An overview of topological, C*-algebraic, and algebraic K-theory

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What is K-theory?

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This is a superficial overview of topological, C*-algebraic, and algebraic K-theory, with an emphasis on how they are related. There are other kinds of K-theory, which will not be mentioned.

A rough outline

- Algebraic K_0. (This is essentially the same in all three versions of K-theory.)
- ~~Topoloogical~~ Topological K_0.
- ~~Topoloogical~~ Topological K-theory and K-homology.
- K-theory of Banach algebras and C*-algebras; KK-theory.
- Brief comments on algebraic K-theory.

Finitely generated projective modules

Let A be a unital ring. (A could be a unital C*-algebra.) Recall finitely generated projective modules over A. Use right modules. (We will see why later, but it doesn't really matter.) Here we only need: an A-module P is finitely generated and projective if and only if there are $n \in \mathbb{Z}_{\geqslant 0}$ and an A-module Q such that $P \oplus Q \cong A^n$, the rank n free module.

Definition

$V(A)$ is the set of isomorphism classes of finitely generated projective right A-modules. Write $\langle P \rangle$ for the class of P.

As stated, $V(A)$ isn't really a set, because $\langle P \rangle$ isn't a set. There are standard ways to deal with this, omitted here, and in later occurrences in these notes. The standard notation is $[P]$, but that is also the standard for the class in $K_0(A)$.

Proposition

The operation $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$ on $V(A)$ is well defined, and makes $V(A)$ a commutative semigroup with identity the class of the zero module.

Vector bundles and projections

A is a unital ring.

$V(A)$ is the commutative semigroup of isomorphism classes of finitely generated projective right A-modules, with $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$.

The notation $V(A)$ is standard in C*-algebras. I don't know what is common elsewhere.

For topological K-theory of X, we will use finite dimensional vector bundles over X instead of finitely generated projective A-modules.

For C*-algebras, it is common to use Murray-von Neumann equivalence classes of projections in matrix algebras over A.

We will see that these are all really the same thing.

$K_0(A)$

A is a unital ring.

$V(A)$ is the commutative semigroup of isomorphism classes of finitely generated projective right A-modules, with $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$.

For any commutative semigroup S, one can adjoin additive inverses to get a "universal enveloping group" $G(S)$, the Grothendieck group of S. It has a semigroup homomorphism $S \to G(S)$.

Related constructions you have seen: getting $\mathbb{Z}$ from $\mathbb{Z}_{\geqslant 0}$; localization of a commutative ring at a multiplicative system.

Warning: if S does not have cancellation (that is, $r + t = s + t$ does not imply $r = s$), then $S \to G(S)$ need not be injective.

Definition

$K_0(A) = G(V(A))$.

We write $[P]$ for the class of the module P in $K_0(A)$.

Examples

A is a unital ring.

$V(A)$ is the commutative semigroup of isomorphism classes of finitely generated projective right A-modules, with $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$.

$K_0(A)$ results from forcibly making $V(A)$ into a group.

Example

Suppose A is a field. Then the finitely generated projective right A-modules are just the finite dimensional vector spaces over A. So $\langle P \rangle \mapsto \dim(P)$ is an isomorphism $V(A) \to \mathbb{Z}_{\geqslant 0}$. Therefore $K_0 A) \cong \mathbb{Z}$.

It is tricky to give other examples at this stage. But here is an illustration.

Example

Let $H = l^2(\mathbb{Z}_{>0})$, a separable infinite dimensional Hilbert space. Let $L(H)$ be the algebra of bounded operators on H. It turns out that if P is any finitely generated projective right $L(H)$-module, then, as $L(H)$-modules, $P \oplus L(H) \cong L(H)$. In a group, $L(H)$ can be cancelled. So $K_0(L(H)) = 0$.

The calculations are more easily done using idempotents. See below.

Ring structure

A is a unital ring.

$V(A)$ is the commutative semigroup of isomorphism classes of finitely generated projective right A-modules, with $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$.

$K_0(A)$ results from forcibly making $V(A)$ into a group. $[P]$ is the class of the module P in $K_0(A)$.

