Lisboa Summer School Course on Crossed Product C*-Algebras

N. Christopher Phillips

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Reminder: Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v\colon G \to U(H)$ (the unitary group of H) and a representation $\pi\colon A \to L(H)$ (the algebra of all bounded operators on H), satisfying the *covariance condition*

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if π is nondegenerate.

Reminder: Twisted convolution

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We let $C_{\mathrm{c}}(G, A, \alpha)$ be the *-algebra of continuous functions $a\colon G \to A$, with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following "twisted convolution":

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\, dh.$$

Let Δ be the modular function of G. We define the adjoint by

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*).$$

We define a norm $\|\cdot\|_1$ on $C_{\mathrm{c}}(G, A, \alpha)$ by $\|a\|_1 = \int_G \|a(g)\|\, dg$. One checks that $\|ab\|_1 \leq \|a\|_1\|b\|_1$ and $\|a^*\|_1 = \|a\|_1$. Then $L^1(G, A, \alpha)$ is the Banach *-algebra obtained by completing $C_{\mathrm{c}}(G, A, \alpha)$ in $\|\cdot\|_1$.

Reminder: Twisted convolution (continued)

Exercise

Assuming suitable versions of Fubini's Theorem for Banach space valued integrals, check that that multiplication in $C_{\mathrm{c}}(G, A, \alpha)$ is associative. Further check for $a, b \in C_{\mathrm{c}}(G, A, \alpha)$ that $\|ab\|_1 \leq \|a\|_1\|b\|_1$, that $(ab)^* = b^*a^*$, and that $\|a^*\|_1 = \|a\|_1$.

Reminder: Twisted convolution (continued)

Exercise

Suppose $A = C_0(X)$, and α comes from an action of G on X. Since we complete in a suitable norm later on, it suffices to use only the dense subalgebra $C_c(X)$ in place of $C_0(X)$. There is an obvious identification of $C_c(G,\, C_c(X))$ with $C_c(G \times X)$. Check that, on $C_c(G \times X)$, the formulas for multiplication and adjoint become

$$(f_1 f_2)(g, x) = \int_G f_1(h, x) f_2(h^{-1}g,\, h^{-1}x)\, dh$$

and

$$f^*(g, x) = \Delta(g)^{-1}\overline{f(g^{-1},\, g^{-1}x)}.$$

When G is discrete

If G is discrete, we choose Haar measure to be counting measure. In this case, $C_c(G, A, \alpha)$ is, as a vector space, the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by

$$(a{\cdot}g)(b{\cdot}h) = (a[gbg^{-1}]){\cdot}(gh) = (a\alpha_g(b)){\cdot}(gh) \quad \text{and} \quad (a{\cdot}g)^* = \alpha_g^{-1}(a^*){\cdot}g^{-1}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. This definition makes sense in the purely algebraic situation, where it is called the *skew group ring*.

We also often write $l^1(G, A, \alpha)$ instead of $L^1(G, A, \alpha)$.

When G is discrete (continued)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. For $g \in G$, we let u_g be the element of $C_c(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G. We use the same notation for its image in $l^1(G, A, \alpha)$ (above) and in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$ (defined below). It is unitary, and we call it the canonical unitary associated with g.

In particular, $l^1(G, A, \alpha)$ is the set of all sums $\sum_{g\in G} a_g u_g$ with $a_g \in A$ and $\sum_{g\in G} \|a_g\| < \infty$. These sums converge in $l^1(G, A, \alpha)$, and hence also in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$. A general element of $C^*_{\mathrm{r}}(G, A, \alpha)$ has such an expansion, but unfortunately the series one writes down generally does not converge. See the discussion later.

The integrated form of a covariant representation

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, π) is the representation $\sigma\colon C_c(G, A, \alpha) \to L(H)$ given by

$$\sigma(a)\xi = \int_G \pi(a(g))v(g)\xi\, dg.$$

(This representation is sometimes called $v \times \pi$ or $\pi \times v$.)

One needs to be more careful with the integral here, because v is generally only strong operator continuous, not norm continuous. Nevertheless, one gets $\|\sigma(a)\| \le \|a\|_1$, so σ extends to a representation of $L^1(G, A, \alpha)$. We use the same notation σ for this extension.

The integrated form of a covariant representation (continued)

One needs to check that σ is a representation. When G is discrete and A is unital, the formula for σ comes down to $\sigma(au_g) = \pi(a)v(g)$ for $a \in A$ and $g \in G$. Then

$$\begin{aligned}\sigma(au_g)\sigma(bu_h) &= \pi(a)v(g)\pi(b)v(g)^*v(g)v(h) = \pi(a)\pi(\alpha_g(b))v(g)v(h)\\ &= \pi(a\alpha_g(b))v(gh) = \sigma\big([a\alpha_g(b)]u_{gh}\big) = \sigma\big((au_g)(bu_h)\big).\end{aligned}$$

Exercise

Starting from this computation, fill in the details of the proof that the integrated form representation σ really is a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$.

