A FEW REFERENCES ON CONTINUOUS FIELDS OF C*-ALGEBRAS

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This is a collection of a few references to work on continuous fields of C*-algebras, limited by what I remember having seen. There are probably many further references in [10], but not everything here is there. Essentially no proofreading has been done.

Continuous fields of C*-algebras are defined in Chapter 10 of the textbook [5]. They appear in the theory of type I C*-algebras, since a type I C*-algebra with Hausdorff primitive ideal space and satisfying another natural condition is the section algebra of a continuous field of C*-algebras with fiber $K(H)$. These are the "continuous trace C*-algebras". They are not necessarily locally trivial, although local triviality is automatic if either the algebra is stable or the the fibers are all separable infinite dimensional and the primitive ideal space is finite dimensional.

There is an obstruction $\delta \in \check{H}^3(\mathrm{Prim}(A); \mathbb{Z})$ to triviality of a locally trivial continuous trace algebra, called the Dixmier-Douady invariant, originally treated in (I think) [6] and [4], and found in [5]. Standard modern homotopy theory gives a construction which takes a few lines. See [11]. Also see the book [10], which should contain all this and a lot more, but generally in the same direction. The method of construction of the continuous trace algebra with fiber $K(l^2)$ and infinite dimensional primitive ideal space which is not locally trivial (which is in [5] and surely others of the references above) has been adapted to construct other bizarre examples, but I don't remember the reference for the one I am thinking of.

One application of this theory is in [9], following up on earlier work of Raeburn.

There has been work on generalization of the Dixmier-Douady invariant to locally trivial continuous fields with other fibers, mainly strongly selfabsorbing fibers. See, for example, [3], [2], and [1] (apparently published in reverse order of having been written).

All continuous fields in the papers mentioned so far (beyond the basic definitions) have been at least somewhat close to being locally trivial. But many interesting examples are far from locally trivial. There is, for example, a continuous field over S^1 whose fiber over $\exp(2\pi i\theta)$ is the (rational or irrational) rotation algebra A_θ. Its fibers are pairwise nonisomorphic except when $\theta_1 = \pm\theta_1 \pmod{\mathbb{Z}}$. Continuous fields like this (and, more generally, bundles) play a large role in [7] and [8].

Section algebras of continuous fields over a compact space X are a special case of $C(X)$-algebras: C*-algebras with are algebraically also algebras over $C(X)$, with suitable compatibility conditions. A $C(X)$-algebra can also be thought of as the section algebra of a bundle, but the sections $x \mapsto a(x)$ now only have the property

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Date: 14 May 2024.

This material is based upon work supported by the US National Science Foundation under Grant DMS-2055771.

that $x \mapsto \|a(x)\|$ is upper semicontinuous, rather than being continuous. For example, for any X and any $x_0 \in X$, any C*-algebra A is a $C(X)$-algebra with the operation $f \cdot a = f(x_0)a$. Here, the fiber over x_0 is A, and all the other fibers are zero. This setup has the advantage that quotient by ideals which are also $C(X)$-algebras are again $C(X)$-algebras. I have not thought of references for work on these (except [7] and [8]), but certainly they have been used.

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