If A is commutative, and P and Q are right A-modules, then $P \otimes_A Q$ makes sense as a right A-module. The operation $[P][Q] = [P \otimes_A Q]$ extends to $K_0(A)$ and makes $K_0(A)$ a unital ring, with multiplicative identity $[A]$.

If A is not commutative, one can only form $P \otimes_A Q$ when P is a right A-module and P is a left A-module; the result is not an A-module on either side. Thus, generally no ring structure on $K_0(A)$.

It is still true that $K_0(A)$ is a partially ordered abelian group, with the positive elements being the image of $V(A)$. This order structure can be complicated, and is very important in C*-algebras.

Functoriality

Let A and B be unital rings. (They could be unital C*-algebras.) Let $\varphi \colon A \to B$ be a unital homomorphism. Use φ to make B into an A–A bimodule. If P is a finitely generated projective right A-module, then $P \otimes_A B$ is a right B-module. It is finitely generated and projective.

Theorem

The operations $\langle P \rangle \mapsto \langle P \otimes_A B \rangle$ and $[P] \mapsto [P \otimes_A B]$ are well defined. Moreover:

1. $V(-)$ is a covariant functor from unital rings to commutative semigroups with identity.
2. $K_0(-)$ is a covariant functor from unital rings to partially ordered abelian groups.
3. $K_0(-)$ is a covariant functor from commutative unital rings to commutative unital rings.

The maps are usually written φ_*.

Nonunital rings

If A is a unital ring, then $V(A)$ is the commutative semigroup of isomorphism classes of finitely generated projective right A-modules, with $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$, and $K_0(A)$ results from forcibly making $V(A)$ into a group. $[P]$ is the class of the module P in $K_0(A)$.

What about nonunital rings?

If A is a nonunital Banach algebra, there is a unitization $A^+ = A + \mathbb{C} \cdot 1$, again a Banach algebra, with a unitization map $\kappa \colon A^+ \to \mathbb{C}$. It works properly if one defines $K_0(A)$ to be the kernel of $\kappa_* \colon K_0(A^+) \to K_0(\mathbb{C})$.

In the general algebraic situation, one might use $A + \mathbb{Z} \cdot 1$, but for an F-algebra maybe one wants $A + F \cdot 1$. Unfortunately, one might get different answers, and the situation is quite complicated.

Topological K_0

Let X be a compact Hausdorff space. Let $C(X)$ be the set of all complex valued continuous functions on X. It is a unital ring with the pointwise operations, in fact, a unital C*-algebra. We take $K^0(X) = K_0(C(X))$. (This is not the usual way to define $K^0(X)$; the usual definition uses vector bundles.)

$X \mapsto C(X)$ is a contravariant functor from compact Hausdorff spaces to commutative unital C*-algebras. If $h \colon X \to Y$, then $C(h) \colon C(Y) \to C(X)$ is given by $G(h)(f) = f \circ h$.

Then $h^* \colon K^0(Y) \to K^0(X)$ is $h^* = C(h)_*$. The index is conventionally a superscript since K^0 is contravariant on spaces.

In fact, $X \mapsto C(X)$ is a contravariant category equivalence from compact Hausdorff spaces and continuous maps to commutative unital C*-algebras and unital *-homomorphisms.

Vector bundles

Recall that a (finite dimensional, complex) vector bundle E over a topological space X has fibers E_x over $x \in X$, which are finite dimensional complex vector spaces, and the whole structure is "locally trivial".

Examples of real vector bundles: the tangent space of a manifold; the Möbius strip as a bundle over its center line.

If E is a vector bundle over X, its section space $\Gamma(E)$ is the set of continuous functions $s \colon X \to E$ such that $s(x) \in E_x$ for all $x \in X$. It is a module over $C(X)$, with $(sf)(x) = f(x)s(x)$ for $x \in X$, $s \in \Gamma(E)$, and $f \in C(X)$.

