes_{n=1}^\infty M_4$.

Exercise: Prove this. (Hint: Look at the tensor flip on $M_2 \otimes M_2$.)

Another interesting example is gotten by taking $A$ to be the Jiang-Su algebra $Z$. It is simple, separable, unital, and nuclear. It has no nontrivial projections, its Elliott invariant is the same as for $\mathbb{C}$, and $Z \otimes Z \cong Z$. 
Another example: The tensor flip

Assume (for convenience) that $A$ is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the “tensor flip” $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group $S_n$ acts on $A \otimes^n$.

The tensor flip on the $2^\infty$ UHF algebra $A = \bigotimes_{n=1}^{\infty} M_2$ turns out to be essentially the product type action generated by

$$\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
on $\bigotimes_{n=1}^{\infty} M_4$.

Exercise: Prove this. (Hint: Look at the tensor flip on $M_2 \otimes M_2$.)

Another interesting example is gotten by taking $A$ to be the Jiang-Su algebra $Z$. It is simple, separable, unital, and nuclear. It has no nontrivial projections, its Elliott invariant is the same as for $\mathbb{C}$, and $Z \otimes Z \cong Z$. 
Another example: The tensor flip

Assume (for convenience) that $A$ is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the “tensor flip” $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group $S_n$ acts on $A^\otimes n$.

The tensor flip on the $2^\infty$ UHF algebra $A = \bigotimes_{n=1}^\infty M_2$ turns out to be essentially the product type action generated by

$$\bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ on } \bigotimes_{n=1}^\infty M_4.$$

Exercise: Prove this. (Hint: Look at the tensor flip on $M_2 \otimes M_2$.)

Another interesting example is gotten by taking $A$ to be the Jiang-Su algebra $\mathcal{Z}$. It is simple, separable, unital, and nuclear. It has no nontrivial projections, its Elliott invariant is the same as for $\mathbb{C}$, and $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$. 
The free flip

Let $A$ be a $C^*$-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_C A$ to get a unital $C^*$-algebra.)
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_{\mathbb{C}} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors.
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_\mathbb{C} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \operatorname{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor,
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_{\mathbb{C}} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor.
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_C A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For \( a \in A \), it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_C A$. 
The free flip

Let $A$ be a $C^*$-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_\mathbb{C} A$ to get a unital $C^*$-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates a actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_\mathbb{C} A$.

There are many generalizations.
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_\mathbb{C} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates a actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_\mathbb{C} A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of $A$),
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_\mathbb{C} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates a actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_\mathbb{C} A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of $A$), or the reduced free product $A \star_r A$ (using the same state on both copies of $A$).
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_C A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates a actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_C A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of $A$), or the reduced free product $A \star_r A$ (using the same state on both copies of $A$). There is a permutation action of $S_n$ on the free product of $n$ copies of $A$. 
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_\mathbb{C} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates a actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_\mathbb{C} A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of $A$), or the reduced free product $A \star_r A$ (using the same state on both copies of $A$). There is a permutation action of $S_n$ on the free product of $n$ copies of $A$. And one can make any combination of these generalizations.
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_{\mathbb{C}} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates a actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_{\mathbb{C}} A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of $A$), or the reduced free product $A \star_r A$ (using the same state on both copies of $A$). There is a permutation action of $S_n$ on the free product of $n$ copies of $A$. And one can make any combination of these generalizations.

See the appendix for some actions on Cuntz algebras, along with a reminder of the definition of Cuntz algebras.
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star \mathbb{C} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star \mathbb{C} A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of $A$), or the reduced free product $A \star_r A$ (using the same state on both copies of $A$). There is a permutation action of $S_n$ on the free product of $n$ copies of $A$. And one can make any combination of these generalizations.

See the appendix for some actions on Cuntz algebras, along with a reminder of the definition of Cuntz algebras.
The free flip

Let $A$ be a C*-algebra, and let $A \star A$ be the free product of two copies of $A$. (Use $A \star_\mathbb{C} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \text{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of $a$ in the first free factor to the copy of the same element in the second free factor, and similarly the copy of $a$ in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the “free flip”. It generates a actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_\mathbb{C} A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of $A$), or the reduced free product $A \star_r A$ (using the same state on both copies of $A$). There is a permutation action of $S_n$ on the free product of $n$ copies of $A$. And one can make any combination of these generalizations.

See the appendix for some actions on Cuntz algebras, along with a reminder of the definition of Cuntz algebras.
Recall: Construction of the crossed product by a finite group

Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a unital C*-algebra $A$.
Recall: Construction of the crossed product by a finite group

Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group $G$ on a unital $C^*$-algebra $A$. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in $G$ with coefficients in $A$: 

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.$$ 

The multiplication and adjoint are given by:

$$(a \cdot u_g)(b \cdot u_h) = (a \alpha_g(b)) \cdot u_{gh}$$

$$(a \cdot u_g)^* = \alpha^{-1}_g(a^*) \cdot u_{g^{-1}}$$

for $a, b \in A$ and $g, h \in G$, extended linearly.

We saw that there is a unique norm which makes this a $C^*$-algebra.
Recall: Construction of the crossed product by a finite group

Let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on a unital C*-algebra $A$. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in $G$ with coefficients in $A$:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.$$
Recall: Construction of the crossed product by a finite group

Let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on a unital C*-algebra $A$. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in $G$ with coefficients in $A$:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.$$

The multiplication and adjoint are given by:
Recall: Construction of the crossed product by a finite group

Let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on a unital C*-algebra $A$. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in $G$ with coefficients in $A$:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.$$ 

The multiplication and adjoint are given by:

$$(a \cdot u_g)(b \cdot u_h) = (a[u_gbu_g^{-1}]) \cdot u_gu_h = (a\alpha_g(b)) \cdot u_{gh}$$
Recall: Construction of the crossed product by a finite group

Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a unital C*-algebra \( A \). As a vector space, \( C^*(G, A, \alpha) \) is the group ring \( A[G] \), consisting of all formal linear combinations of elements in \( G \) with coefficients in \( A \):

\[
A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.
\]

The multiplication and adjoint are given by:

\[
(a \cdot u_g)(b \cdot u_h) = (a[u_gbu_g^{-1}]) \cdot u_gu_h = (a\alpha_g(b)) \cdot u_{gh}
\]

\[
(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}
\]
Recall: Construction of the crossed product by a finite group

Let \( \alpha: G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a unital C*-algebra \( A \). As a vector space, \( C^*(G, A, \alpha) \) is the group ring \( A[G] \), consisting of all formal linear combinations of elements in \( G \) with coefficients in \( A \):

\[
A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.
\]

The multiplication and adjoint are given by:

\[
(a \cdot u_g)(b \cdot u_h) = (a[u_gbu_g^{-1}]) \cdot u_gu_h = (a\alpha_g(b)) \cdot u_{gh}
\]

\[
(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}
\]

for \( a, b \in A \) and \( g, h \in G \), extended linearly.
Recall: Construction of the crossed product by a finite group

Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a unital C*-algebra $A$. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in $G$ with coefficients in $A$:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.$$

The multiplication and adjoint are given by:

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g bu_g^{-1}]) \cdot u_g u_h = (a\alpha_g(b)) \cdot u_{gh}$$

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. We saw that there is a unique norm which makes this a C*-algebra.
Recall: Construction of the crossed product by a finite group

Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a unital C*-algebra $A$. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in $G$ with coefficients in $A$:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.$$

The multiplication and adjoint are given by:

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g b u_g^{-1}]) \cdot u_g u_h = (a\alpha_g(b)) \cdot u_{gh}$$

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. We saw that there is a unique norm which makes this a C*-algebra.
Recall: Construction of the crossed product by a finite group

Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a unital C*-algebra $A$. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in $G$ with coefficients in $A$:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.$$  

The multiplication and adjoint are given by:

$$(a \cdot u_g)(b \cdot u_h) = \left( a[u_g bu_g^{-1}] \right) \cdot u_g u_h = (a\alpha_g(b)) \cdot u_{gh}$$

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. We saw that there is a unique norm which makes this a C*-algebra.
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota: G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of $G$. (So far, $G$ could be any locally compact group.)

If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$.

Now let $A$ be any C*-algebra, and let $\iota_A: G \to \text{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, \mathbb{C}, \iota) \cong C^*(G) \otimes A$. The elements of $A$ "factor out", since $A[G]$ is just the ordinary group ring: $(a \cdot u_g)(b \cdot u_h) = (a \iota_g(b)) \cdot u_{gh} = (ab) \cdot u_{gh}$. Exercise: prove this. (Since $C^*(G)$ is finite dimensional, $C^*(G) \otimes A$ is just the algebraic tensor product.)
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota : G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\text{dim}(C^*(G)) = \text{card}(G)$. 
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota : G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$.
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota : G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$. 

Exercise: prove this. (Since $C^*(G)$ is finite dimensional, $C^*(G) \otimes A$ is just the algebraic tensor product.)
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota : G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$.

Now let $A$ be any C*-algebra, and let $\iota_A : G \to \text{Aut}(A)$ be the trivial action.
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota : G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$.

Now let $A$ be any C*-algebra, and let $\iota_A : G \to \text{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, A, \iota_A) \cong C^*(G) \otimes A$. 
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota: G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $\mathcal{C}^*(G, \mathbb{C}, \iota) = \mathcal{C}^*(G)$, the group $C^*$-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $\mathcal{C}^*(G)$ is a finite dimensional $C^*$-algebra, with $\dim(\mathcal{C}^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $\mathcal{C}^*(G)$, so $\mathcal{C}^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $\mathcal{C}^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$.

Now let $A$ be any $C^*$-algebra, and let $\iota_A: G \to \text{Aut}(A)$ be the trivial action. It is not hard to see that $\mathcal{C}^*(G, A, \iota_A) \cong \mathcal{C}^*(G) \otimes A$. The elements of $A$ “factor out”, since $A[G]$ is just the ordinary group ring:
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota : G \rightarrow \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group $C^*$-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional $C^*$-algebra, with $\dim(C^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$.

Now let $A$ be any $C^*$-algebra, and let $\iota_A : G \rightarrow \text{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, A, \iota_A) \cong C^*(G) \otimes A$. The elements of $A$ “factor out”, since $A[G]$ is just the ordinary group ring:

$$(a \cdot u_g)(b \cdot u_h) = (a \iota_g(b)) \cdot u_{gh} = (ab) \cdot u_{gh}.$$
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota: G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$.

Now let $A$ be any C*-algebra, and let $\iota_A: G \to \text{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, A, \iota_A) \cong C^*(G) \otimes A$. The elements of $A$ “factor out”, since $A[G]$ is just the ordinary group ring:

$$(a \cdot u_g)(b \cdot u_h) = (a\iota_g(b)) \cdot u_{gh} = (ab) \cdot u_{gh}.$$  

Exercise: prove this. (Since $C^*(G)$ is finite dimensional, $C^*(G) \otimes A$ is just the algebraic tensor product.)
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota : G \rightarrow \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$.

Now let $A$ be any C*-algebra, and let $\iota_A : G \rightarrow \text{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, A, \iota_A) \cong C^*(G) \otimes A$. The elements of $A$ “factor out”, since $A[G]$ is just the ordinary group ring:

$$(a \cdot u_g)(b \cdot u_h) = (a\iota_g(b)) \cdot u_{gh} = (ab) \cdot u_{gh}.$$ 

Exercise: prove this. (Since $C^*(G)$ is finite dimensional, $C^*(G) \otimes A$ is just the algebraic tensor product.)
Examples of crossed products by finite groups

Let $G$ be a finite group, and let $\iota: G \to \text{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group $C^*$-algebra of $G$. (So far, $G$ could be any locally compact group.)

Since we are assuming that $G$ is finite, $C^*(G)$ is a finite dimensional $C^*$-algebra, with $\dim(C^*(G)) = \text{card}(G)$. If $G$ is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\text{card}(G)}$. If $G$ is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand $M_k$ for each unitary equivalence class of $k$-dimensional irreducible representations of $G$.

Now let $A$ be any $C^*$-algebra, and let $\iota_A: G \to \text{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, A, \iota_A) \cong C^*(G) \otimes A$. The elements of $A$ “factor out”, since $A[G]$ is just the ordinary group ring:

$$(a \cdot u_g)(b \cdot u_h) = (a\iota_g(b)) \cdot u_{gh} = (ab) \cdot u_{gh}.$$

Exercise: prove this. (Since $C^*(G)$ is finite dimensional, $C^*(G) \otimes A$ is just the algebraic tensor product.)
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$.

That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a(h)) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$.

This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u^g)(b \cdot u^h) = (ab) \cdot u^{gh}$.

For $g \in G$, we let $u^g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi_{\{g\}}$.

Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$.

Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$.

(Exercise: Prove this.)

For $g, h \in G$, set $v_{g,h} = \delta_g u^{gh} - 1 \in C^*(G, C(G), \alpha)$.

These elements form a system of matrix units. We calculate:

$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} u^{g_1h_1} - 1 \delta_{g_2} u^{g_2h_2} - 1 = \delta_{g_1} \alpha_{g_1} h_1^{-1} (\delta_{g_2}) u^{g_1h_1} - 1 \delta_{g_2} u^{g_2h_2} - 1 = \delta_{g_1} \delta_{g_1} h_1^{-1} g_2 u^{g_1h_1} - 1 \delta_{g_2} u^{g_2h_2} - 1$. 

N. C. Phillips (U of Oregon)
Crossed Products; the Rokhlin Property
15 July 2016 13 / 41
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. 

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi_{\{g\}}$.

Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$.

Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$.

(Exercise: Prove this.)

For $g, h \in G$, set $v_{g, h} = \delta_g u_{gh} - 1 \in C^*(G, C(G))$. These elements form a system of matrix units. We calculate:

$v_{g_1, h_1} v_{g_2, h_2} = \delta_{g_1} u_{g_1 h_1} \delta_{g_2} u_{g_2 h_2} = \delta_{g_1} \alpha_{g_1}(h_1) \delta_{g_2} u_{g_1 h_1} u_{g_2 h_2} = \delta_{g_1} \delta_{g_2} u_{g_1 h_1} u_{g_2 h_2} = \delta_{g_1} \delta_{g_2} u_{g_1 h_2} u_{g_2 h_1} = v_{g_1, h_1} v_{g_2, h_2}$.
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi\{g\}$.

Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$.

Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$.

(Exercise: Prove this.)

For $g, h \in G$, set $v_{g, h} = \delta_g u_{gh} - 1 \in C^*(G, C(G), \alpha)$.

These elements form a system of matrix units. We calculate:

$v_{g_1, h_1} v_{g_2, h_2} = \delta_{g_1} u_{g_1 h_1 - 1} \delta_{g_2} u_{g_2 h_2 - 1} = \delta_{g_1} \alpha_{g_1 h_1 - 1}(\delta_{g_2}) u_{g_1 h_1 - 1} u_{g_2 h_2 - 1}$.

N. C. Phillips (U of Oregon)
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$. 

Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above),
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi\{g\}$.
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi\{g\}$. Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$. 
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi\{g\}$. Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. 
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi\{g\}$. Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. (Exercise: Prove this.)
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi\{g\}$. Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. (Exercise: Prove this.) For $g, h \in G$, set $v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha)$. 

N. C. Phillips (U of Oregon) Crossed Products; the Rokhlin Property 15 July 2016 13 / 41
Examples of crossed products (continued)

Let \( G \) be a finite group, acting on \( C(G) \) via the translation action on \( G \). That is, the action \( \alpha : G \to \text{Aut}(C(G)) \) is \( \alpha_g(a)(h) = a(g^{-1}h) \) for \( g, h \in G \) and \( a \in C(G) \).

Set \( n = \text{card}(G) \). We describe how to prove that \( C^*(G, C(G)) \cong M_n \). This calculation plays a key role later.

Recall multiplication in the crossed product: \((a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}\).

For \( g \in G \), we let \( u_g \) be the standard unitary (as above), and we let \( \delta_g \in C(G) \) be the function \( \chi\{g\} \). Thus \( \sum_{g \in G} \delta_g = 1 \) in \( C(G) \). Then \( \alpha_g(\delta_h) = \delta_{gh} \) for \( g, h \in G \). (Exercise: Prove this.) For \( g, h \in G \), set

\[ v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha). \]

These elements form a system of matrix units. We calculate:
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi\{g\}$. Thus $\sum_{g\in G} \delta_g = 1$ in $C(G)$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. (Exercise: Prove this.) For $g, h \in G$, set

$$v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha).$$

These elements form a system of matrix units. We calculate:

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}} = \delta_{g_1} \alpha_{g_1 h_1^{-1}} \delta_{g_2} u_{g_1 h_1^{-1} u_{g_2 h_2^{-1}}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*\left(G, C(G)\right) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi\{g\}$. Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. (Exercise: Prove this.) For $g, h \in G$, set

$$v_{g,h} = \delta_g u_{gh^{-1}} \in C^*\left(G, C(G), \alpha\right).$$

These elements form a system of matrix units. We calculate:

$$v_{g_1,h_1}v_{g_2,h_2} = \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}}$$

$$= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1}g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1}g_2} u_{g_1 h_1^{-1}g_2 h_2^{-1}}.$$
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. That is, the action $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \text{card}(G)$. We describe how to prove that $C^*(G, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let $u_g$ be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi_{\{g\}}$. Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. (Exercise: Prove this.) For $g, h \in G$, set

$$v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha).$$

These elements form a system of matrix units. We calculate:

$$v_{g_1, h_1} v_{g_2, h_2} = \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}}$$

$$= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1} g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \rightarrow \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$. 

Similarly (do it as an exercise), $v^*g, h = v h, g$. Since the elements $\delta_g$ span $C(C(G))$, the elements $v g, h$ span $C^*(G, C(G), \alpha)$. 

So $C^*(G, C(G), \alpha) \cong M_n$ with $n = \text{card}(G)$. 

Exercise: Write out a complete proof.
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \quad \text{and} \quad \nu_{g, h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \quad \text{and} \quad \nu_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$

and we got

$$\nu_{g_1,h_1} \nu_{g_2,h_2} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha: G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \simeq M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \quad \text{and} \quad v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$

and we got

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$ 

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1,h_2}$. 

Similarly (do it as an exercise), $v^*_{g,h} = v_{h,g}$. Since the elements $\delta_g$ span $C(G)$, the elements $v_{g,h}$ span $C^*(G, C(G), \alpha)$. So $C^*(G, C(G), \alpha) \simeq M_n$ with $n = \text{card}(G)$.

Exercise: Write out a complete proof.
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G)$$

and

$$v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$

and we got

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} \delta_{g_1h_1^{-1}g_2} u_{g_1h_1^{-1}g_2h_2^{-1}}.$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1,h_2}$. This is what matrix units are supposed to do.
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \rightarrow \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \quad \text{and} \quad v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$

and we got

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$ 

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1,h_2}$. This is what matrix units are supposed to do. Similarly (do it as an exercise), $v_{g,h}^* = v_{h,g}$. 

N. C. Phillips (U of Oregon) 
Crossed Products; the Rokhlin Property 
15 July 2016 14 / 41
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \quad \text{and} \quad v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$

and we got

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1,h_2}$. This is what matrix units are supposed to do. Similarly (do it as an exercise), $v_{g,h}^* = v_{h,g}$.