The integrated form of a covariant representation (continued)

Theorem (Proposition 7.6.4 of Pedersen's book)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate continuous representations of $L^1(G, A, \alpha)$ on the same Hilbert space.

In particular, since integrated form representations of $L^1(G, A, \alpha)$ are necessarily contractive, *all* continuous representations of $L^1(G, A, \alpha)$ are necessarily contractive.

The integrated form when G is discrete

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given by $g \mapsto u_g$ and $a \mapsto au_1$, so it is clear how to get a covariant representation of (G, A, α) from a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$. In general, one must use the multiplier algebra of $L^1(G, A, \alpha)$, which contains copies of $M(A)$ and $M(L^1(G))$. The point is that $M(L^1(G))$ is the measure algebra of G, and therefore contains the group elements as point masses.

Exercise

Prove the theorem on the previous slide when G is discrete and A is unital.

For a small taste of the general case, use approximate identities in A to generalize to the case in which A is not necessarily unital.

The universal representation and the crossed product

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces. Then we define the *crossed product* $C^*(G, A, \alpha)$ to be the norm closure of $\sigma(L^1(G, A, \alpha))$.

One could of course equally well use the norm closure of $\sigma(C_{\mathrm{c}}(G, A, \alpha))$.

There is a minor set theoretic detail: the collection of all nondegenerate representations of $L^1(G, A, \alpha)$ is not a set. There are several standard ways to deal with this problem, but in these notes we will ignore the issue.

Exercise

Give a set theoretically correct definition of the crossed product.

The important point is to preserve the universal property below.

The universal representation and the crossed product (continued)

It follows that every covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$. (Take the integrated form, and restrict elements of $C^*(G, A, \alpha)$ to the appropriate summand in the direct sum in the definition above.) The crossed product is, essentially by construction, the universal C*-algebra for covariant representations of (G, A, α), in the same sense that if G is a locally compact group, then $C^*(G)$ is the universal C*-algebra for unitary representations of G.

There are many notations in use for crossed products, including:

- $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$.
- $C^*(A, G, \alpha)$ and $C^*_{\mathrm{r}}(A, G, \alpha)$.
- $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,\mathrm{r}} G$ (used in Williams' book).
- $A \times_\alpha G$ and $A \times_{\alpha,\mathrm{r}} G$ (used in Davidson's book).
- $G \times_\alpha A$ and $G \times_{\alpha,\mathrm{r}} A$ (used in Pedersen's book).

The universal representation and the crossed product when G is discrete

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. Then $C^*(G, A, \alpha)$ is the universal C*-algebra generated by a unital copy of A (that is, the identity of A is supposed to be the identity of the generated C*-algebra) and unitaries u_g, for $g \in G$, subject to the relations $u_g u_h = u_{gh}$ for $g, h \in G$ and $u_g a u_g^* = \alpha_g(a)$ for $a \in A$ and $g \in G$.

Corollary

Let A be a unital C*-algebra, and let $\alpha \in \mathrm{Aut}(A)$. Then the crossed product $C^*(\mathbb{Z}, A, \alpha)$ is the universal C*-algebra generated by a copy of A and a unitary u, subject to the relations $uau^* = \alpha(a)$ for $a \in A$.

The universal representation and the crossed product when G is discrete (continued)

Exercise

Based on the discussion above, write down a careful proof of the theorem.

Regular covariant representations

So far, it is not clear that there are any covariant representations.

Definition (7.7.1 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation. We define the *regular covariant representation* (v, π) of (G, A, α) on the Hilbert space $H = L^2(G, H_0)$ of L^2 functions from G to H as follows. For $g, h \in G$, set

$$(v(g)\xi)(h) = \xi(g^{-1}h).$$

For $a \in A$ and $g \in G$, set

$$(\pi(a)\xi)(h) = \pi_0(\alpha_{h^{-1}}(a))(\xi(h)).$$

The integrated form of σ, will be called a regular representation of any of $C_{\mathrm{c}}(G, A, \alpha)$, $L^1(G, A, \alpha)$, $C^*(G, A, \alpha)$, and (when defined) $C^*_{\mathrm{r}}(G, A, \alpha)$.

The Hilbert space of the regular covariant representation

The easy way to construct $L^2(G, H)$ is to take it to be the completion of $C_c(G, H)$ in the norm coming from the scalar product

$$\langle \xi, \eta \rangle = \int_G \langle \xi(g), \eta(g) \rangle \, dg.$$

Reduced crossed products

Exercise

Suppose that G is discrete. Prove that a regular representation really is a covariant representation.

If $A = \mathbb{C}$, $H_0 = \mathbb{C}$, and π_0 is the obvious representation of A on H_0, then the regular representation is the usual left regular representation of G.

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\lambda \colon L^1(G, A, \alpha) \to L(H)$ be the direct sum of all regular representations of $L^1(G, A, \alpha)$. We define the *reduced crossed product* $C^*_{\mathrm{r}}(G, A, \alpha)$ to be the norm closure of $\lambda(L^1(G, A, \alpha))$.