Theorem (Swan's Theorem)

Let X be a compact Hausdorff space. Then $E \mapsto \Gamma(E)$ is a category equivalence from finite dimensional complex vector bundles over X to finitely generated projective right $C(X)$-modules.

K-theory from vector bundles

If A is a unital ring, then $V(A)$ is the commutative semigroup of isomorphism classes of finitely generated projective right A-modules, with $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$, and $K_0(A)$ results from forcibly making $V(A)$ into a group.

Swan's Theorem: $E \mapsto \Gamma(E)$ is a category equivalence from finite dimensional complex vector bundles over X to finitely generated projective right $C(X)$-modules.

$K^0(X) = K_0(C(X))$

Therefore $K^0(X)$ has the following description: let $V(X)$ be the commutative semigroup of isomorphism classes of complex vector bundles over X, with the operation induced by direct sum. Then $K^0(X) = G(V(X))$, the universal enveloping group of $V(X)$.

Functoriality of $K^0(X)$

For a compact Hausdorff space X, let $V(X)$ be the commutative semigroup of isomorphism classes of complex vector bundles over X. Then $K^0(X)$ is the universal enveloping group of $V(X)$.

Since $C(X)$ is commutative, $K^0(X)$ is a ring.

Let $h\colon X \to Y$ be continuous, and let F be a vector bundle over Y. There is a topologically very natural definition of the pullback $h^*(F)$, which is a vector bundle over X. Moreover, $[F] \mapsto [h^*(F)]$, as a map from $K^0(Y)$ to $K^0(X)$, is the same as the map $C(h)_*\colon K_0(C(Y)) \to K_0(C(X))$.

That is, $K^0(-)$, as a functor from compact Hausdorff spaces to commutative unital rings, has a natural purely topological description.

Homotopy invariance

Theorem

Let X and Y be compact Hausdorff spaces, and let $h_0, h_1 \colon X \to Y$ be homotopic. Let F be a vector bundle over Y. Then $h_0^*(F) \cong h_1^*(F)$.

Corollary

Let X and Y be compact Hausdorff spaces, and let $h_0, h_1 \colon X \to Y$ be homotopic. Then $h_0^* = h_1^* \colon K^0(Y) \to K^0(X)$.

Homotopy invariance is one of the crucial properties of a cohomology theory.

Locally compact spaces

If X is locally compact, we let $C_0(X)$ be the set of all complex valued continuous functions on X which vanish at infinity. It is a nonunital commutative C*-algebra. We take $K^0(X) = K_0(C_0(X))$. Recall that this is the kernel of $K_0(C_0(X)^+) \to K_0(\mathbb{C})$

If X^+ is the one point compactification of X, then $C_0(X)^+ \cong C(X^+)$, so $K^0(X)$ is the kernel of $K^0(X^+) \to K^0(\{\infty\})$.

Warning: this is cohomology *with compact supports*, better thought of as $K^0(X) = K^0(X^+, \{\infty\})$, and the functoriality is a little strange in topological terms. (It is easily expressed in C*-algebras: from not necessarily unital C*-algebras and not necessarily unital homomorphisms to abelian groups.)

Theorem

Let X be a locally compact Hausdorff space, and let $Y \subset X$ be closed. Then the sequence $K^0(X \setminus Y) \longrightarrow K^0(X) \longrightarrow K^0(Y)$ is exact in the middle.

Topological K-theory

K^0 is a homotopy invariant functor, and middle exact: if $Y \subset X$ be closed, then $K^0(X \backslash Y) \longrightarrow K^0(X) \longrightarrow K^0(Y)$ is exact in the middle.

In topology, this is more conventionally written using the relative group: $K^0(X, Y) \longrightarrow K^0(X) \longrightarrow K^0(Y)$.

This, plus maybe a little more, is enough to give a cohomology theory with long exact sequences. It starts as follows.

Take the suspension SX to be $(0,1) \times X$. This is the right version for theories with compact supports. To compare with the usual suspension: $(SX)^+ = (0,1)^+ \wedge X^+ = S^1 \wedge X^+$. Then mimic $H^n(\Sigma X; \mathbb{Z}) \cong H^{n-1}(X; \mathbb{Z})$ by defining $K^{-1}(X) = K^0(SX)$. (Yes, K^{-1} is correct.)