Since the elements $\delta_g$ span $C(G)$, the elements $v_{g,h}$ span $C^*(G, C(G), \alpha)$. 

Examples of crossed products (continued)

Let $G$ be a finite group, $n = \text{card}(G)$, and $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \quad \text{and} \quad \nu_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$

and we got

$$\nu_{g_1,h_1} \nu_{g_2,h_2} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $\nu_{g_1,h_2}$. This is what matrix units are supposed to do. Similarly (do it as an exercise), $\nu_{g,h}^* = \nu_{h,g}$.

Since the elements $\delta_g$ span $C(G)$, the elements $\nu_{g,h}$ span $C^*(G, C(G), \alpha)$. So $C^*(G, C(G), \alpha) \cong M_n$ with $n = \text{card}(G)$. 

Exercise: Write out a complete proof.
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \quad \text{and} \quad v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$

and we got

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$ 

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1,h_2}$. This is what matrix units are supposed to do. Similarly (do it as an exercise), $v^*_{g,h} = v_{h,g}$.

Since the elements $\delta_g$ span $C(G)$, the elements $v_{g,h}$ span $C^*(G, C(G), \alpha)$. So $C^*(G, C(G), \alpha) \cong M_n$ with $n = \text{card}(G)$. Exercise: Write out a complete proof.
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \rightarrow \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in \text{C}(G) \quad \text{and} \quad v_{g,h} = \delta_g u_{gh^{-1}} \in \text{C}^*(G, C(G), \alpha),$$

and we got

$$v_{g_1,h_1}v_{g_2,h_2} = \delta_{g_1} \delta_{g_1h_1^{-1}g_2} u_{g_1h_1^{-1}g_2h_2^{-1}}.$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1,h_2}$. This is what matrix units are supposed to do. Similarly (do it as an exercise), $v_{g,h}^* = v_{h,g}$.

Since the elements $\delta_g$ span $C(G)$, the elements $v_{g,h}$ span $C^*(G, C(G), \alpha)$. So $C^*(G, C(G), \alpha) \cong M_n$ with $n = \text{card}(G)$. Exercise: Write out a complete proof.
Examples of crossed products (continued)

$G$ is a finite group, $n = \text{card}(G)$, and $\alpha : G \to \text{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \quad \text{and} \quad v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C(G), \alpha),$$

and we got

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} \delta_{g_1h_1^{-1}g_2} u_{g_1h_1^{-1}g_2h_2^{-1}}.$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1,h_2}$. This is what matrix units are supposed to do. Similarly (do it as an exercise), $v_{g,h}^* = v_{h,g}$.

Since the elements $\delta_g$ span $C(G)$, the elements $v_{g,h}$ span $C^*(G, C(G), \alpha)$. So $C^*(G, C(G), \alpha) \cong M_n$ with $n = \text{card}(G)$. Exercise: Write out a complete proof.
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. (That is, $\alpha_g(f)(h) = f(g^{-1}h)$.) Set $n = \text{card}(G)$. Then $C^*(G, C(G)) \cong M_n$. 

Exercise: Use the same method to prove that $C^*(G, C^0(G \times X)) \cong C^0(X, M_n)$ (which is $M_n \otimes C^0(X)$).

A harder exercise: Prove that for any action of $G$ on $X$, and using the diagonal action on $G \times X$, we still have $C^*(G, C^0(G \times X)) \cong C^0(X, M_n)$.

Hint: A trick reduces this to the previous exercise. 

This result generalizes greatly: for any locally compact group $G$, one gets $C^*(G, C^0(G)) \cong K(L_2(G))$, etc.
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. (That is, $\alpha_g(f)(h) = f(g^{-1}h)$.) Set $n = \text{card}(G)$. Then $C^*(G, C(G)) \cong M_n$.

Now consider $G$ acting on $G \times X$, by translation on $G$ and trivially on $X$. 
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. (That is, $\alpha_g(f)(h) = f(g^{-1}h)$.) Set $n = \text{card}(G)$. Then $C^*(G, C(G)) \cong M_n$.

Now consider $G$ acting on $G \times X$, by translation on $G$ and trivially on $X$. Exercise: Use the same method to prove that $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$ (which is $M_n \otimes C_0(X)$).
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. (That is, $\alpha_g(f)(h) = f(g^{-1}h)$.) Set $n = \text{card}(G)$. Then $C^*(G, C(G)) \cong M_n$.

Now consider $G$ acting on $G \times X$, by translation on $G$ and trivially on $X$. Exercise: Use the same method to prove that $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$ (which is $M_n \otimes C_0(X)$).

A harder exercise: Prove that for any action of $G$ on $X$, and using the diagonal action on $G \times X$, we still have $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$. 

Hint: A trick reduces this to the previous exercise. 

This result generalizes greatly: for any locally compact group $G$, one gets $C^*(G, C_0(G)) \cong K(L^2(G))$, etc.
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. (That is, $\alpha_g(f)(h) = f(g^{-1}h)$.) Set $n = \text{card}(G)$. Then $C^*(G, C(G)) \cong M_n$.

Now consider $G$ acting on $G \times X$, by translation on $G$ and trivially on $X$. Exercise: Use the same method to prove that $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$ (which is $M_n \otimes C_0(X)$).

A harder exercise: Prove that for any action of $G$ on $X$, and using the diagonal action on $G \times X$, we still have $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$. Hint: A trick reduces this to the previous exercise.
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. (That is, $\alpha_g(f)(h) = f(g^{-1}h).$) Set $n = \text{card}(G)$. Then $C^*(G, C(G)) \cong M_n$.

Now consider $G$ acting on $G \times X$, by translation on $G$ and trivially on $X$. Exercise: Use the same method to prove that $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$ (which is $M_n \otimes C_0(X)$).

A harder exercise: Prove that for any action of $G$ on $X$, and using the diagonal action on $G \times X$, we still have $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$. Hint: A trick reduces this to the previous exercise.

This result generalizes greatly: for any locally compact group $G$, one gets $C^*(G, C_0(G)) \cong K(L^2(G))$, etc.
Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. (That is, $\alpha_g(f)(h) = f(g^{-1}h)$.) Set $n = \text{card}(G)$. Then $C^*(G, C(G)) \cong M_n$.

Now consider $G$ acting on $G \times X$, by translation on $G$ and trivially on $X$. Exercise: Use the same method to prove that $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$ (which is $M_n \otimes C_0(X)$).

A harder exercise: Prove that for any action of $G$ on $X$, and using the diagonal action on $G \times X$, we still have $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$. Hint: A trick reduces this to the previous exercise.

This result generalizes greatly: for any locally compact group $G$, one gets $C^*(G, C_0(G)) \cong K(L^2(G))$, etc.
Examples of crossed products (continued)

Let $G$ be a finite group, acting on $C(G)$ via the translation action on $G$. (That is, $\alpha_g(f)(h) = f(g^{-1}h)$. ) Set $n = \text{card}(G)$. Then $C^*(G, C(G)) \cong M_n$.

Now consider $G$ acting on $G \times X$, by translation on $G$ and trivially on $X$. Exercise: Use the same method to prove that $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$ (which is $M_n \otimes C_0(X)$).

A harder exercise: Prove that for any action of $G$ on $X$, and using the diagonal action on $G \times X$, we still have $C^*(G, C_0(G \times X)) \cong C_0(X, M_n)$. Hint: A trick reduces this to the previous exercise.

This result generalizes greatly: for any locally compact group $G$, one gets $C^*(G, C_0(G)) \cong K(L^2(G))$, etc.
Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.
Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For $\alpha: G \to \text{Aut}(A)$ and $\beta: G \to \text{Aut}(B)$, we say that a homomorphism $\varphi: A \to B$ is \textit{equivariant} if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. 
Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$, we say that a homomorphism $\varphi : A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. That is, for all $g \in G$, the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\alpha_g} & & \downarrow{\alpha_g} \\
A & \xrightarrow{\varphi} & B
\end{array}
$$
Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$, we say that a homomorphism $\varphi : A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. That is, for all $g \in G$, the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\alpha_g} & & \downarrow{\alpha_g} \\
A & \xrightarrow{\varphi} & B
\end{array}
$$

An equivariant homomorphism $\varphi : A \to B$ induces a homomorphism

$$
\overline{\varphi} : C^*(G, A, \alpha) \to C^*(G, B, \beta),
$$
Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$, we say that a homomorphism $\varphi : A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. That is, for all $g \in G$, the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\alpha_g} & & \downarrow{\alpha_g} \\
A & \xrightarrow{\varphi} & B
\end{array}
\]

An equivariant homomorphism $\varphi : A \to B$ induces a homomorphism $\overline{\varphi} : C^*(G, A, \alpha) \to C^*(G, B, \beta)$, just by applying $\varphi$ to the algebra elements.
Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For \( \alpha: G \to \text{Aut}(A) \) and \( \beta: G \to \text{Aut}(B) \), we say that a homomorphism \( \varphi: A \to B \) is equivariant if \( \varphi(\alpha_g(a)) = \beta_g(\varphi(a)) \) for all \( g \in G \) and \( a \in A \). That is, for all \( g \in G \), the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\alpha_g} & & \downarrow{\alpha_g} \\
A & \xrightarrow{\varphi} & B
\end{array}
\]

An equivariant homomorphism \( \varphi: A \to B \) induces a homomorphism

\[
\overline{\varphi}: C^*(G, A, \alpha) \to C^*(G, B, \beta),
\]

just by applying \( \varphi \) to the algebra elements.
Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$, we say that a homomorphism $\varphi : A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. That is, for all $g \in G$, the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\alpha_g} & & \downarrow{\alpha_g} \\
A & \xrightarrow{\varphi} & B
\end{array}
$$

An equivariant homomorphism $\varphi : A \to B$ induces a homomorphism $\overline{\varphi} : C^*(G, A, \alpha) \to C^*(G, B, \beta)$, just by applying $\varphi$ to the algebra elements.
Equivariant homomorphisms (continued)

φ: A → B is equivariant if φ(α_g(a)) = β_g(φ(a)) for all g ∈ G and a ∈ A.
Equivariant homomorphisms (continued)

ϕ: A → B is equivariant if ϕ(α_g(a)) = β_g(ϕ(a)) for all g ∈ G and a ∈ A. We get

\[ \overline{\varphi}: C^*(G, A, \alpha) \to C^*(G, B, \beta) \]

by applying ϕ to the algebra elements.

Exercises: Assume that G is finite. Prove that ϕ is a *-homomorphism, that if ϕ is injective then so is ϕ, and that if ϕ is surjective then so is ϕ. (Warning: the surjectivity result is true for general G, but the injectivity result can fail if G is not amenable.)
Equivariant homomorphisms (continued)

φ: A → B is equivariant if φ(α_g(a)) = β_g(φ(a)) for all g ∈ G and a ∈ A. We get

ψ: C*(G, A, α) → C*(G, B, β)

by applying φ to the algebra elements.

Thus, if G is discrete, the standard unitaries in C*(G, A, α) are called u_g, and the standard unitaries in C*(G, B, β) are called v_g, then
Equivariant homomorphisms (continued)

$\varphi: A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. We get

$$\overline{\varphi}: C^*(G, A, \alpha) \to C^*(G, B, \beta)$$

by applying $\varphi$ to the algebra elements.

Thus, if $G$ is discrete, the standard unitaries in $C^*(G, A, \alpha)$ are called $u_g$, and the standard unitaries in $C^*(G, B, \beta)$ are called $v_g$, then

$$\overline{\varphi}\left(\sum_{g \in G} c_g u_g\right) = \sum_{g \in G} \varphi(c_g) v_g.$$
Equivariant homomorphisms (continued)

$\varphi: A \to B$ is **equivariant** if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. We get

$$\overline{\varphi}: C^*(G, A, \alpha) \to C^*(G, B, \beta)$$

by applying $\varphi$ to the algebra elements.

Thus, if $G$ is discrete, the standard unitaries in $C^*(G, A, \alpha)$ are called $u_g$, and the standard unitaries in $C^*(G, B, \beta)$ are called $v_g$, then

$$\overline{\varphi} \left( \sum_{g \in G} c_g u_g \right) = \sum_{g \in G} \varphi(c_g) v_g.$$

Exercises: Assume that $G$ is finite. Prove that $\overline{\varphi}$ is a *-homomorphism,
Equivariant homomorphisms (continued)

$\varphi : A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. We get

$$\overline{\varphi} : C^*(G, A, \alpha) \to C^*(G, B, \beta)$$

by applying $\varphi$ to the algebra elements.

Thus, if $G$ is discrete, the standard unitaries in $C^*(G, A, \alpha)$ are called $u_g$, and the standard unitaries in $C^*(G, B, \beta)$ are called $v_g$, then

$$\overline{\varphi} \left( \sum_{g \in G} c_g u_g \right) = \sum_{g \in G} \varphi(c_g) v_g.$$ 

Exercises: Assume that $G$ is finite. Prove that $\overline{\varphi}$ is a $*$-homomorphism, that if $\varphi$ is injective then so is $\overline{\varphi}$,
Equivariant homomorphisms (continued)

\( \varphi: A \to B \) is *equivariant* if \( \varphi(\alpha_g(a)) = \beta_g(\varphi(a)) \) for all \( g \in G \) and \( a \in A \).

We get

\( \overline{\varphi}: C^*(G, A, \alpha) \to C^*(G, B, \beta) \)

by applying \( \varphi \) to the algebra elements.

Thus, if \( G \) is discrete, the standard unitaries in \( C^*(G, A, \alpha) \) are called \( u_g \), and the standard unitaries in \( C^*(G, B, \beta) \) are called \( v_g \), then

\[
\overline{\varphi} \left( \sum_{g \in G} c_g u_g \right) = \sum_{g \in G} \varphi(c_g) v_g.
\]

Exercises: Assume that \( G \) is finite. Prove that \( \overline{\varphi} \) is a *-homomorphism, that if \( \varphi \) is injective then so is \( \overline{\varphi} \), and that if \( \varphi \) is surjective then so is \( \overline{\varphi} \).
Equivariant homomorphisms (continued)

\( \varphi: A \to B \) is **equivariant** if \( \varphi(\alpha_g(a)) = \beta_g(\varphi(a)) \) for all \( g \in G \) and \( a \in A \). We get

\[ \overline{\varphi}: C^*(G, A, \alpha) \to C^*(G, B, \beta) \]

by applying \( \varphi \) to the algebra elements.

Thus, if \( G \) is discrete, the standard unitaries in \( C^*(G, A, \alpha) \) are called \( u_g \), and the standard unitaries in \( C^*(G, B, \beta) \) are called \( v_g \), then

\[ \overline{\varphi}\left( \sum_{g \in G} c_g u_g \right) = \sum_{g \in G} \varphi(c_g) v_g. \]

Exercises: Assume that \( G \) is finite. Prove that \( \overline{\varphi} \) is a *-homomorphism, that if \( \varphi \) is injective then so is \( \overline{\varphi} \), and that if \( \varphi \) is surjective then so is \( \overline{\varphi} \). (Warning: the surjectivity result is true for general \( G \), but the injectivity result can fail if \( G \) is not amenable.)
Equivariant homomorphisms (continued)

ϕ: A → B is equivariant if ϕ(α_g(a)) = β_g(ϕ(a)) for all g ∈ G and a ∈ A.

We get

\[ \overline{\varphi}: C^*(G, A, \alpha) \to C^*(G, B, \beta) \]

by applying ϕ to the algebra elements.

Thus, if G is discrete, the standard unitaries in C^*(G, A, \alpha) are called u_g, and the standard unitaries in C^*(G, B, \beta) are called v_g, then

\[ \overline{\varphi} \left( \sum_{g \in G} c_g u_g \right) = \sum_{g \in G} \varphi(c_g) v_g. \]

Exercises: Assume that G is finite. Prove that \overline{\varphi} is a *-homomorphism, that if ϕ is injective then so is \overline{\varphi}, and that if ϕ is surjective then so is \overline{\varphi}.

(Warning: the surjectivity result is true for general G, but the injectivity result can fail if G is not amenable.)
Equivariant homomorphisms (continued)

φ: A → B is equivariant if φ(α_g(a)) = β_g(φ(a)) for all g ∈ G and a ∈ A. We get

\[ \overline{\varphi}: C^*(G, A, \alpha) \to C^*(G, B, \beta) \]

by applying φ to the algebra elements.

Thus, if G is discrete, the standard unitaries in \( C^*(G, A, \alpha) \) are called \( u_g \), and the standard unitaries in \( C^*(G, B, \beta) \) are called \( v_g \), then

\[ \overline{\varphi}\left( \sum_{g \in G} c_g u_g \right) = \sum_{g \in G} \varphi(c_g) v_g. \]

Exercises: Assume that G is finite. Prove that \( \overline{\varphi} \) is a *-homomorphism, that if \( \varphi \) is injective then so is \( \overline{\varphi} \), and that if \( \varphi \) is surjective then so is \( \overline{\varphi} \). (Warning: the surjectivity result is true for general G, but the injectivity result can fail if G is not amenable.)
Digression: Conjugacy

For $\alpha: G \to \text{Aut}(A)$ and $\beta: G \to \text{Aut}(B)$, we say that a homomorphism $\varphi: A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. If $\varphi$ is an isomorphism, we say it is a conjugacy. If there is such a map, the $\text{C}^*$ dynamical systems $(G, A, \alpha)$ and $(G, B, \beta)$ are conjugate. This is the right version of isomorphism for $\text{C}^*$ dynamical systems. Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if $G$ is locally compact and $\varphi$ is a conjugacy, then $\varphi$ induces an isomorphism from $\text{C}^*(G, A, \alpha)$ to $\text{C}^*(G, B, \beta)$.
Digression: Conjugacy

For $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$, we say that a homomorphism $\varphi : A \to B$ is \textit{equivariant} if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. If $\varphi$ is an isomorphism, we say it is a \textit{conjugacy}. If there is such a map, the $C^*$ dynamical systems $(G, A, \alpha)$ and $(G, B, \beta)$ are \textit{conjugate}. 

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of $\mathbb{Z}_2$ on $A = \bigotimes_{\infty}^{\infty} M_2 \cong \bigotimes_{\infty}^{\infty} \text{Ad}(1,0;0,-1)$ and $\bigotimes_{\infty}^{\infty} \text{Ad}(0,1;1,0)$ are "essentially the same". The correct statement is that these actions are \textit{conjugate}. Exercise: prove this. Hint: Find a unitary $w \in M_2$ such that $w(1,0;0,-1)w^* = (0,1;1,0)$, and take $\varphi = \bigotimes_{\infty}^{\infty} \text{Ad}(w)$. 

N. C. Phillips (U of Oregon) 
Crossed Products; the Rokhlin Property 
15 July 2016 18 / 41
Digression: Conjugacy

For $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$, we say that a homomorphism $\varphi : A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

If $\varphi$ is an isomorphism, we say it is a conjugacy. If there is such a map, the C* dynamical systems $(G, A, \alpha)$ and $(G, B, \beta)$ are conjugate. This is the right version of isomorphism for C* dynamical systems.
Digression: Conjugacy

For \( \alpha : G \to \text{Aut}(A) \) and \( \beta : G \to \text{Aut}(B) \), we say that a homomorphism \( \varphi : A \to B \) is equivariant if \( \varphi(\alpha_g(a)) = \beta_g(\varphi(a)) \) for all \( g \in G \) and \( a \in A \).