Iterate: $K^{-n}(X) = K^0(S^n X)$. It is at least immediate that K^{-n} is a homotopy invariant functor and is middle exact. Long exact sequence: the slide after the next slide.

Bott periodicity

For $n \in \mathbb{Z}_{\geqslant 0}$, K^{-n} is a homotopy invariant functor and is middle exact.

Theorem (Bott periodicity)

There is a natural isomorphism $K^0(S^2 X) \cong K^0(X)$, for all locally compact Hausdorff spaces.

There are many different proofs of this, ~~and~~ some long but elementary, some short and slick, using machinery of C*-algebras, and some of which say somewhat more. There also other formulations, in terms of homotopy groups of unitary groups.

The theorem says $K^{-n-2}(X) = K^{-n}(X)$. Use this to *define* a periodic cohomology theory, with period 2, by $K^n(X) = K^0(X)$ when $n \in \mathbb{Z}$ is even, and $K^n(X) = K^{-1}(X)$ when $n \in \mathbb{Z}$ is odd.

Long exact sequence

For $n \in \mathbb{Z}$, K^n is a homotopy invariant functor and is middle exact.

$K^n(X) = K^0(X)$ when $n \in \mathbb{Z}$ is even, and $K^n(X) = K^{-1}(X)$ when $n \in \mathbb{Z}$ is odd.

The standard topological construction gives long exact sequences, here, for $Y \subset X$ closed:

$$\begin{array}{ccccc} K^0(X \setminus Y) & \longrightarrow & K^0(X) & \longrightarrow & K^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(X \setminus Y) & \longleftarrow & K^1(Y) & \longleftarrow & K^1(X). \end{array}$$

If one uses $\mathbb{R}$ instead of $\mathbb{C}$, real vector bundles instead of complex vector bundles, and $C(X, \mathbb{R})$, the set of all *real* valued continuous functions on X, instead of $C(X)$ (but the same definition of algebraic K_0, now applied to $C(X, \mathbb{R})$, one gets the real K-theory of compact Hausdorff spaces. It has period 8 instead of 2.

Chern character

Rationally, $K^0(X) \otimes \mathbb{Q}$ is the direct sum of all even rational (Čech) cohomology (with compact supports), and $K^1(X) \otimes \mathbb{Q}$ is the direct sum of all odd rational (Čech) cohomology. (The isomorphism is natural; it is the Chern character.)

So there seems to be be not so much use for topological K-theory. Three reasons it is actually useful:

- K-theory treats torsion differently.
- Some topological invariants, such as the index of a family of elliptic differential operators, naturally live in K-theory.
- K-theory generalizes nicely to Banach algebras and C*-algebras, but ordinary cohomology doesn't.

K-homology

A (generalized) cohomology theory has a dual (generalized) homology theory, at least on finite complexes. (One construction uses Alexander-Spanier duality.)

For K-theory, the dual is called K-homology. It was known to exist, but for some time no natural constructions were known.

Natural definitions, for general compact metric spaces, were first discovered by C*-algebraists, using extensions of C*-algebras and, later, abstract elliptic pseudodifferential operators.

Concrete description of $K_0(A)$

If A is a unital ring, then $V(A)$ is the commutative semigroup of isomorphism classes of finitely generated projective right A-modules, with $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$, and $K_0(A)$ results from forcibly making $V(A)$ into a group. $[P]$ is the class of the module P in $K_0(A)$.

P is finitely generated and projective if and only if there are $n \in \mathbb{Z}_{\geqslant 0}$ and an A-module Q such that $P \oplus Q \cong A^n$, the rank n free module.

Rewriting the characterization: $\text{End}(A^n) \cong M_n(A)$, acting on the left via the usual formula for applying a matrix to a vector. (This is why we used right modules.)

So P is finitely generated and projective if and only if there are $n \in \mathbb{Z}_{\geqslant 0}$ and an idempotent $e \in M_n(A)$ such that $P \cong eA^n$. (Then $Q \cong (1 - e)A^n$.)

Therefore $V(A)$ is the set of equivalence classes of idempotents in $\bigcup_{n=1}^{\infty} M_n(A)$, for a suitable equivalence relation.

Concrete description of $K_0(A)$ (continued)

Therefore $V(A)$ is the set of equivalence classes of idempotents in $\bigcup_{n=1}^{\infty} M_n(A)$, for a suitable equivalence relation.

The relation should be: $e \sim f$ if and only if $eA^m \cong fA^n$.

The right way to do this is to let $M_\infty(A)$ be the (algebraic) direct limit $\varinjlim_n M_n(A)$, with maps

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

(a nonunital ring), and define $e \sim f$ if there exist $v, w \in M_\infty(A)$ such that $wv = e$ and $vw = f$. This is ~~algebraiv~~ algebraic *Murray-von Neumann equivalence*.

If A is a C*-algebra (but not for a general Banach *-algebra), one can "normalize": use the projections (selfadjoint idempotents) in $M_\infty(A)$, and $p \sim q$ if there is $s \in M_\infty(A)$ such that $s^*s = p$ and $ss^* = q$. This is much easier to work with.

Functoriality is $\varphi_*([p]) = [(\mathrm{id}_{M_\infty} \otimes \varphi)(p)]$.

$K_0(L(l^2))$

Addition in the concrete description of $V(A)$ is as follows. Let $e \in M_m(A)$ and $f \in M_n(A)$ be idempotents. Then $[e] + [f]$ is the class of

$$e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in M_{m+n}(A). \tag{1}$$

(One checks that $(e \oplus f)A^{m+n} = eA^m \oplus fA^n$.)

Back to $K_0(L(H))$, with $H = l^2$.. First, $M_n(L(H)) \cong L(H^n)$, and $H^n \cong H$. Second, let's use projections. The projections $p \in L(H)$ are exactly the orthogonal projections onto closed subspaces of H. Moreover, if pH and qH have the same dimension as Hilbert spaces, then $pH \cong qH$, which means that there is a unitary $s \in L(pH, qH)$, with inverse s^*. These can be regarded as operators in $L(H)$ satisfying $ss^* = q$ and $s^*s = p$. Thus $p \sim q$. It is easy to see that if $p \sim q$ then $\dim(pH) = \dim(qH)$

Combining this with (1) one sees that $V(L(H))$ is the semigroup of possible dimensions, as Hilbert spaces, of closed subspaces of H^n for $n \in \mathbb{Z}$, with the "obvious" addition, that is, $V(L(H)) \cong \{0, 1, 2, \ldots, \infty\}$. Since $n + \infty = \infty$ for any $n \in V(L(H))$, it is easy to see that $K_0(L(H)) = 0$.

Concrete description of $K_1(A)$

Let A be a unital Banach algebra.

There is then also a concrete description of $K_1(A)$, which gives $K_1(C(X)) \cong K^1(X)$.

Define:

- $\mathrm{inv}(A)$ is the set of invertible elements of A.
- $\mathrm{inv}_0(A)$ is the connected component of $\mathrm{inv}(A)$ which contains 1.
- $K_1(A) = \varinjlim_n \mathrm{inv}(M_n(A))/\mathrm{inv}_0(M_n(A))$, using the maps

$$u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

This matches the topological theorem $K^1(X) \cong \left[X, \varinjlim_n \mathrm{GL}_n(\mathbb{C})\right]$.

Again, there is a C* normalization: use the unitary group $U(A)$ and its identity component $U_0(A)$. Then $K_1(A) \cong \varinjlim_n U(M_n(A))/U_0(M_n(A))$.

Homotopy

$V(A)$ is the ~~set~~ semigroup of Murray-von Neumann equivalence classes of idempotents in ~~$\bigcup_{n=1}^{\infty} M_n(A)$~~ $M_{\infty}(A)$, and $K_0(A)$ ~~results from forcibly making $V(A)$ into a~~ is its universal enveloping group.