If \( \varphi \) is an isomorphism, we say it is a conjugacy. If there is such a map, the C* dynamical systems \( (G, A, \alpha) \) and \( (G, B, \beta) \) are conjugate. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if \( G \) is locally compact and \( \varphi \) is a conjugacy, then \( \varphi \) induces an isomorphism from \( C^*(G, A, \alpha) \) to \( C^*(G, B, \beta) \).
Digression: Conjugacy

For $\alpha: G \to \text{Aut}(A)$ and $\beta: G \to \text{Aut}(B)$, we say that a homomorphism $\varphi: A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

If $\varphi$ is an isomorphism, we say it is a conjugacy. If there is such a map, the C* dynamical systems $(G, A, \alpha)$ and $(G, B, \beta)$ are conjugate. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if $G$ is locally compact and $\varphi$ is a conjugacy, then $\varphi$ induces an isomorphism from $C^*(G, A, \alpha)$ to $C^*(G, B, \beta)$.

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of $\mathbb{Z}_2$ on $A = \bigotimes_{n=1}^\infty M_2$ generated by

$$\bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are “essentially the same”.

Exercise: prove this. Hint: Find a unitary $w \in M_2$ such that $w \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and take $\varphi = \bigotimes_{n=1}^\infty \text{Ad}(w)$. 

N. C. Phillips (U of Oregon)
Digression: Conjugacy

For \( \alpha : G \to \text{Aut}(A) \) and \( \beta : G \to \text{Aut}(B) \), we say that a homomorphism \( \varphi : A \to B \) is equivariant if \( \varphi(\alpha_g(a)) = \beta_g(\varphi(a)) \) for all \( g \in G \) and \( a \in A \).

If \( \varphi \) is an isomorphism, we say it is a conjugacy. If there is such a map, the C* dynamical systems \((G, A, \alpha)\) and \((G, B, \beta)\) are conjugate. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if \( G \) is locally compact and \( \varphi \) is a conjugacy, then \( \varphi \) induces an isomorphism from \( C^*(G, A, \alpha) \) to \( C^*(G, B, \beta) \).

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of \( \mathbb{Z}_2 \) on \( A = \bigotimes_{n=1}^{\infty} M_2 \) generated by

\[
\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

are “essentially the same”. The correct statement is that these actions are conjugate.
Digression: Conjugacy

For $\alpha: G \to \text{Aut}(A)$ and $\beta: G \to \text{Aut}(B)$, we say that a homomorphism $\varphi: A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

If $\varphi$ is an isomorphism, we say it is a conjugacy. If there is such a map, the C* dynamical systems $(G, A, \alpha)$ and $(G, B, \beta)$ are conjugate. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if $G$ is locally compact and $\varphi$ is a conjugacy, then $\varphi$ induces an isomorphism from $C^*(G, A, \alpha)$ to $C^*(G, B, \beta)$.

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of $\mathbb{Z}_2$ on $A = \bigotimes_{n=1}^{\infty} M_2$ generated by

$$
\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

are “essentially the same”. The correct statement is that these actions are conjugate. Exercise: prove this.
Digression: Conjugacy

For $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$, we say that a homomorphism $\varphi : A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

If $\varphi$ is an isomorphism, we say it is a conjugacy. If there is such a map, the C* dynamical systems $(G, A, \alpha)$ and $(G, B, \beta)$ are conjugate. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if $G$ is locally compact and $\varphi$ is a conjugacy, then $\varphi$ induces an isomorphism from $C^*(G, A, \alpha)$ to $C^*(G, B, \beta)$.

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of $\mathbb{Z}_2$ on $A = \bigotimes_{n=1}^{\infty} M_2$ generated by

$$
\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

are “essentially the same”. The correct statement is that these actions are conjugate. Exercise: prove this. Hint: Find a unitary $w \in M_2$ such that $w \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and take $\varphi = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$. 

N. C. Phillips (U of Oregon) | Crossed Products; the Rokhlin Property | 15 July 2016 | 18 / 41
Digression: Conjugacy

For $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$, we say that a homomorphism $\varphi : A \to B$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

If $\varphi$ is an isomorphism, we say it is a conjugacy. If there is such a map, the C* dynamical systems $(G, A, \alpha)$ and $(G, B, \beta)$ are conjugate. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if $G$ is locally compact and $\varphi$ is a conjugacy, then $\varphi$ induces an isomorphism from $C^*(G, A, \alpha)$ to $C^*(G, B, \beta)$.

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of $\mathbb{Z}_2$ on $A = \bigotimes_{n=1}^{\infty} M_2$ generated by

$$\bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$

and

$$\bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$$

are “essentially the same”. The correct statement is that these actions are conjugate. Exercise: prove this. Hint: Find a unitary $w \in M_2$ such that

$$w \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) w^* = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),$$

and take $\varphi = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$. 

N. C. Phillips (U of Oregon)

Crossed Products; the Rokhlin Property

15 July 2016 18 / 41
Digression: Conjugacy

For $\alpha: G \to \text{Aut}(A)$ and $\beta: G \to \text{Aut}(B)$, we say that a homomorphism $\varphi: A \to B$ is **equivariant** if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

If $\varphi$ is an isomorphism, we say it is a **conjugacy**. If there is such a map, the C* dynamical systems $(G, A, \alpha)$ and $(G, B, \beta)$ are **conjugate**. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if $G$ is locally compact and $\varphi$ is a conjugacy, then $\varphi$ induces an isomorphism from $C^*(G, A, \alpha)$ to $C^*(G, B, \beta)$.

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of $\mathbb{Z}_2$ on $A = \bigotimes_{n=1}^{\infty} M_2$ generated by

$$
\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

and

$$
\bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

are “essentially the same”. The correct statement is that these actions are conjugate. Exercise: prove this. Hint: Find a unitary $w \in M_2$ such that $w \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and take $\varphi = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$. 
Equivariant exact sequences

The homomorphism \( \varphi \) is equivariant if \( \varphi(\alpha_g(a)) = \beta_g(\varphi(a)) \) for all \( g \in G \) and \( a \in A \).

Recall that equivariant homomorphisms induce homomorphisms of crossed products.
Equivariant exact sequences

The homomorphism \( \varphi \) is equivariant if \( \varphi(\alpha_g(a)) = \beta_g(\varphi(a)) \) for all \( g \in G \) and \( a \in A \).

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

**Theorem**

Let \( G \) be a locally compact group. Let \( 0 \to J \to A \to B \to 0 \) be an exact sequence of \( C^* \)-algebras with actions \( \gamma \) of \( G \) on \( J \), \( \alpha \) of \( G \) on \( A \), and \( \beta \) of \( G \) on \( B \), and equivariant maps.
Equivariant exact sequences

The homomorphism \( \varphi \) is equivariant if \( \varphi(\alpha_g(a)) = \beta_g(\varphi(a)) \) for all \( g \in G \) and \( a \in A \).

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

Theorem

Let \( G \) be a locally compact group. Let \( 0 \to J \to A \to B \to 0 \) be an exact sequence of \( \text{C}^* \)-algebras with actions \( \gamma \) of \( G \) on \( J \), \( \alpha \) of \( G \) on \( A \), and \( \beta \) of \( G \) on \( B \), and equivariant maps. Then the sequence

\[
0 \to C^*(G, J, \gamma) \to C^*(G, A, \alpha) \to C^*(G, B, \beta) \to 0
\]

is exact.
Equivariant exact sequences

The homomorphism $\varphi$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

**Theorem**

Let $G$ be a locally compact group. Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be an exact sequence of C*-algebras with actions $\gamma$ of $G$ on $J$, $\alpha$ of $G$ on $A$, and $\beta$ of $G$ on $B$, and equivariant maps. Then the sequence

$$0 \rightarrow C^*(G, J, \gamma) \rightarrow C^*(G, A, \alpha) \rightarrow C^*(G, B, \beta) \rightarrow 0$$

is exact.

When $G$ is finite, the proof is easy: consider

$$0 \rightarrow J[G] \rightarrow A[G] \rightarrow B[G] \rightarrow 0$$
Equivariant exact sequences

The homomorphism $\varphi$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

**Theorem**

Let $G$ be a locally compact group. Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be an exact sequence of C*-algebras with actions $\gamma$ of $G$ on $J$, $\alpha$ of $G$ on $A$, and $\beta$ of $G$ on $B$, and equivariant maps. Then the sequence

$$0 \rightarrow C^*(G, J, \gamma) \rightarrow C^*(G, A, \alpha) \rightarrow C^*(G, B, \beta) \rightarrow 0$$

is exact.

When $G$ is finite, the proof is easy: consider

$$0 \rightarrow J[G] \rightarrow A[G] \rightarrow B[G] \rightarrow 0$$

Exercise: Do it.
Equivariant exact sequences

The homomorphism $\varphi$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

**Theorem**

Let $G$ be a locally compact group. Let $0 \to J \to A \to B \to 0$ be an exact sequence of $C^*$-algebras with actions $\gamma$ of $G$ on $J$, $\alpha$ of $G$ on $A$, and $\beta$ of $G$ on $B$, and equivariant maps. Then the sequence

$$0 \to C^*(G, J, \gamma) \to C^*(G, A, \alpha) \to C^*(G, B, \beta) \to 0$$

is exact.

When $G$ is finite, the proof is easy: consider

$$0 \to J[G] \to A[G] \to B[G] \to 0$$

Exercise: Do it. (You already proved exactness at $J[G]$ and $B[G]$ in a previous exercise.)
Equivariant exact sequences

The homomorphism $\varphi$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

**Theorem**

Let $G$ be a locally compact group. Let $0 \to J \to A \to B \to 0$ be an exact sequence of C*-algebras with actions $\gamma$ of $G$ on $J$, $\alpha$ of $G$ on $A$, and $\beta$ of $G$ on $B$, and equivariant maps. Then the sequence

$$0 \to C^*(G, J, \gamma) \to C^*(G, A, \alpha) \to C^*(G, B, \beta) \to 0$$

is exact.

When $G$ is finite, the proof is easy: consider

$$0 \to J[G] \to A[G] \to B[G] \to 0$$

Exercise: Do it. (You already proved exactness at $J[G]$ and $B[G]$ in a previous exercise.)
Equivariant exact sequences

The homomorphism $\varphi$ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

**Theorem**

Let $G$ be a locally compact group. Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be an exact sequence of $C^*$-algebras with actions $\gamma$ of $G$ on $J$, $\alpha$ of $G$ on $A$, and $\beta$ of $G$ on $B$, and equivariant maps. Then the sequence

$$0 \rightarrow C^*(G, J, \gamma) \rightarrow C^*(G, A, \alpha) \rightarrow C^*(G, B, \beta) \rightarrow 0$$

is exact.

When $G$ is finite, the proof is easy: consider

$$0 \rightarrow J[G] \rightarrow A[G] \rightarrow B[G] \rightarrow 0$$

Exercise: Do it. (You already proved exactness at $J[G]$ and $B[G]$ in a previous exercise.)
Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_n$ acts on the circle $S^1$ by rotation, with the standard generator acting by multiplication by $\omega = e^{2\pi i/n}$.

For any point $x \in S^1$, let $L_x = \{\omega^k x : k = 0, 1, \ldots, n-1\}$ and $U_x = S^1 \setminus L_x$.

Then $L_x$ is equivariantly homeomorphic to $\mathbb{Z}_n$ with translation, and $U_x$ is equivariantly homeomorphic to $\mathbb{Z}_n \times \{e^{2\pi i t/x} : 0 < t < 1\} \sim \mathbb{Z}_n \times (0, 1)$.

The equivariant exact sequence $0 \to C_0(U_x) \to C(S^1) \to C(L_x) \to 0$ gives the following exact sequence of crossed products:

$0 \to C_0((0, 1), M_n) \to C^*(\mathbb{Z}_n, C(S^1)) \to M_n \to 0$.

With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_n, C(S^1)) \sim C(S^1, M_n)$. The copy of $S^1$ on the right arises as the orbit space $S^1 / \mathbb{Z}_n$. 

N. C. Phillips (U of Oregon)

Crossed Products; the Rokhlin Property
Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_n$ acts on the circle $S^1$ by rotation, with the standard generator acting by multiplication by $\omega = e^{2\pi i/n}$.

For any point $x \in S^1$, let

$$L_x = \{\omega^k x : k = 0, 1, \ldots, n - 1\} \quad \text{and} \quad U_x = S^1 \setminus L_x.$$
Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_n$ acts on the circle $S^1$ by rotation, with the standard generator acting by multiplication by $\omega = e^{2\pi i / n}$.

For any point $x \in S^1$, let

$$L_x = \{\omega^k x : k = 0, 1, \ldots, n - 1\}$$

and

$$U_x = S^1 \setminus L_x.$$

Then $L_x$ is equivariantly homeomorphic to $\mathbb{Z}_n$ with translation,
Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_n$ acts on the circle $S^1$ by rotation, with the standard generator acting by multiplication by $\omega = e^{2\pi i/n}$.

For any point $x \in S^1$, let

$$L_x = \{\omega^k x : k = 0, 1, \ldots, n-1\} \quad \text{and} \quad U_x = S^1 \setminus L_x.$$  

Then $L_x$ is equivariantly homeomorphic to $\mathbb{Z}_n$ with translation, and $U_x$ is equivariantly homeomorphic to

$$\mathbb{Z}_n \times \{e^{2\pi it/n} x : 0 < t < 1\} \cong \mathbb{Z}_n \times (0, 1).$$
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_n \) acts on the circle \( S^1 \) by rotation, with the standard generator acting by multiplication by \( \omega = e^{2\pi i/n} \).

For any point \( x \in S^1 \), let

\[
L_x = \{ \omega^k x : k = 0, 1, \ldots, n - 1 \} \quad \text{and} \quad U_x = S^1 \setminus L_x.
\]

Then \( L_x \) is equivariantly homeomorphic to \( \mathbb{Z}_n \) with translation, and \( U_x \) is equivariantly homeomorphic to

\[
\mathbb{Z}_n \times \{ e^{2\pi it/n} x : 0 < t < 1 \} \cong \mathbb{Z}_n \times (0, 1).
\]

The equivariant exact sequence

\[
0 \longrightarrow C_0(U_x) \longrightarrow C(S^1) \longrightarrow C(L_x) \longrightarrow 0
\]
Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_n$ acts on the circle $S^1$ by rotation, with the standard generator acting by multiplication by $\omega = e^{2\pi i/n}$.

For any point $x \in S^1$, let

$$L_x = \{\omega^k x : k = 0, 1, \ldots, n - 1\} \quad \text{and} \quad U_x = S^1 \setminus L_x.$$  

Then $L_x$ is equivariantly homeomorphic to $\mathbb{Z}_n$ with translation, and $U_x$ is equivariantly homeomorphic to

$$\mathbb{Z}_n \times \{e^{2\pi it/n} x : 0 < t < 1\} \cong \mathbb{Z}_n \times (0, 1).$$

The equivariant exact sequence

$$0 \to C_0(U_x) \to C(S^1) \to C(L_x) \to 0$$

gives the following exact sequence of crossed products:

$$0 \to C_0((0, 1), M_n) \to C^*(\mathbb{Z}_n, C(S^1)) \to M_n \to 0.$$
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_n \) acts on the circle \( S^1 \) by rotation, with the standard generator acting by multiplication by \( \omega = e^{2\pi i/n} \).

For any point \( x \in S^1 \), let
\[
L_x = \{ \omega^k x : k = 0, 1, \ldots, n - 1 \}
\]
and
\[
U_x = S^1 \setminus L_x.
\]
Then \( L_x \) is equivariantly homeomorphic to \( \mathbb{Z}_n \) with translation, and \( U_x \) is equivariantly homeomorphic to
\[
\mathbb{Z}_n \times \{ e^{2\pi it/n} x : 0 < t < 1 \} \cong \mathbb{Z}_n \times (0, 1).
\]

The equivariant exact sequence
\[
0 \to C_0(U_x) \to C(S^1) \to C(L_x) \to 0
\]
gives the following exact sequence of crossed products:
\[
0 \to C_0((0, 1), M_n) \to C^*(\mathbb{Z}_n, C(S^1)) \to M_n \to 0.
\]

With more work (details are in my crossed product notes), one can show that \( C^*(\mathbb{Z}_n, C(S^1)) \cong C(S^1, M_n) \).
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_n \) acts on the circle \( S^1 \) by rotation, with the standard generator acting by multiplication by \( \omega = e^{2\pi i/n} \).

For any point \( x \in S^1 \), let
\[
L_x = \{ \omega^k x : k = 0, 1, \ldots, n - 1 \} \quad \text{and} \quad U_x = S^1 \setminus L_x.
\]
Then \( L_x \) is equivariantly homeomorphic to \( \mathbb{Z}_n \) with translation, and \( U_x \) is equivariantly homeomorphic to
\[
\mathbb{Z}_n \times \{ e^{2\pi it/n} x : 0 < t < 1 \} \cong \mathbb{Z}_n \times (0, 1).
\]

The equivariant exact sequence
\[
0 \longrightarrow C_0(U_x) \longrightarrow C(S^1) \longrightarrow C(L_x) \longrightarrow 0
\]
gives the following exact sequence of crossed products:
\[
0 \longrightarrow C_0((0, 1), M_n) \longrightarrow C^*(\mathbb{Z}_n, C(S^1)) \longrightarrow M_n \longrightarrow 0.
\]
With more work (details are in my crossed product notes), one can show that \( C^*(\mathbb{Z}_n, C(S^1)) \cong C(S^1, M_n) \). The copy of \( S^1 \) on the right arises as the orbit space \( S^1 / \mathbb{Z}_n \).
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_n \) acts on the circle \( S^1 \) by rotation, with the standard generator acting by multiplication by \( \omega = e^{2\pi i/n} \).

For any point \( x \in S^1 \), let

\[
L_x = \{ \omega^k x : k = 0, 1, \ldots, n - 1 \} \quad \text{and} \quad U_x = S^1 \setminus L_x.
\]

Then \( L_x \) is equivariantly homeomorphic to \( \mathbb{Z}_n \) with translation, and \( U_x \) is equivariantly homeomorphic to

\[
\mathbb{Z}_n \times \{ e^{2\pi it/n} x : 0 < t < 1 \} \cong \mathbb{Z}_n \times (0, 1).
\]

The equivariant exact sequence

\[
0 \longrightarrow C_0(U_x) \longrightarrow C(S^1) \longrightarrow C(L_x) \longrightarrow 0
\]

gives the following exact sequence of crossed products:

\[
0 \longrightarrow C_0((0, 1), M_n) \longrightarrow C^*(\mathbb{Z}_n, C(S^1)) \longrightarrow M_n \longrightarrow 0.
\]

With more work (details are in my crossed product notes), one can show that \( C^*(\mathbb{Z}_n, C(S^1)) \cong C(S^1, M_n) \). The copy of \( S^1 \) on the right arises as the orbit space \( S^1/\mathbb{Z}_n \).
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_n \) acts on the circle \( S^1 \) by rotation, with the standard generator acting by multiplication by \( \omega = e^{2\pi i/n} \).

For any point \( x \in S^1 \), let

\[
L_x = \{ \omega^k x : k = 0, 1, \ldots, n - 1 \}
\]

and

\[
U_x = S^1 \setminus L_x.
\]

Then \( L_x \) is equivariantly homeomorphic to \( \mathbb{Z}_n \) with translation, and \( U_x \) is equivariantly homeomorphic to

\[
\mathbb{Z}_n \times \{ e^{2\pi it/n} x : 0 < t < 1 \} \cong \mathbb{Z}_n \times (0, 1).
\]

The equivariant exact sequence

\[
0 \to C_0(U_x) \to C(S^1) \to C(L_x) \to 0
\]

gives the following exact sequence of crossed products:

\[
0 \to C_0((0, 1), M_n) \to C^*(\mathbb{Z}_n, C(S^1)) \to M_n \to 0.
\]

With more work (details are in my crossed product notes), one can show that \( C^*(\mathbb{Z}_n, C(S^1)) \cong C(S^1, M_n) \). The copy of \( S^1 \) on the right arises as the orbit space \( S^1/\mathbb{Z}_n \).
Examples of crossed products (continued)

We use the standard abbreviation $C^* (G, X) = C^* (G, C_0 (X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$C^* (\mathbb{Z}_n, S^1) \sim C (S^1 / \mathbb{Z}_n, \mathbb{M}_n)$.

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one might hope that $C^* (\mathbb{Z}_2, S^n)$ would be isomorphic to $C (S^n / \mathbb{Z}_2, \mathbb{M}_2)$.

This is almost right, but not quite. In fact, $C^* (\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n / \mathbb{Z}_2$ with fiber $\mathbb{M}_2$, and the bundle is locally trivial—but not trivial.

We still have the general principle: A closed orbit $Gx \sim G / H$ in $X$ gives a quotient in the crossed product isomorphic to $K (L^2 (G / H)) \otimes C^* (H)$.

We illustrate this when $G$ is finite (so that all orbits are closed) and $H$ is either $\{1\}$ (above) or $G$ (for $C^* (G)$, and see the next slide).
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$. 

Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation,
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, M_2)$. 
Examples of crossed products (continued)

We use the standard abbreviation \( C^*(G, X) = C^*(G, C_0(X)) \).

For the action of \( \mathbb{Z}_n \) on the circle \( S^1 \) by rotation, we got

\[
C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).
\]

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^n \) via the order two homeomorphism \( x \mapsto -x \).

Based on what happened with \( \mathbb{Z}_n \) acting on the circle \( S^1 \) by rotation, one might hope that \( C^*(\mathbb{Z}_2, S^n) \) would be isomorphic to \( C(S^n/\mathbb{Z}_2, M_2) \). This is almost right, but not quite.
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber $M_2$. 
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two
homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one
might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, M_2)$. This
is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the
section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber $M_2$, and the bundle is
locally trivial—but not trivial.
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber $M_2$, and the bundle is locally trivial—but not trivial.

We still have the general principle: A closed orbit $Gx \cong G/H$ in $X$ gives a quotient in the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$. 
Examples of crossed products (continued)
We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber $M_2$, and the bundle is locally trivial—but not trivial.

We still have the general principle: A closed orbit $Gx \cong G/H$ in $X$ gives a quotient in the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$. We illustrate this when $G$ is finite (so that all orbits are closed)
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber $M_2$, and the bundle is locally trivial—but not trivial.

We still have the general principle: A closed orbit $Gx \cong G/H$ in $X$ gives a quotient in the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$. We illustrate this when $G$ is finite (so that all orbits are closed) and $H$ is either $\{1\}$ (above) or $G$ ($C^*(G)$, and see the next slide).
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber $M_2$, and the bundle is locally trivial—but not trivial.

We still have the general principle: A closed orbit $Gx \cong G/H$ in $X$ gives a quotient in the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$. We illustrate this when $G$ is finite (so that all orbits are closed) and $H$ is either $\{1\}$ (above) or $G$ ($C^*(G)$, and see the next slide).
Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle $S^1$ by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^n$ via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle $S^1$ by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber $M_2$, and the bundle is locally trivial—but not trivial.

We still have the general principle: A closed orbit $Gx \cong G/H$ in $X$ gives a quotient in the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$. We illustrate this when $G$ is finite (so that all orbits are closed) and $H$ is either $\{1\}$ (above) or $G$ ($C^*(G)$, and see the next slide).
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^1 \) via the order two homeomorphism \( \zeta \mapsto \bar{\zeta} \).

Set \( L = \{-1, 1\} \subset S^1 \) and \( U = S^1 \setminus L \).

Then the action on \( L \) is trivial, and \( U \) is equivariantly homeomorphic to \( \mathbb{Z}_2 \times \{ x \in U : \text{Im}(x) > 0 \} \sim \mathbb{Z}_2 \times (-1, 1) \).

The equivariant exact sequence

\[
0 \rightarrow C_0(U) \rightarrow C(S^1) \rightarrow C(L) \rightarrow 0
\]

gives the following exact sequence of crossed products:

\[
0 \rightarrow C_0((-1, 1), M_2) \rightarrow C^*(\mathbb{Z}_2, C(S^1)) \rightarrow C(L) \otimes C^*(\mathbb{Z}_2) \rightarrow 0,
\]

in which \( C(L) \otimes C^*(\mathbb{Z}_2) \sim C_4 \).

With more work (details are in my crossed product notes), one can show that \( C^*(\mathbb{Z}_2, C(S^1)) \) is isomorphic to \( \{ f \in C([-1, 1], M_2) : f(1) \text{ and } f(-1) \text{ are diagonal matrices} \} \).
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^1 \) via the order two homeomorphism \( \zeta \mapsto \overline{\zeta} \).

Set

\[
L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.
\]
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^1 \) via the order two homeomorphism \( \zeta \mapsto \overline{\zeta} \).

Set

\[
L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.
\]

Then the action on \( L \) is trivial,
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^1 \) via the order two homeomorphism \( \zeta \mapsto \bar{\zeta} \).

Set

\[
L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.
\]

Then the action on \( L \) is trivial, and \( U \) is equivariantly homeomorphic to

\[
\mathbb{Z}_2 \times \{x \in U: \text{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).
\]
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^1 \) via the order two homeomorphism \( \zeta \mapsto \bar{\zeta} \).

Set

\[
L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.
\]

Then the action on \( L \) is trivial, and \( U \) is equivariantly homeomorphic to

\[
\mathbb{Z}_2 \times \{x \in U: \text{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).
\]

The equivariant exact sequence

\[
0 \longrightarrow C_0(U) \longrightarrow C(S^1) \longrightarrow C(L) \longrightarrow 0
\]
Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^1$ via the order two homeomorphism $\zeta \mapsto \bar{\zeta}$.

Set

$$L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.$$ 

Then the action on $L$ is trivial, and $U$ is equivariantly homeomorphic to

$$\mathbb{Z}_2 \times \{x \in U : \text{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C(S^1) \longrightarrow C(L) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((-1, 1), M_2) \longrightarrow C^*(\mathbb{Z}_2, C(S^1)) \longrightarrow C(L) \otimes C^*(\mathbb{Z}_2) \longrightarrow 0,$$
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^1 \) via the order two homeomorphism \( \zeta \mapsto -\bar{\zeta} \).

Set

\[
L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.
\]

Then the action on \( L \) is trivial, and \( U \) is equivariantly homeomorphic to

\[
\mathbb{Z}_2 \times \{x \in U : \text{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).
\]

The equivariant exact sequence

\[
0 \rightarrow C_0(U) \rightarrow C(S^1) \rightarrow C(L) \rightarrow 0
\]

gives the following exact sequence of crossed products:

\[
0 \rightarrow C_0((-1, 1), M_2) \rightarrow C^*(\mathbb{Z}_2, C(S^1)) \rightarrow C(L) \otimes C^*(\mathbb{Z}_2) \rightarrow 0,
\]

in which \( C(L) \otimes C^*(\mathbb{Z}_2) \cong \mathbb{C}^4 \).
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^1 \) via the order two homeomorphism \( \zeta \mapsto \bar{\zeta} \).

Set

\[
L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.
\]

Then the action on \( L \) is trivial, and \( U \) is equivariantly homeomorphic to

\[
\mathbb{Z}_2 \times \{ x \in U : \text{Im}(x) > 0 \} \cong \mathbb{Z}_2 \times (-1, 1).
\]

The equivariant exact sequence

\[
0 \longrightarrow C_0(U) \longrightarrow C(S^1) \longrightarrow C(L) \longrightarrow 0
\]

gives the following exact sequence of crossed products:

\[
0 \longrightarrow C_0((-1, 1), M_2) \longrightarrow C^*(\mathbb{Z}_2, C(S^1)) \longrightarrow C(L) \otimes C^*(\mathbb{Z}_2) \longrightarrow 0,
\]

in which \( C(L) \otimes C^*(\mathbb{Z}_2) \cong \mathbb{C}^4 \). With more work (details are in my crossed product notes), one can show that \( C^*(\mathbb{Z}_2, C(S^1)) \) is isomorphic to
Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^1$ via the order two homeomorphism $\zeta \mapsto \bar{\zeta}$.

Set

$$L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.$$  

Then the action on $L$ is trivial, and $U$ is equivariantly homeomorphic to

$$\mathbb{Z}_2 \times \{x \in U : \text{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C(S^1) \longrightarrow C(L) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((-1, 1), M_2) \longrightarrow C^*(\mathbb{Z}_2, C(S^1)) \longrightarrow C(L) \otimes C^*(\mathbb{Z}_2) \longrightarrow 0,$$

in which $C(L) \otimes C^*(\mathbb{Z}_2) \cong \mathbb{C}^4$. With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_2, C(S^1))$ is isomorphic to

$$\{ f \in C([-1, 1], M_2) : f(1) \text{ and } f(-1) \text{ are diagonal matrices} \}.$$
Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_2$ acts on $S^1$ via the order two homeomorphism $\zeta \mapsto \overline{\zeta}$.

Set

$$L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.$$  

Then the action on $L$ is trivial, and $U$ is equivariantly homeomorphic to

$$\mathbb{Z}_2 \times \{x \in U: \text{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C(S^1) \longrightarrow C(L) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((-1, 1), M_2) \longrightarrow C^*(\mathbb{Z}_2, C(S^1)) \longrightarrow C(L) \otimes C^*(\mathbb{Z}_2) \longrightarrow 0,$$

in which $C(L) \otimes C^*(\mathbb{Z}_2) \cong \mathbb{C}^4$. With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_2, C(S^1))$ is isomorphic to

$$\{f \in C([-1, 1], M_2): f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.$$
Examples of crossed products (continued)

Recall the example from earlier: \( \mathbb{Z}_2 \) acts on \( S^1 \) via the order two homeomorphism \( \zeta \mapsto \overline{\zeta} \).

Set

\[
L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.
\]

Then the action on \( L \) is trivial, and \( U \) is equivariantly homeomorphic to

\[
\mathbb{Z}_2 \times \{x \in U : \text{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).
\]

The equivariant exact sequence

\[
0 \rightarrow C_0(U) \rightarrow C(S^1) \rightarrow C(L) \rightarrow 0
\]

gives the following exact sequence of crossed products:

\[
0 \rightarrow C_0((-1, 1), M_2) \rightarrow C^*(\mathbb{Z}_2, C(S^1)) \rightarrow C(L) \otimes C^*(\mathbb{Z}_2) \rightarrow 0,
\]

in which \( C(L) \otimes C^*(\mathbb{Z}_2) \cong \mathbb{C}^4 \). With more work (details are in my crossed product notes), one can show that \( C^*(\mathbb{Z}_2, C(S^1)) \) is isomorphic to

\[
\{f \in C([-1, 1], M_2) : f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.
\]
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way. Assume $G$ is finite. Let $\iota: G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in \mathbb{C}^*(G, A, \alpha)$ and $v_g \in \mathbb{C}^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\phi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings. We check the most important part of showing that $\phi$ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b) u_g$ (and $v_g b = \iota_g(b) v_g = bv_g$). So we need $\phi(u_g) \phi(b) = \phi(u_g b)$. We have $\phi(u_g b) = \phi(\alpha_g(b) u_g) = \alpha_g(b) z_g v_g$ and, using $z_g b = \alpha_g(b) z_g$, $\phi(u_g) \phi(b) = z_g v_g b = z_g bv_g = \alpha_g(b) z_g v_g$. Exercise: When $G$ is finite, give a detailed proof that $\phi$ is an isomorphism. (This is written out in my crossed product notes.)
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota: G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. 

Exercise: When $G$ is finite, give a detailed proof that $\phi$ is an isomorphism. (This is written out in my crossed product notes.)
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota: G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements.
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota : G \rightarrow \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\varphi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota : G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\varphi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that $\varphi$ is an algebra homomorphism.
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota : G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\varphi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that $\varphi$ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b)u_g$ (and $v_g b = \iota_g(b)v_g = bv_g$).
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota : G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\varphi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that $\varphi$ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b)u_g$ (and $v_g b = \iota_g(b)v_g = bv_g$). So we need $\varphi(u_g)\varphi(b) = \varphi(u_g b)$. 

Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota : G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\varphi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that $\varphi$ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b) u_g$ (and $v_g b = \iota_g(b) v_g = bv_g$).

So we need $\varphi(u_g) \varphi(b) = \varphi(u_g b)$. We have

$$\varphi(u_g b) = \varphi(\alpha_g(b) u_g) = \alpha_g(b) z_g v_g$$
Crossed products by inner actions

Recall the inner action \( \alpha_g = \text{Ad}(z_g) \) for a continuous homomorphism \( g \mapsto z_g \) from \( G \) to the unitary group of a C*-algebra \( A \). The crossed product is the same as for the trivial action, in a canonical way.

Assume \( G \) is finite. Let \( \iota : G \to \text{Aut}(A) \) be the trivial action of \( G \) on \( A \). Let \( u_g \in C^*(G, A, \alpha) \) and \( v_g \in C^*(G, A, \iota) \) be the unitaries corresponding to the group elements. The isomorphism \( \varphi \) sends \( a \cdot u_g \) to \( az_g \cdot v_g \). This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that \( \varphi \) is an algebra homomorphism. Recall that \( u_g b = \alpha_g(b)u_g \) (and \( v_g b = \iota_g(b)v_g = bv_g \)). So we need \( \varphi(u_g)\varphi(b) = \varphi(u_g b) \). We have

\[
\varphi(u_g b) = \varphi(\alpha_g(b)u_g) = \alpha_g(b)z_g v_g
\]

and, using \( z_g b = \alpha_g(b)z_g \),

\[
\varphi(u_g)\varphi(b) = z_g v_g b = z_g bv_g = \alpha_g(b)z_g v_g.
\]
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota : G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\varphi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that $\varphi$ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b)u_g$ (and $v_g b = \iota_g(b)v_g = bv_g$). So we need $\varphi(u_g)\varphi(b) = \varphi(u_g b)$. We have

$$\varphi(u_g b) = \varphi(\alpha_g(b)u_g) = \alpha_g(b)z_g v_g$$

and, using $z_g b = \alpha_g(b)z_g$,

$$\varphi(u_g)\varphi(b) = z_g v_g b = z_g bv_g = \alpha_g(b)z_g v_g.$$

Exercise: When $G$ is finite, give a detailed proof that $\varphi$ is an isomorphism. (This is written out in my crossed product notes.)
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota : G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\varphi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that $\varphi$ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b) u_g$ (and $v_g b = \iota_g(b) v_g = b v_g$). So we need $\varphi(u_g) \varphi(b) = \varphi(u_g b)$. We have

$$\varphi(u_g b) = \varphi(\alpha_g(b) u_g) = \alpha_g(b) z_g v_g$$

and, using $z_g b = \alpha_g(b) z_g$,

$$\varphi(u_g) \varphi(b) = z_g v_g b = z_g b v_g = \alpha_g(b) z_g v_g.$$

Exercise: When $G$ is finite, give a detailed proof that $\varphi$ is an isomorphism.
(This is written out in my crossed product notes.)
Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from $G$ to the unitary group of a C*-algebra $A$. The crossed product is the same as for the trivial action, in a canonical way.

Assume $G$ is finite. Let $\iota : G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism $\varphi$ sends $a \cdot u_g$ to $az_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that $\varphi$ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b) u_g$ (and $v_g b = \iota_g(b) v_g = bv_g$). So we need $\varphi(u_g) \varphi(b) = \varphi(u_g b)$. We have

$$\varphi(u_g b) = \varphi(\alpha_g(b) u_g) = \alpha_g(b) z_g v_g$$

and, using $z_g b = \alpha_g(b) z_g$,

$$\varphi(u_g) \varphi(b) = z_g v_g b = z_g b v_g = \alpha_g(b) z_g v_g.$$

Exercise: When $G$ is finite, give a detailed proof that $\varphi$ is an isomorphism. (This is written out in my crossed product notes.)
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$ 

Reminder: $\text{Ad}(\nu)(a) = \nu a \nu^*$. 
Crossed products by product type actions

Recall the action of \( \mathbb{Z}_2 \) on the \( 2^\infty \) UHF algebra generated by

\[
\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.
\]

Reminder: \( \text{Ad}(\nu)(a) = \nu a \nu^* \). Set

\[
\nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z_n = \nu \otimes^n \in (M_2) \otimes^n \cong M_{2^n}.
\]

Then write this action as \( \alpha = \lim_{\to n} \\text{Ad}(z_n) \) on \( A = \lim_{\to n} M_{2^n} \).
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ on } A = \bigotimes_{n=1}^{\infty} M_2.$$ 

Reminder: $\text{Ad}(\nu)(a) = \nu a \nu^*$. Set

$$\nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } z_n = \nu^\otimes n \in (M_2)^\otimes n \cong M_{2^n}.$$ 

Then write this action as $\alpha = \lim_{\rightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\rightarrow n} M_{2^n}$. It is not hard to show that crossed products commute with direct limits.
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$ 

Reminder: $\text{Ad}(\nu)(a) = \nu a \nu^*$. Set

$$\nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z_n = \nu \otimes^n \in (M_2) \otimes^n \cong M_{2^n}.$$ 

Then write this action as $\alpha = \lim_{\longrightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\longrightarrow n} M_{2^n}$.

It is not hard to show that crossed products commute with direct limits. (Exercise: Do it for finite $G$.)
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \text{ on } A = \bigotimes_{n=1}^{\infty} M_2.$$ 

Reminder: $\text{Ad}(\nu)(a) = \nu a \nu^*$. Set

$$\nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z_n = \nu \otimes^n \in (M_2)^\otimes^n \cong M_{2^n}.$$ 

Then write this action as $\alpha = \lim_{\longrightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\longrightarrow n} M_{2^n}$.