There is a (quite naive) definition of what it means for two homomorphisms of Banach algebras $\varphi_0, \varphi_1 \colon A \to B$ to be homotopic~~.~~ ~~Moreover, homotopic~~: there are homomorphisms $\varphi_t \colon A \to B$ for $t \in [0, 1]$ such that $t \mapsto \varphi_t(a)$ is continuous for all $a \in A$. (With this definition, if X and Y are compact, then $h_0, h_1 \colon X \to Y$ are homotopic if and only if $C(h_0), C(h_1) \colon C(Y) \to C(X)$ are homotopic.)

Homotopic idempotents are Murray-von Neumann equivalent. (Given standard Banach algebra machinery, the proof is easier than for vector bundles.)

~~Conclusion:~~ Therefore, if $\varphi_0, \varphi_1 \colon A \to B$ are homotopic homomorphisms of Banach algebras, then $(\varphi_0)_* = (\varphi_1)_* \colon K_0(A) \to K_0(B)$.

$K_1(A) = \varinjlim_n \operatorname{inv}(M_n(A))/\operatorname{inv}_0(M_n(A))$.

So, if $\varphi_0, \varphi_1 \colon A \to B$ are homotopic homomorphisms of Banach algebras,

Bott periodicity

There is a very naive version of the suspension for a Banach algebra, satisfying $SC_0(X) \cong C_0(SX)$. It is: SA is the algebra of continuous functions from $[0,1]$ to A which vanish at 0 and 1. (This can be identified as $C_0((0,1), A)$, A-valued continuous functions vanishing at infinity.)

Theorem (Bott periodicity)

There are natural isomorphisms $K_0(SA) \cong K_1(A)$ and $K_1(SA) \cong K_0(A)$, for all Banach algebras A.

The long exact sequence takes the form: if J is a closed ideal in A and $B = A/J$, with maps $\iota\colon J \to A$ and $\pi\colon A \to B$, there is an exact sequence

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(B) \\ \uparrow \scriptstyle{\exp} & & & & \downarrow \scriptstyle{\partial} \\ K_1(B) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[\iota_*]{} & K_1(J). \end{array}$$

The hypothesis says:

Example theorem from C*-algebras

I won't define the terms in this theorem: that would go too far afield.

Theorem

Let A and B be purely infinite simple separable nuclear nonunital C*-algebras satisfying the Universal Coefficient Theorem. If $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$, then $A \cong B$.

There is an analogous theorem for the unital case. In the unital stably finite case with real rank zero, one must add Jiang-Su stability and include the structure of K_0 as a scaled ordered group.

K-homology and KK-theory

There are also concrete descriptions of K-homology for C*-algebras.

~~There~~ For C*-algebras, there is a two variable version, $KK(A, B)$, which for A and B separable and A nuclear is heuristically "K-homolology ~~of~~ in the first variable and K-theory ~~of~~ in the second variable". It has several quite ~~difference constructions, and~~ different constructions. In the following, $K \otimes A$ is a norm completion of $M_\infty(A)$. The constructions include: equivalence classes of extensions of C*-algebras

$$0 \longrightarrow K \otimes B \longrightarrow E \longrightarrow A \longrightarrow 0$$

(addition is not the same as for extensions of modules), homotopy classes of abstract elliptic operators, and homotopy classes of approximate (in a suitable sense) homomorphisms $K \otimes SA \to K \otimes SB$.

It has a product $KK(A, B) \times KK(B, C) \to KK(A, C)$. ~~This theory plays~~

These groups play a key role in classification theorems for C*-algebras.

Algebraic K_1

Recall that if A is a unital Banach algebra, then $K_1(A) = \varinjlim_n \mathrm{inv}(M_n(A))/\mathrm{inv}_0(M_n(A))$. If A does not have a topology, then $\mathrm{inv}_0(M_n(A))$ does not make sense.

$SA = C_0((0,1), A)$ does not make sense, so defining $K_1(A) = K_0(SA)$ will not work, even independently of the problem of algebraic K-theory for nonunital algebras.