It is not hard to show that crossed products commute with direct limits. (Exercise: Do it for finite $G$.) Since $\text{Ad}(z_n)$ is inner,
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Reminder: $\text{Ad}(\nu)(a) = \nu a \nu^*$. Set

$$\nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z_n = \nu^{\otimes n} \in (M_2)^{\otimes n} \cong M_{2^n}.$$

Then write this action as $\alpha = \lim_{\longrightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\longrightarrow n} M_{2^n}$.

It is not hard to show that crossed products commute with direct limits.

(Exercise: Do it for finite $G$.) Since $\text{Ad}(z_n)$ is inner, we get

$$C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \cong C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong M_{2^n} \oplus M_{2^n}.$$
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

\[ \alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2. \]

Reminder: $\text{Ad}(v)(a) = vav^*$. Set

\[ v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z_n = v^\otimes n \in (M_2)^\otimes n \cong M_{2^n}. \]

Then write this action as $\alpha = \lim_n \text{Ad}(z_n)$ on $A = \lim_n M_{2^n}$.

It is not hard to show that crossed products commute with direct limits. (Exercise: Do it for finite $G$.) Since $\text{Ad}(z_n)$ is inner, we get

\[ C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \cong C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong M_{2^n} \oplus M_{2^n}. \]

Now we have to use the explicit form of these isomorphisms to compute the maps in the direct system of crossed products, and then find the direct limit.
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on $A = \bigotimes_{n=1}^\infty M_2$.

Reminder: $\text{Ad}(v)(a) = vav^*$. Set

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$z_n = v \otimes v \in (M_2)^\otimes n \cong M_{2^n}.$$

Then write this action as $\alpha = \lim_{\rightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\rightarrow n} M_{2^n}$.

It is not hard to show that crossed products commute with direct limits. (Exercise: Do it for finite $G$.) Since $\text{Ad}(z_n)$ is inner, we get

$$C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \cong C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong M_{2^n} \oplus M_{2^n}.$$

Now we have to use the explicit form of these isomorphisms to compute the maps in the direct system of crossed products, and then find the direct limit.
Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Reminder: $\text{Ad}(\nu)(a) = \nu a \nu^*$. Set

$$\nu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z_n = \nu^\otimes n \in (M_2)^\otimes n \cong M_{2^n}.$$

Then write this action as $\alpha = \lim_{\rightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\rightarrow n} M_{2^n}$.

It is not hard to show that crossed products commute with direct limits. (Exercise: Do it for finite $G$.) Since $\text{Ad}(z_n)$ is inner, we get

$$C^*\left(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)\right) \cong C^*\left(\mathbb{Z}_2\right) \otimes M_{2^n} \cong M_{2^n} \oplus M_{2^n}.$$

Now we have to use the explicit form of these isomorphisms to compute the maps in the direct system of crossed products, and then find the direct limit.
Crossed products by product type actions (continued)

The action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra is generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \text{ on } A = \bigotimes_{n=1}^{\infty} M_2.$$

We wrote this action as $\alpha = \lim_{n \to -} \text{Ad}(z_n) \text{ on } A = \lim_{n \to -} M_{2^n}$. Then

$$C^*(\mathbb{Z}_2, A, \alpha) \cong \lim_{n \to -} C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n))$$

$$\cong \lim_{n \to -} C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong \lim_{n \to -}(M_{2^n} \oplus M_{2^n}).$$

The maps turn out to be unitarily equivalent to

$$(a, b) \mapsto \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right),$$
Crossed products by product type actions (continued)

The action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra is generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ on } A = \bigotimes_{n=1}^{\infty} M_2.$$

We wrote this action as $\alpha = \lim_{n \to \infty} \text{Ad}(z_n)$ on $A = \lim_{n \to \infty} M_{2^n}$. Then

$$C^*(\mathbb{Z}_2, A, \alpha) \cong \lim_{n \to \infty} C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \cong \lim_{n \to \infty} C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong \lim_{n \to \infty} (M_{2^n} \oplus M_{2^n}).$$

The maps turn out to be unitarily equivalent to

$$(a, b) \mapsto \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right),$$

and a computation with Bratteli diagrams shows that the direct limit is again the $2^\infty$ UHF algebra.
The action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra is generated by

$$\alpha = \bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ on } A = \bigotimes_{n=1}^\infty M_2.$$ 

We wrote this action as $\alpha = \lim_{\rightarrow} \text{Ad}(z_n)$ on $A = \lim_{\rightarrow} M_{2^n}$. Then

$$C^*(\mathbb{Z}_2, A, \alpha) \cong \lim_{\rightarrow} C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \cong \lim_{\rightarrow} C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong \lim_{\rightarrow} (M_{2^n} \oplus M_{2^n}).$$

The maps turn out to be unitarily equivalent to

$$(a, b) \mapsto \begin{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \end{pmatrix},$$

and a computation with Bratteli diagrams shows that the direct limit is again the $2^\infty$ UHF algebra. (For general product type actions, the direct limit will be more complicated, and usually not a UHF algebra.)
Crossed products by product type actions (continued)

The action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra is generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ on } A = \bigotimes_{n=1}^{\infty} M_2.$$ 

We wrote this action as $\alpha = \lim_{\rightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\rightarrow n} M_{2^n}$. Then

$$C^*(\mathbb{Z}_2, A, \alpha) \cong \lim_{\rightarrow n} C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n))$$

$$\cong \lim_{\rightarrow n} C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong \lim_{\rightarrow n}(M_{2^n} \oplus M_{2^n}).$$

The maps turn out to be unitarily equivalent to

$$(a, b) \mapsto \begin{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \end{pmatrix},$$

and a computation with Bratteli diagrams shows that the direct limit is again the $2^\infty$ UHF algebra. (For general product type actions, the direct limit will be more complicated, and usually not a UHF algebra.)
Crossed products by product type actions (continued)

The action of $\mathbb{Z}_2$ on the $2^{\infty}$ UHF algebra is generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$ 

We wrote this action as $\alpha = \lim_{n \to \infty} \text{Ad}(z_n)$ on $A = \lim_{n \to \infty} M_{2^n}$. Then

$$C^*(\mathbb{Z}_2, A, \alpha) \cong \lim_{n \to \infty} C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \cong \lim_{n \to \infty} C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong \lim_{n \to \infty} (M_{2^n} \oplus M_{2^n}).$$ 

The maps turn out to be unitarily equivalent to

$$(a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

and a computation with Bratteli diagrams shows that the direct limit is again the $2^{\infty}$ UHF algebra. (For general product type actions, the direct limit will be more complicated, and usually not a UHF algebra.)
Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
on $A = \bigotimes_{n=1}^{\infty} M_2$. 

Write it as

$$\alpha = \lim_{n \to -} \text{Ad}(z^n)$$
on $A = \lim_{n \to -} M_2^n$, with maps $\phi_n : M_2^n \to M_2^{n+1}$.

Exercise: Find isomorphisms $\sigma_n : C^*(\mathbb{Z}_2, M_2^n, \text{Ad}(z^n)) \to M_2^n \oplus M_2^n$ and homomorphisms $\psi_n : M_2^n \oplus M_2^n \to M_2^{n+1} \oplus M_2^{n+1}$ such that, with $\phi_n$ being the map induced by $\phi_n$ on the crossed products, the following diagram commutes for all $n$:

$$
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M_2^n, \text{Ad}(z^n)) & \sigma_n & M_2^n \oplus M_2^n \\
\downarrow \quad \downarrow \psi_n & & \downarrow \\
C^*(\mathbb{Z}_2, M_2^{n+1}, \text{Ad}(z^{n+1})) & \sigma_{n+1} & M_2^{n+1} \oplus M_2^{n+1}
\end{array}
$$

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism $C^*(\mathbb{Z}_2) \to C \oplus C$).

Then prove that, using the maps $\psi_n$, one gets $\lim_{n \to -} (M_2^n \oplus M_2^n) \sim = A$.

(This part doesn't have anything to do with crossed products.)

Conclude that $C^*(\mathbb{Z}_2, A, \alpha) \sim = A$. 

Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by 
$\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Write it as $\alpha = \lim_{\rightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\rightarrow n} M_{2^n}$, with maps $\varphi_n: M_{2^n} \rightarrow M_{2^{n+1}}$. 

Exercise: Find isomorphisms $\sigma_n: C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \rightarrow M_{2^n} \oplus M_{2^n}$ and homomorphisms $\psi_n: M_{2^n} \oplus M_{2^n} \rightarrow M_{2^n+1} \oplus M_{2^n+1}$ such that, with $\varphi_n$ being the map induced by $\varphi_n$ on the crossed products, the following diagram commutes for all $n$:

\[
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2^n} \oplus M_{2^n} \\
\downarrow \varphi_n & & \downarrow \psi_n \\
C^*(\mathbb{Z}_2, M_{2^{n+1}}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2^{n+1}} \oplus M_{2^{n+1}}
\end{array}
\]

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism $C^*(\mathbb{Z}_2) \rightarrow C \oplus C$.) Then prove that, using the maps $\psi_n$, one gets $\lim_{\rightarrow n} (M_{2^n} \oplus M_{2^n}) \sim = A$.

(This part doesn't have anything to do with crossed products.) Conclude that $C^*(\mathbb{Z}_2, A, \alpha) \sim = A$. 

N. C. Phillips (U of Oregon)

Crossed Products; the Rokhlin Property

15 July 2016 27 / 41
Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by $\alpha = \bigotimes_{n=1}^{\infty} \text{Ad}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Write it as $\alpha = \lim_{n \to} \text{Ad}(z_n)$ on $A = \lim_{n \to} M_{2n}$, with maps $\varphi_n: M_{2n} \to M_{2n+1}$.

Exercise: Find isomorphisms $\sigma_n: C^*(\mathbb{Z}_2, M_{2n}, \text{Ad}(z_n)) \to M_{2n} \oplus M_{2n}$ and homomorphisms $\psi_n: M_{2n} \oplus M_{2n} \to M_{2n+1} \oplus M_{2n+1}$ such that, with $\overline{\varphi}_n$ being the map induced by $\varphi_n$ on the crossed products, the following diagram commutes for all $n$:

\[
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M_{2n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2n} \oplus M_{2n} \\
\downarrow{\overline{\varphi}_n} & & \downarrow{\overline{\varphi}_n} \\
C^*(\mathbb{Z}_2, M_{2n+1}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2n+1} \oplus M_{2n+1}
\end{array}
\]
Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by $\alpha = \bigotimes_{n=1}^\infty \text{Ad} \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ on $A = \bigotimes_{n=1}^\infty M_2$. Write it as $\alpha = \lim_{\to n} \text{Ad}(z_n)$ on $A = \lim_{\to n} M_{2^n}$, with maps $\varphi_n: M_{2^n} \to M_{2^{n+1}}$.

Exercise: Find isomorphisms $\sigma_n: C^*\left(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)\right) \to M_{2^n} \oplus M_{2^n}$ and homomorphisms $\psi_n: M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$ such that, with $\overline{\varphi}_n$ being the map induced by $\varphi_n$ on the crossed products, the following diagram commutes for all $n$:

$$
\begin{array}{ccc}
C^*\left(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)\right) & \longrightarrow & M_{2^n} \oplus M_{2^n} \\
\downarrow \overline{\varphi}_n & & \downarrow \psi_n \\
C^*\left(\mathbb{Z}_2, M_{2^{n+1}}, \text{Ad}(z_{n+1})\right) & \longrightarrow & M_{2^{n+1}} \oplus M_{2^{n+1}}
\end{array}
$$
Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by $\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Write it as $\alpha = \lim_{\rightarrow n} \text{Ad}(z_n)$ on $A = \lim_{\rightarrow n} M_{2^n}$, with maps $\varphi_n : M^{2n} \rightarrow M^{2n+1}$.

Exercise: Find isomorphisms $\sigma_n : C^*(\mathbb{Z}_2, M^{2n}, \text{Ad}(z_n)) \rightarrow M^{2n} \oplus M^{2n}$ and homomorphisms $\psi_n : M^{2n} \oplus M^{2n} \rightarrow M^{2n+1} \oplus M^{2n+1}$ such that, with $\overline{\varphi}_n$ being the map induced by $\varphi_n$ on the crossed products, the following diagram commutes for all $n$:

$$
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M^{2n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M^{2n} \oplus M^{2n} \\
\downarrow \overline{\varphi}_n & & \downarrow \psi_n \\
C^*(\mathbb{Z}_2, M^{2n+1}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M^{2n+1} \oplus M^{2n+1}
\end{array}
$$

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism $C^*(\mathbb{Z}_2) \rightarrow \mathbb{C} \oplus \mathbb{C}$.)
Recall the action of \( \mathbb{Z}_2 \) on the \( 2^\infty \) UHF algebra generated by 
\[ \alpha = \bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] on 
\[ A = \bigotimes_{n=1}^\infty M_2. \] Write it as 
\[ \alpha = \lim_{\to} \text{Ad}(z_n) \] on 
\[ A = \lim_{\to} M_{2n}, \] with maps \( \varphi_n : M_{2n} \to M_{2n+1}. \)

Exercise: Find isomorphisms \( \sigma_n : C^*(\mathbb{Z}_2, M_{2n}, \text{Ad}(z_n)) \to M_{2n} \oplus M_{2n} \) and homomorphisms \( \psi_n : M_{2n} \oplus M_{2n} \to M_{2n+1} \oplus M_{2n+1} \) such that, with \( \overline{\varphi}_n \) being the map induced by \( \varphi_n \) on the crossed products, the following diagram commutes for all \( n \):

\[
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M_{2n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2n} \oplus M_{2n} \\
\downarrow \overline{\varphi}_n & & \downarrow \psi_n \\
C^*(\mathbb{Z}_2, M_{2n+1}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2n+1} \oplus M_{2n+1}.
\end{array}
\]

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism \( C^*(\mathbb{Z}_2) \to \mathbb{C} \oplus \mathbb{C} \).) Then prove that, using the maps \( \psi_n \), one gets \( \lim_{\to} (M_{2n} \oplus M_{2n}) \cong A. \)
Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by $\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Write it as $\alpha = \lim_{\rightarrow} \text{Ad}(z_n)$ on $A = \lim_{\rightarrow} M_{2n}$, with maps $\varphi_n: M_{2n} \to M_{2n+1}$.

Exercise: Find isomorphisms $\sigma_n: C^*(\mathbb{Z}_2, M_{2n}, \text{Ad}(z_n)) \to M_{2n} \oplus M_{2n}$ and homomorphisms $\psi_n: M_{2n} \oplus M_{2n} \to M_{2n+1} \oplus M_{2n+1}$ such that, with $\overline{\varphi}_n$ being the map induced by $\varphi_n$ on the crossed products, the following diagram commutes for all $n$:

\[
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M_{2n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2n} \oplus M_{2n} \\
& \downarrow{\overline{\varphi}_n} & \downarrow{\psi_n} \\
C^*(\mathbb{Z}_2, M_{2n+1}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2n+1} \oplus M_{2n+1} \\
\end{array}
\]

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism $C^*(\mathbb{Z}_2) \to \mathbb{C} \oplus \mathbb{C}$.) Then prove that, using the maps $\psi_n$, one gets $\lim_{\rightarrow} (M_{2n} \oplus M_{2n}) \cong A$. (This part doesn’t have anything to do with crossed products.)
Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by
\[ \alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \] on $A = \bigotimes_{n=1}^{\infty} M_2$. Write it as $\alpha = \lim_n \text{Ad}(z_n)$ on $A = \lim_n M_{2^n}$, with maps $\varphi_n : M_{2^n} \to M_{2^{n+1}}$.

Exercise: Find isomorphisms $\sigma_n : C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \to M_{2^n} \oplus M_{2^n}$ and homomorphisms $\psi_n : M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$ such that, with $\bar{\varphi}_n$ being the map induced by $\varphi_n$ on the crossed products, the following diagram commutes for all $n$:

\[
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2^n} \oplus M_{2^n} \\
\bar{\varphi}_n & & \downarrow \psi_n \\
C^*(\mathbb{Z}_2, M_{2^{n+1}}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2^{n+1}} \oplus M_{2^{n+1}}.
\end{array}
\]

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism $C^*(\mathbb{Z}_2) \to \mathbb{C} \oplus \mathbb{C}$.) Then prove that, using the maps $\psi_n$, one gets $\lim_n (M_{2^n} \oplus M_{2^n}) \cong A$. (This part doesn’t have anything to do with crossed products.) Conclude that $C^*(\mathbb{Z}_2, A, \alpha) \cong A$. 

N. C. Phillips (U of Oregon)
Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the $2^\infty$ UHF algebra generated by 
$\alpha = \bigotimes_{n=1}^\infty \text{Ad} \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ on $A = \bigotimes_{n=1}^\infty M_2$. Write it as $\alpha = \lim_{n \to} \text{Ad}(z_n)$ on $A = \lim_{n \to} M_{2n}$, with maps $\varphi_n : M_{2n} \to M_{2n+1}$.

Exercise: Find isomorphisms $\sigma_n : C^*(\mathbb{Z}_2, M_{2n}, \text{Ad}(z_n)) \to M_{2n} \oplus M_{2n}$ and homomorphisms $\psi_n : M_{2n} \oplus M_{2n} \to M_{2n+1} \oplus M_{2n+1}$ such that, with $\overline{\varphi}_n$ being the map induced by $\varphi_n$ on the crossed products, the following diagram commutes for all $n$:

$$
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M_{2n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2n} \oplus M_{2n} \\
\downarrow \overline{\varphi}_n & & \downarrow \psi_n \\
C^*(\mathbb{Z}_2, M_{2n+1}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2n+1} \oplus M_{2n+1}.
\end{array}
$$

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism $C^*(\mathbb{Z}_2) \to \mathbb{C} \oplus \mathbb{C}$.) Then prove that, using the maps $\psi_n$, one gets $\lim_{n \to} (M_{2n} \oplus M_{2n}) \cong A$. (This part doesn’t have anything to do with crossed products.) Conclude that $C^*(\mathbb{Z}_2, A, \alpha) \cong A$. 
Crossed products by product type actions (continued)

Recall the action of \( \mathbb{Z}_2 \) on the \( 2^\infty \) UHF algebra generated by 
\[
\alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \text{ on } A = \bigotimes_{n=1}^{\infty} M_2.
\]
Write it as \( \alpha = \lim_{\rightarrow n} \text{Ad}(z_n) \) on 
\[
A = \lim_{\rightarrow n} M_{2^n},
\]
with maps \( \varphi_n : M_{2^n} \rightarrow M_{2^{n+1}} \).

Exercise: Find isomorphisms \( \sigma_n : C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) \rightarrow M_{2^n} \oplus M_{2^n} \) and homomorphisms \( \psi_n : M_{2^n} \oplus M_{2^n} \rightarrow M_{2^{n+1}} \oplus M_{2^{n+1}} \) such that, with \( \overline{\varphi}_n \) being the map induced by \( \varphi_n \) on the crossed products, the following diagram commutes for all \( n \):

\[
\begin{array}{ccc}
C^*(\mathbb{Z}_2, M_{2^n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2^n} \oplus M_{2^n} \\
\overline{\varphi}_n \downarrow & & \downarrow \psi_n \\
C^*(\mathbb{Z}_2, M_{2^{n+1}}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2^{n+1}} \oplus M_{2^{n+1}}
\end{array}
\]

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism \( C^*(\mathbb{Z}_2) \rightarrow \mathbb{C} \oplus \mathbb{C} \).) Then prove that, using the maps \( \psi_n \), one gets \( \lim_{\rightarrow n} (M_{2^n} \oplus M_{2^n}) \cong A \). (This part doesn’t have anything to do with crossed products.) Conclude that 
\[
C^*(\mathbb{Z}_2, A, \alpha) \cong A.
\]
Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is \textit{free} if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points.
Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is free if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points. Equivalently, whenever \(g \in G\) and \(x \in X\) satisfy \(gx = x\), then \(g = 1\).
Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is \textit{free} if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points. Equivalently, whenever \(g \in G\) and \(x \in X\) satisfy \(gx = x\), then \(g = 1\). Equivalently, every orbit is isomorphic to \(G\) acting on \(G\) by translation.
Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is free if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points. Equivalently, whenever \(g \in G\) and \(x \in X\) satisfy \(gx = x\), then \(g = 1\). Equivalently, every orbit is isomorphic to \(G\) acting on \(G\) by translation. (Examples: \(G\) acting on \(G\) by translation, \(\mathbb{Z}_n\) acting on \(S^1\) by rotation by \(e^{2\pi i/n}\), and \(\mathbb{Z}\) acting on \(S^1\) by an irrational rotation.)
Motivation for the Rokhlin property

Recall that an action $(g, x) \mapsto gx$ of a group $G$ on a set $X$ is free if every $g \in G \setminus \{1\}$ acts on $X$ with no fixed points. Equivalently, whenever $g \in G$ and $x \in X$ satisfy $gx = x$, then $g = 1$. Equivalently, every orbit is isomorphic to $G$ acting on $G$ by translation. (Examples: $G$ acting on $G$ by translation, $\mathbb{Z}_n$ acting on $S^1$ by rotation by $e^{2\pi i/n}$, and $\mathbb{Z}$ acting on $S^1$ by an irrational rotation.)

Let $X$ be the Cantor set, let $G$ be a finite group, and let $G$ act freely on $X$. 

Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is free if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points. Equivalently, whenever \(g \in G\) and \(x \in X\) satisfy \(gx = x\), then \(g = 1\). Equivalently, every orbit is isomorphic to \(G\) acting on \(G\) by translation. (Examples: \(G\) acting on \(G\) by translation, \(\mathbb{Z}_n\) acting on \(S^1\) by rotation by \(e^{2\pi i/n}\), and \(\mathbb{Z}\) acting on \(S^1\) by an irrational rotation.)

Let \(X\) be the Cantor set, let \(G\) be a finite group, and let \(G\) act freely on \(X\). Fix \(x_0 \in X\). Then the points \(gx_0\), for \(g \in G\), are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set \(K \subset X\) such that \(x_0 \in K\) and the sets \(gK\), for \(g \in G\), are all disjoint.

Exercise: Carry out the details. (It isn't quite trivial.)
Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is free if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points. Equivalently, whenever \(g \in G\) and \(x \in X\) satisfy \(gx = x\), then \(g = 1\). Equivalently, every orbit is isomorphic to \(G\) acting on \(G\) by translation. (Examples: \(G\) acting on \(G\) by translation, \(\mathbb{Z}_n\) acting on \(S^1\) by rotation by \(e^{2\pi i/n}\), and \(\mathbb{Z}\) acting on \(S^1\) by an irrational rotation.)

Let \(X\) be the Cantor set, let \(G\) be a finite group, and let \(G\) act freely on \(X\). Fix \(x_0 \in X\). Then the points \(gx_0\), for \(g \in G\), are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set \(K \subset X\) such that \(x_0 \in K\) and the sets \(gK\), for \(g \in G\), are all disjoint.

By repeating this process, one can find a compact open set \(L \subset X\) such that the sets \(L_g = gL\), for \(g \in G\), are all disjoint, and such that their union is \(X\).
Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is free if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points. Equivalently, whenever \(g \in G\) and \(x \in X\) satisfy \(gx = x\), then \(g = 1\). Equivalently, every orbit is isomorphic to \(G\) acting on \(G\) by translation. (Examples: \(G\) acting on \(G\) by translation, \(\mathbb{Z}_n\) acting on \(S^1\) by rotation by \(e^{2\pi i/n}\), and \(\mathbb{Z}\) acting on \(S^1\) by an irrational rotation.)

Let \(X\) be the Cantor set, let \(G\) be a finite group, and let \(G\) act freely on \(X\). Fix \(x_0 \in X\). Then the points \(gx_0\), for \(g \in G\), are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set \(K \subset X\) such that \(x_0 \in K\) and the sets \(gK\), for \(g \in G\), are all disjoint.

By repeating this process, one can find a compact open set \(L \subset X\) such that the sets \(L_g = gL\), for \(g \in G\), are all disjoint, and such that their union is \(X\).

Exercise: Carry out the details. (It isn’t quite trivial.)
Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is free if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points. Equivalently, whenever \(g \in G\) and \(x \in X\) satisfy \(gx = x\), then \(g = 1\). Equivalently, every orbit is isomorphic to \(G\) acting on \(G\) by translation. (Examples: \(G\) acting on \(G\) by translation, \(\mathbb{Z}_n\) acting on \(S^1\) by rotation by \(e^{2\pi i/n}\), and \(\mathbb{Z}\) acting on \(S^1\) by an irrational rotation.)

Let \(X\) be the Cantor set, let \(G\) be a finite group, and let \(G\) act freely on \(X\). Fix \(x_0 \in X\). Then the points \(gx_0\), for \(g \in G\), are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set \(K \subset X\) such that \(x_0 \in K\) and the sets \(gK\), for \(g \in G\), are all disjoint.

By repeating this process, one can find a compact open set \(L \subset X\) such that the sets \(L_g = gL\), for \(g \in G\), are all disjoint, and such that their union is \(X\).

Exercise: Carry out the details. (It isn’t quite trivial.)
Motivation for the Rokhlin property

Recall that an action \((g, x) \mapsto gx\) of a group \(G\) on a set \(X\) is \textit{free} if every \(g \in G \setminus \{1\}\) acts on \(X\) with no fixed points. Equivalently, whenever \(g \in G\) and \(x \in X\) satisfy \(gx = x\), then \(g = 1\). Equivalently, every orbit is isomorphic to \(G\) acting on \(G\) by translation. (Examples: \(G\) acting on \(G\) by translation, \(\mathbb{Z}_n\) acting on \(S^1\) by rotation by \(e^{2\pi i / n}\), and \(\mathbb{Z}\) acting on \(S^1\) by an irrational rotation.)

Let \(X\) be the Cantor set, let \(G\) be a finite group, and let \(G\) act freely on \(X\). Fix \(x_0 \in X\). Then the points \(gx_0\), for \(g \in G\), are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set \(K \subset X\) such that \(x_0 \in K\) and the sets \(gK\), for \(g \in G\), are all disjoint.

By repeating this process, one can find a compact open set \(L \subset X\) such that the sets \(L_g = gL\), for \(g \in G\), are all disjoint, and such that their union is \(X\).

Exercise: Carry out the details. (It isn’t quite trivial.)
The Rokhlin property

Definition

Let $A$ be a unital $C^*$-algebra, and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. For every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\| \alpha_g(e_h) - e_{gh} \| < \varepsilon$ for all $g, h \in G$.
2. $\| e_ga - ae_g \| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For $C^*$-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further. The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.
The Rokhlin property

Definition

Let $A$ be a unital C*-algebra, and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the \textit{Rokhlin property} if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further. The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.
The Rokhlin property

Definition

Let $A$ be a unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$. 

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further. The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.
The Rokhlin property

Definition

Let $A$ be a unital C*-algebra, and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
The Rokhlin property

Definition

Let $A$ be a unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.
The Rokhlin property

Definition

Let $A$ be a unital C*-algebra, and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further.
The Rokhlin property

Definition

Let $A$ be a unital C*-algebra, and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further.

The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.
The Rokhlin property

Definition

Let $A$ be a unital C*-algebra, and let $\alpha: G \to \operatorname{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further.

The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.
The Rokhlin property

Definition

Let $A$ be a unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further.

The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \epsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \epsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

The projections $e_g$ are the analogs of the characteristic functions of the compact open sets $gL$ from the Cantor set example.
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

The projections $e_g$ are the analogs of the characteristic functions of the compact open sets $gL$ from the Cantor set example.

Condition (1) is an approximate version of $gL_h = L_{gh}$. (Recall that $L_g = gL$.)
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. \( \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \) for all \( g, h \in G \).
2. \( \| e_g a - ae_g \| < \varepsilon \) for all \( g \in G \) and all \( a \in F \).
3. \( \sum_{g \in G} e_g = 1 \).

The projections \( e_g \) are the analogs of the characteristic functions of the compact open sets \( gL \) from the Cantor set example.

Condition (1) is an approximate version of \( gL_h = L_{gh} \). (Recall that \( L_g = gL \).)

Condition (3) is the requirement that \( X \) be the disjoint union of the \( L_g \).
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - ae_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

The projections \( e_g \) are the analogs of the characteristic functions of the compact open sets \( gL \) from the Cantor set example.

Condition (1) is an approximate version of \( gL_h = L_{gh} \). (Recall that \( L_g = gL \).)

Condition (3) is the requirement that \( X \) be the disjoint union of the \( L_g \).

Condition (2) is vacuous for a commutative \( C^* \)-algebra. In the noncommutative case, one needs something more than (1) and (3).
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]

2. \[ \| e_g a - a e_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]

3. \[ \sum_{g \in G} e_g = 1. \]

The projections \( e_g \) are the analogs of the characteristic functions of the compact open sets \( gL \) from the Cantor set example.

Condition (1) is an approximate version of \( gL_h = L_{gh} \). (Recall that \( L_g = gL \).)

Condition (3) is the requirement that \( X \) be the disjoint union of the \( L_g \).

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, one needs something more than (1) and (3).

Without (2) the inner action \( \alpha : \mathbb{Z}_2 \to \text{Aut}(M_2) \) generated by \( \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) would have the Rokhlin property.
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - ae_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

The projections \( e_g \) are the analogs of the characteristic functions of the compact open sets \( gL \) from the Cantor set example.

Condition (1) is an approximate version of \( gL_h = L_{gh} \). (Recall that \( L_g = gL \).)

Condition (3) is the requirement that \( X \) be the disjoint union of the \( L_g \).

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, one needs something more than (1) and (3). Without (2) the inner action \( \alpha : \mathbb{Z}_2 \to \text{Aut}(M_2) \) generated by \( \text{Ad}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \) would have the Rokhlin property. (Exercise: Prove this statement.)
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. \( \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \) for all \( g, h \in G \).
2. \( \| e_ga - ae_g \| < \varepsilon \) for all \( g \in G \) and all \( a \in F \).
3. \( \sum_{g \in G} e_g = 1 \).

The projections \( e_g \) are the analogs of the characteristic functions of the compact open sets \( gL \) from the Cantor set example.

Condition (1) is an approximate version of \(gL_h = L_{gh}\). (Recall that \( L_g = gL \).)

Condition (3) is the requirement that \( X \) be the disjoint union of the \( L_g \).

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, one needs something more than (1) and (3).

Without (2) the inner action \( \alpha: \mathbb{Z}_2 \to \text{Aut}(M_2) \) generated by \( \text{Ad} \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \) would have the Rokhlin property. (Exercise: Prove this statement.) We don’t want this. For example, \( M_2 \) is simple but \( C^*(\mathbb{Z}_2, M_2, \alpha) \) isn’t.
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - a e_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

The projections \( e_g \) are the analogs of the characteristic functions of the compact open sets \( gL \) from the Cantor set example.

Condition (1) is an approximate version of \( gL_h = L_{gh} \). (Recall that \( L_g = gL \).)

Condition (3) is the requirement that \( X \) be the disjoint union of the \( L_g \).

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, one needs something more than (1) and (3). Without (2) the inner action \( \alpha : \mathbb{Z}_2 \to \text{Aut}(M_2) \) generated by \( \text{Ad} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \) would have the Rokhlin property. (Exercise: Prove this statement.) We don’t want this. For example, \( M_2 \) is simple but \( C^*(\mathbb{Z}_2, M_2, \alpha) \) isn’t. (There is more on outerness in Lecture 5.)
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - a e_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

The projections \( e_g \) are the analogs of the characteristic functions of the compact open sets \( gL \) from the Cantor set example.

Condition (1) is an approximate version of \( gL_h = L_{gh} \). (Recall that \( L_g = gL \).)

Condition (3) is the requirement that \( X \) be the disjoint union of the \( L_g \).

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, one needs something more than (1) and (3).

Without (2) the inner action \( \alpha : \mathbb{Z}_2 \to \text{Aut}(M_2) \) generated by \( \text{Ad} \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \) would have the Rokhlin property. (Exercise: Prove this statement.) We don’t want this. For example, \( M_2 \) is simple but \( C^*(\mathbb{Z}_2, M_2, \alpha) \) isn’t. (There is more on outerness in Lecture 5.)
The Rokhlin property (continued)

The conditions in the definition of the Rokhlin property:

1. \( \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \) for all \( g, h \in G \).
2. \( \| e_g a - a e_g \| < \varepsilon \) for all \( g \in G \) and all \( a \in F \).
3. \( \sum_{g \in G} e_g = 1 \).

The projections \( e_g \) are the analogs of the characteristic functions of the compact open sets \( gL \) from the Cantor set example.

Condition (1) is an approximate version of \( gL_h = L_{gh} \). (Recall that \( L_g = gL \).)

Condition (3) is the requirement that \( X \) be the disjoint union of the \( L_g \).

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, one needs something more than (1) and (3). Without (2) the inner action \( \alpha : \mathbb{Z}_2 \to \text{Aut}(M_2) \) generated by \( \text{Ad} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \) would have the Rokhlin property. (Exercise: Prove this statement.) We don’t want this. For example, \( M_2 \) is simple but \( C^*(\mathbb{Z}_2, M_2, \alpha) \) isn’t. (There is more on outerness in Lecture 5.)
Examples

The conditions in the definition of the Rokhlin property:

1. \(|\alpha_g(e_h) - e_{gh}| < \varepsilon\) for all \(g, h \in G\).
2. \(|e_g a - ae_g| < \varepsilon\) for all \(g \in G\) and all \(a \in F\).
3. \(\sum_{g \in G} e_g = 1\).
Examples

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - ae_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

Exercise: Let \( G \) be finite. Let act \( G \) on \( G \) by translation. Prove that the action of \( G \) on \( C(G) \)
Examples

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - a e_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

Exercise: Let \( G \) be finite. Let act \( G \) on \( G \) by translation. Prove that the action of \( G \) on \( C(G) \) (namely \( \alpha_g(f)(h) = f(g^{-1}h) \)) has the Rokhlin property.
Examples

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \] for all \( g, h \in G \).

2. \[ \| e_g a - a e_g \| < \varepsilon \] for all \( g \in G \) and all \( a \in F \).

3. \[ \sum_{g \in G} e_g = 1. \]

Exercise: Let \( G \) be finite. Let act \( G \) on \( G \) by translation. Prove that the action of \( G \) on \( C(G) \) (namely \( \alpha_g(f)(h) = f(g^{-1}h) \)) has the Rokhlin property.

Exercise: Let \( G \) be finite. Let \( A \) be any unital C*-algebra. Prove that the action of \( G \) on \( \bigoplus_{g \in G} A \) by translation of the summands has the Rokhlin property.
Examples

The conditions in the definition of the Rokhlin property:

1. \( \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \) for all \( g, h \in G \).
2. \( \| e_g a - a e_g \| < \varepsilon \) for all \( g \in G \) and all \( a \in F \).
3. \( \sum_{g \in G} e_g = 1 \).

Exercise: Let \( G \) be finite. Let act \( G \) on \( G \) by translation. Prove that the action of \( G \) on \( C(G) \) (namely \( \alpha_g(f)(h) = f(g^{-1}h) \)) has the Rokhlin property.

Exercise: Let \( G \) be finite. Let \( A \) be any unital C*-algebra. Prove that the action of \( G \) on \( \bigoplus_{g \in G} A \) by translation of the summands has the Rokhlin property.

Exercise: Let \( G \) be finite, and let \( G \) act freely on the Cantor set \( X \).
Examples

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - a e_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

Exercise: Let \( G \) be finite. Let act \( G \) on \( G \) by translation. Prove that the action of \( G \) on \( C(G) \) (namely \( \alpha_g(f)(h) = f(g^{-1}h) \)) has the Rokhlin property.

Exercise: Let \( G \) be finite. Let \( A \) be any unital C*-algebra. Prove that the action of \( G \) on \( \bigoplus_{g \in G} A \) by translation of the summands has the Rokhlin property.

Exercise: Let \( G \) be finite, and let \( G \) act freely on the Cantor set \( X \). Prove that the corresponding action of \( G \) on \( C(X) \) has the Rokhlin property. (Use the earlier exercise on free actions on the Cantor set.)
Examples

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - ae_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

Exercise: Let \( G \) be finite. Let act \( G \) on \( G \) by translation. Prove that the action of \( G \) on \( C(G) \) (namely \( \alpha_g(f)(h) = f(g^{-1}h) \)) has the Rokhlin property.

Exercise: Let \( G \) be finite. Let \( A \) be any unital C*-algebra. Prove that the action of \( G \) on \( \bigoplus_{g \in G} A \) by translation of the summands has the Rokhlin property.

Exercise: Let \( G \) be finite, and let \( G \) act freely on the Cantor set \( X \). Prove that the corresponding action of \( G \) on \( C(X) \) has the Rokhlin property. (Use the earlier exercise on free actions on the Cantor set.)

In the exercises above, condition (2) is trivial. Can it be satisfied in a nontrivial way? In particular, are there any actions on simple C*-algebras with the Rokhlin property?
Examples

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]

2. \[ \| e_g a - ae_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]

3. \[ \sum_{g \in G} e_g = 1. \]

Exercise: Let $G$ be finite. Let act $G$ on $G$ by translation. Prove that the action of $G$ on $C(G)$ (namely $\alpha_g(f)(h) = f(g^{-1}h)$) has the Rokhlin property.

Exercise: Let $G$ be finite. Let $A$ be any unital C*-algebra. Prove that the action of $G$ on $\bigoplus_{g \in G} A$ by translation of the summands has the Rokhlin property.

Exercise: Let $G$ be finite, and let $G$ act freely on the Cantor set $X$. Prove that the corresponding action of $G$ on $C(X)$ has the Rokhlin property. (Use the earlier exercise on free actions on the Cantor set.)

In the exercises above, condition (2) is trivial. Can it be satisfied in a nontrivial way? In particular, are there any actions on simple C*-algebras with the Rokhlin property?
Examples

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Exercise: Let $G$ be finite. Let act $G$ on $G$ by translation. Prove that the action of $G$ on $C(G)$ (namely $\alpha_g(f)(h) = f(g^{-1}h)$) has the Rokhlin property.

Exercise: Let $G$ be finite. Let $A$ be any unital C*-algebra. Prove that the action of $G$ on $\bigoplus_{g \in G} A$ by translation of the summands has the Rokhlin property.