The replacement is to use the commutator subgroup instead of the identity component. If G is a group, let G' be the subgroup generated by all $ghg^{-1}h^{-1}$ for $g, h \in G$. For a unital ring A, define

$$K_1^{\mathrm{alg}}(A) = \varinjlim_n \mathrm{inv}(M_n(A))/\mathrm{inv}(M_n(A))'.$$

To emphasize the difference, write $K_1^{\mathrm{top}}(A)$ for the group already defined.

$K_1^{\mathrm{alg}}(A)$ vs. $K_1^{\mathrm{top}}(A)$

G' is generated by $\{ghg^{-1}h^{-1} \colon g, h \in G\}$.

$K_1^{\mathrm{alg}}(A) = \varinjlim_n \mathrm{inv}(M_n(A))/\mathrm{inv}(M_n(A))'$.

This is a functor in the same way that $K_1^{\mathrm{top}}(A)$ is.

For a Banach algebra, $K_1^{\mathrm{alg}}(A)$ and $K_1^{\mathrm{top}}(A)$ need not be the same.

Example: $A = \mathbb{C}$. We have $\mathrm{inv}(M_n(\mathbb{C})) = \mathrm{GL}_n(\mathbb{C})$, which is connected for all n. Therefore $K_1^{\mathrm{top}}(\mathbb{C}) - 0$.

However, all commutators have determinant 1. That is, $\mathrm{inv}(M_n(\mathbb{C}))' \subset \mathrm{SL}_n(\mathbb{C})$. In fact, these are equal, and the determinant defines an isomorphism from $K_1^{\mathrm{alg}}(\mathbb{C})$ to the multiplicative group $\mathbb{C} \setminus \{0\}$.

It is not true that $\mathrm{inv}(M_n(\mathbb{C}))' \subset \mathrm{inv}_0(M_n(A))$, but this is true "in the limit", and one therefore gets a surjective comparison map $K_1^{\mathrm{alg}}(A) \to K_1^{\mathrm{top}}(A)$. As we have seen, it can be very far from injective.

Exact sequence; $K_2(A)$

For a unital ring A and an ideal $J \subset A$, there is an exact sequence (for which I say nothing about the definitions of the relative groups which appear):

$$K_1(A, J) \longrightarrow K_1(A) \longrightarrow K_1(A/J) \longrightarrow K_0(A, J) \longrightarrow K_0(A) \longrightarrow K_0(A/J).$$

The last map is normally not surjective.

Milnor defined a group $K_2(A)$ for a unital ring A, in such a way that the exact sequence above at least extends to

$$K_2(A) \longrightarrow K_2(A/J) \longrightarrow K_1(A, J) \longrightarrow K_1(A) \longrightarrow K_1(A/J) \longrightarrow \cdots .$$

However, this construction didn't obviously generalize in an appropriate way to $K_n(A)$ for $n \geqslant 3$.

Higher algebraic K-theory

Defining $K_n(A)$ for $n \geqslant 3$ had to wait for Quillen's work, for which he received a Fields Medal, using almost completely different ideas. Here is a very brief description. Start with the group $\varinjlim_n \mathrm{inv}(M_n(A))$. Form its classifying space. Attach additional cells; among other things, the attached cells change the fundamental group from $\varinjlim_n \mathrm{inv}(M_n(A))$ to its abelianization, which is $\varinjlim_n \mathrm{inv}(M_n(A))/\mathrm{inv}(M_n(A))' = K_1(A)$. Then form the higher homotopy groups of the result. There is in general no periodicity.

There are by now variants, and algebraic K-groups $K_n(A)$ for $n < 0$, about which I can say nothing here. There are even algebraic versions of KK-theory, about which I know very little.

There is always a comparison map $K_n^{\mathrm{alg}}(A) \to K_n^{\mathrm{top}}(A)$ (in any degree, taking n mod 2 on the right). For $n \neq 0$, rarely, but occasionally, these maps are isomorphismss. An example is if A is a unital properly infinite C*-algebra, such as a Cuntz algebra or $L(H)$ for an infinite dimensional Hilbert space H.