Exercise: Let $G$ be finite, and let $G$ act freely on the Cantor set $X$. Prove that the corresponding action of $G$ on $C(X)$ has the Rokhlin property. (Use the earlier exercise on free actions on the Cantor set.)

In the exercises above, condition (2) is trivial. Can it be satisfied in a nontrivial way? In particular, are there any actions on simple C*-algebras with the Rokhlin property?
An example using a simple C*-algebra (more in the next lecture)

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

We want an example in which $A$ is simple. Thus, we won’t be able to satisfy condition (2) by choosing $e_g$ to be in the center of $A$. 

From Lecture 1, recall the product type action of $\mathbb{Z}_2$ generated by $\beta = \infty \otimes \lim_{n \to \infty} Ad((1 \ 0) \ \ 0 \ -1)$ on $A = \infty \otimes \lim_{n \to \infty} M_2$.

We will show that this action has the Rokhlin property. In fact, we will use an action conjugate to this one: we will use $w = (0 \ 1) \ \ 1 \ 0$ in place of $(1 \ 0) \ \ 0 \ -1$.

Reasons for using $(1 \ 0) \ \ 0 \ -1$ will appear in Lecture 4.
An example using a simple C*-algebra (more in the next lecture)

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - a e_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

We want an example in which \( A \) is simple. Thus, we won’t be able to satisfy condition (2) by choosing \( e_g \) to be in the center of \( A \).

From Lecture 1, recall the product type action of \( \mathbb{Z}_2 \) generated by

\[ \beta = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \text{ on } A = \bigotimes_{n=1}^{\infty} M_2. \]

We will show that this action has the Rokhlin property.
An example using a simple C*-algebra (more in the next lecture)

The conditions in the definition of the Rokhlin property:

1. \[ \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \text{ for all } g, h \in G. \]
2. \[ \| e_g a - a e_g \| < \varepsilon \text{ for all } g \in G \text{ and all } a \in F. \]
3. \[ \sum_{g \in G} e_g = 1. \]

We want an example in which \( A \) is simple. Thus, we won’t be able to satisfy condition (2) by choosing \( e_g \) to be in the center of \( A \).

From Lecture 1, recall the product type action of \( \mathbb{Z}_2 \) generated by \( \beta = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \) on \( A = \bigotimes_{n=1}^{\infty} M_2 \).

We will show that this action has the Rokhlin property.

In fact, we will use an action conjugate to this one: we will use \( w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) in place of \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).
An example using a simple C*-algebra (more in the next lecture)

The conditions in the definition of the Rokhlin property:

1. \( \| \alpha_g(e_h) - e_{gh} \| < \varepsilon \) for all \( g, h \in G \).
2. \( \| e_g a - a e_g \| < \varepsilon \) for all \( g \in G \) and all \( a \in F \).
3. \( \sum_{g \in G} e_g = 1 \).

We want an example in which \( A \) is simple. Thus, we won’t be able to satisfy condition (2) by choosing \( e_g \) to be in the center of \( A \).

From Lecture 1, recall the product type action of \( \mathbb{Z}_2 \) generated by

\[
\beta = \bigotimes_{n=1}^\infty \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \quad \text{on} \quad A = \bigotimes_{n=1}^\infty M_2.
\]

We will show that this action has the Rokhlin property.

In fact, we will use an action conjugate to this one: we will use \( w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) in place of \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

Reasons for using \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) will appear in Lecture 4.
An example using a simple C*-algebra (more in the next lecture)

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

We want an example in which $A$ is simple. Thus, we won't be able to satisfy condition (2) by choosing $e_g$ to be in the center of $A$.

From Lecture 1, recall the product type action of $\mathbb{Z}_2$ generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$ on $A = \bigotimes_{n=1}^{\infty} M_2$.

We will show that this action has the Rokhlin property.

In fact, we will use an action conjugate to this one: we will use $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in place of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Reasons for using $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ will appear in Lecture 4.
An example using a simple C*-algebra (more in the next lecture)

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

We want an example in which $A$ is simple. Thus, we won’t be able to satisfy condition (2) by choosing $e_g$ to be in the center of $A$.

From Lecture 1, recall the product type action of $\mathbb{Z}_2$ generated by

$$\beta = \bigotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \text{ on } A = \bigotimes_{n=1}^{\infty} M_2.$$ 

We will show that this action has the Rokhlin property.

In fact, we will use an action conjugate to this one: we will use $w = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ in place of $\left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$.

Reasons for using $\left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ will appear in Lecture 4.
An example (continued)

We had

\[ w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The action \( \alpha \) of \( \mathbb{Z}_2 \) is generated by

\[ \bigotimes_{n=1}^\infty \text{Ad}(w) \text{ on } A = \bigotimes_{n=1}^\infty M_2. \]
An example (continued)

We had

\[ w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The action \( \alpha \) of \( \mathbb{Z}_2 \) is generated by

\[ \bigotimes_{n=1}^\infty \text{Ad}(w) \text{ on } A = \bigotimes_{n=1}^\infty M_2. \]

Define projections \( p_0, p_1 \in M_2 \) by

\[ p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]
An example (continued)

We had

\[ w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The action \( \alpha \) of \( \mathbb{Z}_2 \) is generated by

\[ \bigotimes_{n=1}^{\infty} \text{Ad}(w) \text{ on } A = \bigotimes_{n=1}^{\infty} M_2. \]

Define projections \( p_0, p_1 \in M_2 \) by

\[ p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

Then

\[ wp_0 w^* = p_1, \quad wp_1 w^* = p_0, \quad \text{and} \quad p_0 + p_1 = 1. \]
An example (continued)

We had

\[ w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The action \( \alpha \) of \( \mathbb{Z}_2 \) is generated by

\[ \bigotimes_{n=1}^{\infty} \text{Ad}(w) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2. \]

Define projections \( p_0, p_1 \in M_2 \) by

\[ p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

Then

\[ wp_0 w^* = p_1, \quad wp_1 w^* = p_0, \quad \text{and} \quad p_0 + p_1 = 1. \]
An example (continued)

We had

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

The action $\alpha$ of $\mathbb{Z}_2$ is generated by

$$\bigotimes_{n=1}^{\infty} \text{Ad}(w) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$ 

Define projections $p_0, p_1 \in M_2$ by

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then

$$wp_0w^* = p_1, \quad wp_1w^* = p_0, \quad \text{and} \quad p_0 + p_1 = 1.$$
The action $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$. 
The action $\alpha : \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections $e_g$ such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$. 
The action $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections $e_g$ such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in $A$, we can assume $F \subset A_n$ for some $n$. (Exercise: Check this!)
The action $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections $e_g$ such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in $A$, we can assume $F \subset A_n$ for some $n$. (Exercise: Check this!)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$
The action $\alpha : \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections $e_g$ such that:

1. $\| \beta_g(e_h) - e_{gh} \| < \varepsilon$ for all $g, h \in G$.
2. $\| e_g a - a e_g \| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in $A$, we can assume $F \subset A_n$ for some $n$. (Exercise: Check this!)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$ 

Clearly $e_0 + e_1 = 1$. 

N. C. Phillips (U of Oregon) 
Crossed Products; the Rokhlin Property  
15 July 2016 34 / 41
The action $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections $e_g$ such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in $A$, we can assume $F \subset A_n$ for some $n$. (Exercise: Check this!)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$  

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that $e_0$ and $e_1$ actually commute with everything in $F$. (Proofs: See the next slide.)
The action $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections $e_g$ such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - ae_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in $A$, we can assume $F \subset A_n$ for some $n$. (Exercise: Check this!)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$ 

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that $e_0$ and $e_1$ actually commute with everything in $F$. (Proofs: See the next slide.) This proves the Rokhlin property.
The action $\alpha : \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections $e_g$ such that:

1. $\| \beta_g(e_h) - e_{gh} \| < \varepsilon$ for all $g, h \in G$.
2. $\| e_g a - ae_g \| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in $A$, we can assume $F \subset A_n$ for some $n$. (Exercise: Check this!)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$ 

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that $e_0$ and $e_1$ actually commute with everything in $F$. (Proofs: See the next slide.) This proves the Rokhlin property.
The action $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $wp_0 w^* = p_1$, $wp_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections $e_g$ such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in $A$, we can assume $F \subset A_n$ for some $n$. (Exercise: Check this!)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$ 

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that $e_0$ and $e_1$ actually commute with everything in $F$. (Proofs: See the next slide.) This proves the Rokhlin property.
An example (continued)

The projections $e_0$ and $e_1$ actually commute with everything in $F$, essentially because the nontrival parts are in different tensor factors.
An example (continued)

The projections $e_0$ and $e_1$ actually commute with everything in $F$, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of $F$ have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$
An example (continued)

The projections $e_0$ and $e_1$ actually commute with everything in $F$, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^n+1}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of $F$ have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$ 

Clearly these commute.
An example (continued)

The projections $e_0$ and $e_1$ actually commute with everything in $F$, essentially because the nontrivial parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2n+1}$, which we identify with $M_{2n} \otimes M_2$. In this tensor factorization, elements of $F$ have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \text{Ad}(w^{\otimes n} \otimes w)$,
An example (continued)

The projections $e_0$ and $e_1$ actually commute with everything in $F$, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_2^{2n+1}$, which we identify with $M_{2n} \otimes M_2$. In this tensor factorization, elements of $F$ have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \text{Ad}(w^n \otimes w)$, so

$$\beta(e_0) = (w^n \otimes w)(1 \otimes p_0)(w^n \otimes w)^* = 1 \otimes wp_0 w^* = 1 \otimes p_1 = e_1.$$
An example (continued)

The projections $e_0$ and $e_1$ actually commute with everything in $F$, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of $F$ have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \text{Ad}(w^\otimes n \otimes w)$, so

$$\beta(e_0) = (w^\otimes n \otimes w)(1 \otimes p_0)(w^\otimes n \otimes w)^* = 1 \otimes wp_0 w^* = 1 \otimes p_1 = e_1.$$

The proof that $\beta(e_1) = e_0$ is the same.
An example (continued)

The projections $e_0$ and $e_1$ actually commute with everything in $F$, essentially because the nontrivial parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of $F$ have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \text{Ad}(w^\otimes n \otimes w)$, so

$$\beta(e_0) = (w^\otimes n \otimes w)(1 \otimes p_0)(w^\otimes n \otimes w)^* = 1 \otimes wp_0w^* = 1 \otimes p_1 = e_1.$$  

The proof that $\beta(e_1) = e_0$ is the same.
An example (continued)

The projections $e_0$ and $e_1$ actually commute with everything in $F$, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of $F$ have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \text{Ad}(w^{\otimes n} \otimes w)$, so

$$\beta(e_0) = (w^{\otimes n} \otimes w)(1 \otimes p_0)(w^{\otimes n} \otimes w)^* = 1 \otimes wp_0w^* = 1 \otimes p_1 = e_1.$$

The proof that $\beta(e_1) = e_0$ is the same.
Appendix: Cuntz algebras and some actions on them

We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.

Let \( d \in \{2, 3, \ldots \} \). Recall that the Cuntz algebra \( \mathcal{O}_d \) is the universal C*-algebra on generators \( s_1, s_2, \ldots, s_d \) satisfying the relations

\[
s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1.
\]
Appendix: Cuntz algebras and some actions on them

We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.

Let \( d \in \{2, 3, \ldots \} \). Recall that the Cuntz algebra \( \mathcal{O}_d \) is the universal C*-algebra on generators \( s_1, s_2, \ldots, s_d \) satisfying the relations

\[
s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1.
\]

Thus, \( s_1, s_2, \ldots, s_d \) are isometries with orthogonal ranges which add up to 1.
Appendix: Cuntz algebras and some actions on them

We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.

Let $d \in \{2, 3, \ldots\}$. Recall that the Cuntz algebra $\mathcal{O}_d$ is the universal C*-algebra on generators $s_1, s_2, \ldots, s_d$ satisfying the relations

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$$
and

$$s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.$$

Thus, $s_1, s_2, \ldots, s_d$ are isometries with orthogonal ranges which add up to 1. The Cuntz algebra $\mathcal{O}_\infty$ is the universal C*-algebra generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.
Appendix: Cuntz algebras and some actions on them

We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.

Let \( d \in \{2, 3, \ldots\} \). Recall that the Cuntz algebra \( \mathcal{O}_d \) is the universal C*-algebra on generators \( s_1, s_2, \ldots, s_d \) satisfying the relations

\[
    s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.
\]

Thus, \( s_1, s_2, \ldots, s_d \) are isometries with orthogonal ranges which add up to 1. The Cuntz algebra \( \mathcal{O}_\infty \) is the universal C*-algebra generated by isometries \( s_1, s_2, \ldots \) with orthogonal ranges. Thus, \( s_1^* s_1 = s_2^* s_2 = \cdots = 1 \) and \( s_j^* s_k = 0 \) for \( j \neq k \).
Appendix: Cuntz algebras and some actions on them

We will be more concerned with stably finite simple $C^*$-algebras here, but the basic examples of purely infinite simple $C^*$-algebras should at least be mentioned.

Let $d \in \{2, 3, \ldots \}$. Recall that the Cuntz algebra $\mathcal{O}_d$ is the universal $C^*$-algebra on generators $s_1, s_2, \ldots, s_d$ satisfying the relations

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.$$ 

Thus, $s_1, s_2, \ldots, s_d$ are isometries with orthogonal ranges which add up to 1. The Cuntz algebra $\mathcal{O}_\infty$ is the universal $C^*$-algebra generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges. Thus, $s_1^* s_1 = s_2^* s_2 = \cdots = 1$ and $s_j^* s_k = 0$ for $j \neq k$.

These algebras are purely infinite, simple, and nuclear. Details and other properties are on the next slide.
Appendix: Cuntz algebras and some actions on them

We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.

Let $d \in \{2, 3, \ldots\}$. Recall that the Cuntz algebra $\mathcal{O}_d$ is the universal C*-algebra on generators $s_1, s_2, \ldots, s_d$ satisfying the relations

$$s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1.$$ 

Thus, $s_1, s_2, \ldots, s_d$ are isometries with orthogonal ranges which add up to 1. The Cuntz algebra $\mathcal{O}_\infty$ is the universal C*-algebra generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges. Thus, $s_1^*s_1 = s_2^*s_2 = \cdots = 1$ and $s_j^*s_k = 0$ for $j \neq k$.

These algebras are purely infinite, simple, and nuclear. Details and other properties are on the next slide.
Appendix: Cuntz algebras and some actions on them

We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.

Let \( d \in \{2, 3, \ldots \} \). Recall that the Cuntz algebra \( \mathcal{O}_d \) is the universal C*-algebra on generators \( s_1, s_2, \ldots, s_d \) satisfying the relations

\[
 s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1.
\]

Thus, \( s_1, s_2, \ldots, s_d \) are isometries with orthogonal ranges which add up to 1. The Cuntz algebra \( \mathcal{O}_\infty \) is the universal C*-algebra generated by isometries \( s_1, s_2, \ldots \) with orthogonal ranges. Thus, \( s_1^*s_1 = s_2^*s_2 = \cdots = 1 \) and \( s_j^*s_k = 0 \) for \( j \neq k \).

These algebras are purely infinite, simple, and nuclear. Details and other properties are on the next slide.
Cuntz algebras (continued)
Some standard facts, presented without proof.

\[ O_d \text{ is simple for } d \in \{2, 3, \ldots, \infty\}. \]

For \( d \in \{2, 3, \ldots\} \), for example, this means that whenever elements \( s_1, s_2, \ldots, s_d \) in any unital C*-algebra satisfy
\[
s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \text{ and } s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1,
\]
then they generate a copy of \( O_d \).

\( O_d \) is purely infinite and nuclear.

\[ K_1(O_d) = 0, \quad K_0(O_\infty) \cong \mathbb{Z}, \ \text{generated by } [1], \]

and \( K_0(O_d) \cong \mathbb{Z}^{d-1}, \ \text{generated by } [1], \) for \( d \in \{2, 3, \ldots\} \).

If \( A \) is any simple separable unital nuclear C*-algebra, then
\[ O_2 \otimes A \cong O_2. \]

If \( A \) is any simple separable purely infinite nuclear C*-algebra, then
\[ O_\infty \otimes A \cong A. \]

The last two facts are Kirchberg's absorption theorems. They are much harder.
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $O_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$.
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

$$s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1,$$

then they generate a copy of $\mathcal{O}_d$. 

$\mathcal{O}_d$ is purely infinite and nuclear.

$K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_\infty) \simeq \mathbb{Z}$, generated by [1], and $K_0(\mathcal{O}_d) \simeq \mathbb{Z}^{d-1}$, generated by [1], for $d \in \{2, 3, \ldots\}$.

If $A$ is any simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \simeq \mathcal{O}_2$.

If $A$ is any simple separable purely infinite nuclear C*-algebra, then $\mathcal{O}_\infty \otimes A \simeq A$.

The last two facts are Kirchberg's absorption theorems. They are much harder.
Cuntz algebras (continued)

Some standard facts, presented without proof.

- \( \mathcal{O}_d \) is simple for \( d \in \{2, 3, \ldots, \infty\} \). For \( d \in \{2, 3, \ldots\} \), for example, this means that whenever elements \( s_1, s_2, \ldots, s_d \) in any unital C*-algebra satisfy

\[
s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1,
\]

then they generate a copy of \( \mathcal{O}_d \).

- \( \mathcal{O}_d \) is purely infinite and nuclear.
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

$$s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1,$$

then they generate a copy of $\mathcal{O}_d$.

- $\mathcal{O}_d$ is purely infinite and nuclear.

- $K_1(\mathcal{O}_d) = 0$. 

Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

\[
s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1,
\]

then they generate a copy of $\mathcal{O}_d$.

- $\mathcal{O}_d$ is purely infinite and nuclear.

- $K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_{\infty}) \cong \mathbb{Z}$, generated by [1],
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

\[
s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1,
\]

then they generate a copy of $\mathcal{O}_d$.

- $\mathcal{O}_d$ is purely infinite and nuclear.

- $K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, generated by [1], and $K_0(\mathcal{O}_d) \cong \mathbb{Z}_{d-1}$, generated by [1], for $d \in \{2, 3, \ldots\}$.

If $A$ is any simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

If $A$ is any simple separable purely infinite nuclear C*-algebra, then $\mathcal{O}_\infty \otimes A \cong A$.

The last two facts are Kirchberg's absorption theorems. They are much harder.
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

$$s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1$$

and

$$s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1,$$

then they generate a copy of $\mathcal{O}_d$.

- $\mathcal{O}_d$ is purely infinite and nuclear.

- $K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, generated by $[1]$, and $K_0(\mathcal{O}_d) \cong \mathbb{Z}_{d-1}$, generated by $[1]$, for $d \in \{2, 3, \ldots\}$.

- If $A$ is any simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$. If $A$ is any simple separable purely infinite nuclear C*-algebra, then $\mathcal{O}_\infty \otimes A \cong A$.
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

\[ s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1, \]

then they generate a copy of $\mathcal{O}_d$.

- $\mathcal{O}_d$ is purely infinite and nuclear.

- $K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, generated by $[1]$, and $K_0(\mathcal{O}_d) \cong \mathbb{Z}_{d-1}$, generated by $[1]$, for $d \in \{2, 3, \ldots\}$.

- If $A$ is any simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

- If $A$ is any simple separable purely infinite nuclear C*-algebra, then $\mathcal{O}_\infty \otimes A \cong A$. 
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

$$s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \quad \text{and} \quad s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1,$$

then they generate a copy of $\mathcal{O}_d$.

- $\mathcal{O}_d$ is purely infinite and nuclear.

- $K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, generated by $[1]$, and $K_0(\mathcal{O}_d) \cong \mathbb{Z}_{d-1}$, generated by $[1]$, for $d \in \{2, 3, \ldots\}$.

- If $A$ is any simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

- If $A$ is any simple separable purely infinite nuclear C*-algebra, then $\mathcal{O}_\infty \otimes A \cong A$.

The last two facts are Kirchberg’s absorption theorems. They are much harder.
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1,$$

then they generate a copy of $\mathcal{O}_d$.

- $\mathcal{O}_d$ is purely infinite and nuclear.

- $K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, generated by $[1]$, and $K_0(\mathcal{O}_d) \cong \mathbb{Z}_{d-1}$, generated by $[1]$, for $d \in \{2, 3, \ldots\}$.

- If $A$ is any simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

- If $A$ is any simple separable purely infinite nuclear C*-algebra, then $\mathcal{O}_\infty \otimes A \cong A$.

The last two facts are Kirchberg’s absorption theorems. They are much harder.
Cuntz algebras (continued)

Some standard facts, presented without proof.

- $O_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1,$$

then they generate a copy of $O_d$.

- $O_d$ is purely infinite and nuclear.

- $K_1(O_d) = 0$, $K_0(O_\infty) \cong \mathbb{Z}$, generated by [1], and $K_0(O_d) \cong \mathbb{Z}_{d-1}$, generated by [1], for $d \in \{2, 3, \ldots\}$.

- If $A$ is any simple separable unital nuclear C*-algebra, then $O_2 \otimes A \cong O_2$.

- If $A$ is any simple separable purely infinite nuclear C*-algebra, then $O_\infty \otimes A \cong A$.

The last two facts are Kirchberg’s absorption theorems. They are much harder.
Actions on Cuntz algebras

For \( d \) finite, \( \mathcal{O}_d \) is generated by isometries \( s_1, s_2, \ldots, s_d \) with orthogonal ranges which add up to 1, and \( \mathcal{O}_\infty \) is generated by isometries \( s_1, s_2, \ldots \) with orthogonal ranges.

Let \( \rho: G \to \mathcal{L}(C^d) \) be a unitary representation of \( G \).

Write \( \rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix} \) for \( g \in G \).

Then there exists a unique action \( \beta_\rho: G \to \text{Aut}(\mathcal{O}_d) \) such that

\[
\beta_\rho g(s_k) = \sum_{j=1}^d \rho_{j,k}(g)s_j
\]

for \( j = 1, 2, \ldots, d \).

(This can be checked by a computation.)

For \( d = \infty \), a similar formula works for any unitary representation of \( G \) on \( l^2(\mathbb{N}) \).
Actions on Cuntz algebras

For $d$ finite, $O_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $O_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.
Actions on Cuntz algebras

For $d$ finite, $\mathcal{O}_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $\mathcal{O}_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho: G \to \mathcal{L}(\mathbb{C}^d)$ be a unitary representation of $G$. 

Write $\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$ for $g \in G$. Then there exists a unique action $\beta_{\rho}: G \to \text{Aut}(\mathcal{O}_d)$ such that

$$\beta_{\rho} g (s_k) = \sum_{j=1}^{d} \rho_j (g) s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.)

For $d = \infty$, a similar formula works for any unitary representation of $G$ on $l^2(\mathbb{N})$. 

N. C. Phillips (U of Oregon)
Actions on Cuntz algebras

For $d$ finite, $O_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $O_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho: G \to L(\mathbb{C}^d)$ be a unitary representation of $G$. Write

$$\rho(g) = \begin{pmatrix}
\rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\
\vdots & \ddots & \vdots \\
\rho_{d,1}(g) & \cdots & \rho_{d,d}(g)
\end{pmatrix}$$

for $g \in G$. 
Actions on Cuntz algebras

For $d$ finite, $\mathcal{O}_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $\mathcal{O}_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho: G \rightarrow L(\mathbb{C}^d)$ be a unitary representation of $G$. Write

$$\rho(g) = \begin{pmatrix} 
\rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\
\vdots & \ddots & \vdots \\
\rho_{d,1}(g) & \cdots & \rho_{d,d}(g) 
\end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^\rho: G \rightarrow \text{Aut}(\mathcal{O}_d)$ such that

$$\beta^\rho_g(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g)s_j$$

for $j = 1, 2, \ldots, d$. 
Actions on Cuntz algebras

For $d$ finite, $\mathcal{O}_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $\mathcal{O}_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho: G \to L(\mathbb{C}^d)$ be a unitary representation of $G$. Write

$$
\rho(g) = \begin{pmatrix}
\rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\
\vdots & \ddots & \vdots \\
\rho_{d,1}(g) & \cdots & \rho_{d,d}(g)
\end{pmatrix}
$$

for $g \in G$. Then there exists a unique action $\beta^\rho: G \to \text{Aut}(\mathcal{O}_d)$ such that

$$
\beta^\rho_g(s_k) = \sum_{j=1}^d \rho_{j,k}(g)s_j
$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.)
Actions on Cuntz algebras

For $d$ finite, $\mathcal{O}_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $\mathcal{O}_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho: G \to L(\mathbb{C}^d)$ be a unitary representation of $G$. Write

$$
\rho(g) = \begin{pmatrix}
\rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\
\vdots & \ddots & \vdots \\
\rho_{d,1}(g) & \cdots & \rho_{d,d}(g)
\end{pmatrix}
$$

for $g \in G$. Then there exists a unique action $\beta^\rho: G \to \text{Aut}(\mathcal{O}_d)$ such that

$$
\beta^\rho_g(s_k) = \sum_{j=1}^d \rho_{j,k}(g)s_j
$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.) For $d = \infty$, a similar formula works for any unitary representation of $G$ on $l^2(\mathbb{N})$. 
Actions on Cuntz algebras

For $d$ finite, $\mathcal{O}_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $\mathcal{O}_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho: G \to L(\mathbb{C}^d)$ be a unitary representation of $G$. Write

$$\rho(g) = \begin{pmatrix}
\rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\
\vdots & \ddots & \vdots \\
\rho_{d,1}(g) & \cdots & \rho_{d,d}(g)
\end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^\rho: G \to \text{Aut}(\mathcal{O}_d)$ such that

$$\beta^\rho_g(s_k) = \sum_{j=1}^d \rho_{j,k}(g)s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.) For $d = \infty$, a similar formula works for any unitary representation of $G$ on $l^2(\mathbb{N})$. 

N. C. Phillips (U of Oregon)
Actions on Cuntz algebras

For $d$ finite, $\mathcal{O}_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $\mathcal{O}_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho: G \to L(\mathbb{C}^d)$ be a unitary representation of $G$. Write

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^\rho: G \to \text{Aut}(\mathcal{O}_d)$ such that

$$\beta^\rho_g(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g)s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.) For $d = \infty$, a similar formula works for any unitary representation of $G$ on $l^2(\mathbb{N})$. 

N. C. Phillips (U of Oregon)  
Crossed Products; the Rokhlin Property  
15 July 2016  
39 / 41
Actions on Cuntz algebras (continued)

The Cuntz relations: $s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1$ and $s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1$. (For $d = \infty$, $s_1, s_2, \ldots$ are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:
Actions on Cuntz algebras (continued)

The Cuntz relations: $s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1$ and $s_1s_1^* + s_2s_2^* + \cdots + s_d s_d^* = 1$. (For $d = \infty$, $s_1, s_2, \ldots$ are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For $G = \mathbb{Z}_n$, choose $n$-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$
Actions on Cuntz algebras (continued)

The Cuntz relations: \( s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \) and 
\( s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1. \) (For \( d = \infty \), \( s_1, s_2, \ldots \) are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For \( G = \mathbb{Z}_n \), choose \( n \)-th roots of unity \( \zeta_1, \zeta_2, \ldots, \zeta_d \) and let a generator of the group multiply \( s_j \) by \( \zeta_j \).
Actions on Cuntz algebras (continued)

The Cuntz relations: \( s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \) and
\( s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1. \) (For \( d = \infty \), \( s_1, s_2, \ldots \) are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For \( G = \mathbb{Z}_n \), choose \( n \)-th roots of unity \( \zeta_1, \zeta_2, \ldots, \zeta_d \) and let a generator of the group multiply \( s_j \) by \( \zeta_j \).
- Let \( G \) be a finite group. Take \( d = \text{card}(G) \), and label the generators \( s_g \) for \( g \in G \).
Actions on Cuntz algebras (continued)

The Cuntz relations: 
\[ s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \]  
and 
\[ s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1. \] (For \( d = \infty \), \( s_1, s_2, \ldots \) are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For \( G = \mathbb{Z}_n \), choose \( n \)-th roots of unity \( \zeta_1, \zeta_2, \ldots, \zeta_d \) and let a generator of the group multiply \( s_j \) by \( \zeta_j \).

- Let \( G \) be a finite group. Take \( d = \text{card}(G) \), and label the generators \( s_g \) for \( g \in G \). Then define \( \beta^G : G \to \text{Aut}(O_d) \) by \( \beta_g^G(s_h) = s_{gh} \) for \( g, h \in G \).
Actions on Cuntz algebras (continued)

The Cuntz relations: \( s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \) and
\( s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1 \). (For \( d = \infty \), \( s_1, s_2, \ldots \) are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For \( G = \mathbb{Z}_n \), choose \( n \)-th roots of unity \( \zeta_1, \zeta_2, \ldots, \zeta_d \) and let a generator of the group multiply \( s_j \) by \( \zeta_j \).
- Let \( G \) be a finite group. Take \( d = \text{card}(G) \), and label the generators \( s_g \) for \( g \in G \). Then define \( \beta^G : G \to \text{Aut}(O_d) \) by \( \beta^G_g(s_h) = s_{gh} \) for \( g, h \in G \). (This is the quasifree action coming from regular representation of \( G \).)
Actions on Cuntz algebras (continued)

The Cuntz relations: \( s_1 s_1^* = s_2 s_2^* = \cdots = s_d s_d^* = 1 \) and
\( s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1 \). (For \( d = \infty \), \( s_1, s_2, \ldots \) are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For \( G = \mathbb{Z}_n \), choose \( n \)-th roots of unity \( \zeta_1, \zeta_2, \ldots, \zeta_d \) and let a generator of the group multiply \( s_j \) by \( \zeta_j \).
- Let \( G \) be a finite group. Take \( d = \text{card}(G) \), and label the generators \( s_g \) for \( g \in G \). Then define \( \beta^G : G \to \text{Aut}(\mathcal{O}_d) \) by \( \beta^G_g(s_h) = s_{gh} \) for \( g, h \in G \). (This is the quasifree action coming from regular representation of \( G \).)
- Label the generators of \( \mathcal{O}_\infty \) as \( s_{g,j} \) for \( g \in G \) and \( j \in \mathbb{N} \).
Actions on Cuntz algebras (continued)

The Cuntz relations: $s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1$ and $s_1s_1^* + s_2s_2^* + \cdots + s_d s_d^* = 1$. (For $d = \infty$, $s_1, s_2, \ldots$ are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For $G = \mathbb{Z}_n$, choose $n$-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$ and let a generator of the group multiply $s_j$ by $\zeta_j$.

- Let $G$ be a finite group. Take $d = \text{card}(G)$, and label the generators $s_g$ for $g \in G$. Then define $\beta^G \colon G \to \text{Aut}(\mathcal{O}_d)$ by $\beta^G_g(s_h) = s_{gh}$ for $g, h \in G$. (This is the quasifree action coming from regular representation of $G$.)

- Label the generators of $\mathcal{O}_\infty$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{N}$. Define $\iota : G \to \text{Aut}(\mathcal{O}_\infty)$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{N}$.
Actions on Cuntz algebras (continued)

The Cuntz relations: \( s_1^*s_1 = s_2^*s_2 = \cdots = s_d^*s_d = 1 \) and 
\[ s_1s_1^* + s_2s_2^* + \cdots + s_ds_d^* = 1. \] (For \( d = \infty \), \( s_1, s_2, \ldots \) are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For \( G = \mathbb{Z}_n \), choose \( n \)-th roots of unity \( \zeta_1, \zeta_2, \ldots, \zeta_d \) and let a generator of the group multiply \( s_j \) by \( \zeta_j \).

- Let \( G \) be a finite group. Take \( d = \text{card}(G) \), and label the generators \( s_g \) for \( g \in G \). Then define \( \beta^G : G \to \text{Aut}(\mathcal{O}_d) \) by \( \beta^G_g(s_h) = s_{gh} \) for \( g, h \in G \). (This is the quasifree action coming from regular representation of \( G \).)

- Label the generators of \( \mathcal{O}_\infty \) as \( s_{g,j} \) for \( g \in G \) and \( j \in \mathbb{N} \). Define \( \iota : G \to \text{Aut}(\mathcal{O}_\infty) \) by \( \iota_g(s_{h,j}) = s_{gh,j} \) for \( g \in G \) and \( j \in \mathbb{N} \). (This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation of \( G \).)
Actions on Cuntz algebras (continued)

The Cuntz relations: \( s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \) and \( s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1 \). (For \( d = \infty \), \( s_1, s_2, \ldots \) are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For \( G = \mathbb{Z}_n \), choose \( n \)-th roots of unity \( \zeta_1, \zeta_2, \ldots, \zeta_d \) and let a generator of the group multiply \( s_j \) by \( \zeta_j \).

- Let \( G \) be a finite group. Take \( d = \text{card}(G) \), and label the generators \( s_g \) for \( g \in G \). Then define \( \beta^G : G \to \text{Aut}(\mathcal{O}_d) \) by \( \beta^G_g(s_h) = s_{gh} \) for \( g, h \in G \). (This is the quasifree action coming from regular representation of \( G \).)

- Label the generators of \( \mathcal{O}_\infty \) as \( s_{g,j} \) for \( g \in G \) and \( j \in \mathbb{N} \). Define \( \iota : G \to \text{Aut}(\mathcal{O}_\infty) \) by \( \iota_g(s_{h,j}) = s_{gh,j} \) for \( g \in G \) and \( j \in \mathbb{N} \). (This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation of \( G \).)
Actions on Cuntz algebras (continued)

The Cuntz relations: $s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$ and $s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1$. (For $d = \infty$, $s_1, s_2, \ldots$ are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For $G = \mathbb{Z}_n$, choose $n$-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$ and let a generator of the group multiply $s_j$ by $\zeta_j$.

- Let $G$ be a finite group. Take $d = \text{card}(G)$, and label the generators $s_g$ for $g \in G$. Then define $\beta^G : G \to \text{Aut}(\mathcal{O}_d)$ by $\beta^G_g(s_h) = s_{gh}$ for $g, h \in G$. (This is the quasifree action coming from regular representation of $G$.)

- Label the generators of $\mathcal{O}_\infty$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{N}$. Define $\iota : G \to \text{Aut}(\mathcal{O}_\infty)$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{N}$. (This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation of $G$.)
Actions on Cuntz algebras: The tensor flips on $\mathcal{O}_2$ and $\mathcal{O}_{\infty}$

There are tensor flip actions of $\mathbb{Z}_2$ on $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $\mathcal{O}_{\infty} \otimes \mathcal{O}_{\infty}$. 
Actions on Cuntz algebras: The tensor flips on $O_2$ and $O_\infty$

There are tensor flip actions of $\mathbb{Z}_2$ on $O_2 \otimes O_2$ and $O_\infty \otimes O_\infty$. Since

$$O_2 \otimes O_2 \cong O_2 \quad \text{and} \quad O_\infty \otimes O_\infty \cong O_\infty,$$

one gets actions of $\mathbb{Z}_2$ on $O_2$ and $O_\infty$.

More generally, any subgroup of $S_n$ acts on the $n$-fold tensor products $(O_2)^\otimes n$ and $(O_\infty)^\otimes n$. This gives actions of these groups on $O_2$ and $O_\infty$. 

There are tensor flip actions of $\mathbb{Z}_2$ on $O_2 \otimes O_2$ and $O_\infty \otimes O_\infty$. Since

$$O_2 \otimes O_2 \cong O_2 \quad \text{and} \quad O_\infty \otimes O_\infty \cong O_\infty,$$

one gets actions of $\mathbb{Z}_2$ on $O_2$ and $O_\infty$. 
There are tensor flip actions of $\mathbb{Z}_2$ on $O_2 \otimes O_2$ and $O_\infty \otimes O_\infty$. Since

$$O_2 \otimes O_2 \cong O_2 \quad \text{and} \quad O_\infty \otimes O_\infty \cong O_\infty,$$

one gets actions of $\mathbb{Z}_2$ on $O_2$ and $O_\infty$.

More generally, any subgroup of $S_n$ acts on the $n$-fold tensor products $(O_2)^{\otimes n}$ and $(O_\infty)^{\otimes n}$. 
Actions on Cuntz algebras: The tensor flips on $O_2$ and $O_\infty$

There are tensor flip actions of $\mathbb{Z}_2$ on $O_2 \otimes O_2$ and $O_\infty \otimes O_\infty$. Since $O_2 \otimes O_2 \cong O_2$ and $O_\infty \otimes O_\infty \cong O_\infty$, one gets actions of $\mathbb{Z}_2$ on $O_2$ and $O_\infty$.

More generally, any subgroup of $S_n$ acts on the $n$-fold tensor products $(O_2)^\otimes n$ and $(O_\infty)^\otimes n$. This gives actions of these groups on $O_2$ and $O_\infty$. 
There are tensor flip actions of \( \mathbb{Z}_2 \) on \( O_2 \otimes O_2 \) and \( O_\infty \otimes O_\infty \). Since
\[
O_2 \otimes O_2 \cong O_2 \quad \text{and} \quad O_\infty \otimes O_\infty \cong O_\infty,
\]
one gets actions of \( \mathbb{Z}_2 \) on \( O_2 \) and \( O_\infty \).

More generally, any subgroup of \( S_n \) acts on the \( n \)-fold tensor products \( (O_2)^\otimes n \) and \( (O_\infty)^\otimes n \). This gives actions of these groups on \( O_2 \) and \( O_\infty \).
Actions on Cuntz algebras: The tensor flips on $\mathcal{O}_2$ and $\mathcal{O}_\infty$

There are tensor flip actions of $\mathbb{Z}_2$ on $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$. Since

$$\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2 \quad \text{and} \quad \mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty,$$

one gets actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ and $\mathcal{O}_\infty$.

More generally, any subgroup of $S_n$ acts on the $n$-fold tensor products $(\mathcal{O}_2)^\otimes n$ and $(\mathcal{O}_\infty)^\otimes n$. This gives actions of these groups on $\mathcal{O}_2$ and $\mathcal{O}_\infty$. 