As with crossed products, in these notes we ignore the set theoretic difficulty.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Theorem (Theorem 7.7.7 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. If G is amenable, then $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism.

The converse is true for $A = \mathbb{C}$: if $C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism, then G is amenable. But it is not true in general. For example, if G acts on itself by translation, then $C^*(G, \, C_0(G)) \to C^*_{\mathrm{r}}(G, \, C_0(G))$ is an isomorphism for every G. (We will do this below for a discrete group.)

The crossed product is not too small

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then $C_{\mathrm{c}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective.

We will prove this below in the case of a discrete group. The proof of the general case can be found in Lemma 2.26 of the book of Williams. It is, I believe, true that $L^1(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective, and this can probably be proved by working a little harder in the proof of Lemma 2.26 of the book of Williams, but I have not carried out the details and I do not know a reference.

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G. The main tool is the structure of regular representations. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$, and elements of it can be thought of as families $(\xi_g)_{g \in G}$. The following formula for the integrated form of a regular representation is just a calculation.

Lemma

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $\pi_0\colon A \to L(H_0)$ be a representation, and let $\sigma\colon C^*_{\mathrm{r}}(G, A, \alpha) \to H = L^2(G, H_0)$ be the associated regular representation. Let $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$, with $a_g = 0$ for all but finitely many g. For $\xi \in H$ and $h \in G$, we then have

$$(\sigma(a)\xi)(h) = \sum_{g \in G} \pi_0(\alpha_h^{-1}(a_g))(\xi(g^{-1}h)).$$

When G is discrete: integrated form of a regular representation (continued)

In particular, picking off coordinates in $L^2(G, H_0)$ gives:

Corollary

Let the hypotheses be as in the Lemma, and let $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$. For $g \in G$, let $s_g \in L(H_0, H)$ be the isometry which sends $\eta \in H_0$ to the function $\xi \in L^2(G, H_0)$ given by

$$\xi(h) = \begin{cases} \eta & h = g \\ 0 & h \neq g. \end{cases}$$

Then

$$s_h^* \sigma(a) s_k = \pi_0(\alpha_h^{-1}(a_{hk^{-1}}))$$

for all $h, k \in G$.

Comparing norms

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Define norms on $C_{\mathrm{c}}(G, A, \alpha)$ as follows:

- $\|\cdot\|_\infty$ is the supremum norm.
- $\|\cdot\|_1$ is the l^1 norm.
- $\|\cdot\|$ is the restriction of the C*-algebra norm on $C^*(G, A, \alpha)$.
- $\|\cdot\|_{\mathrm{r}}$ is the restriction of the C*-algebra norm on $C^*_{\mathrm{r}}(G, A, \alpha)$.

Lemma

For every $a \in C_{\mathrm{c}}(G, A, \alpha)$, we have $\|a\|_\infty \leq \|a\|_{\mathrm{r}} \leq \|a\| \leq \|a\|_1$.

Comparing norms: the proof

The middle of this inequality follows from the definitions.
The last part follows from the observation above that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing. Here is a direct proof: for $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, with all but finitely many of the a_g equal to zero, we have

$$\left\| \sum_{g \in G} a_g u_g \right\| \leq \sum_{g \in G} \|a_g\| \cdot \|u_g\| = \sum_{g \in G} \|a_g\| = \left\| \sum_{g \in G} a_g u_g \right\|_1.$$

We prove the first part of this inequality. Let $a = \sum_{g \in G} a_g u_g$, with all but finitely many of the a_g equal to zero, and let $g \in G$. Let $\pi_0\colon A \to L(H_0)$ be an injective nondegenerate representation. With the notation of the previous corollary, we have

$$\|a_g\| = \|\pi_0(a_g)\| = \|s_g^* \sigma(a) s_1\| \leq \|\sigma(a)\| \leq \|a\|_{\mathrm{r}}.$$

This completes the proof.

A is a subalgebra of the reduced crossed product

The lemma implies that the map $a \mapsto a u_1$, from A to $C^*_{\mathrm{r}}(G, A, \alpha)$, is injective. We routinely identify A with its image in $C^*_{\mathrm{r}}(G, A, \alpha)$ under this map, thus treating it as a subalgebra of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Of course, we can do the same with the full crossed product $C^*(G, A, \alpha)$.

For finite groups, no completion is needed

Corollary

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. Then the maps $C_{\mathrm{c}}(G, A, \alpha) \to C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ are bijective.

Proof.

When G is finite, $\| \cdot \|_1$ (the l^1 norm) is equivalent to $\| \cdot \|_\infty$ (the supremum norm), and is complete in both. The lemma now implies that both C* norms are equivalent to these norms, so $C_{\mathrm{c}}(G, A, \alpha)$ is complete in both C* norms. □

Coefficients in reduced crossed products

When G is discrete but not finite, things are much more complicated. We can get started:

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then for each $g \in G$, there is a linear map $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$ such that if $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, then $E_g(a) = a_g$.

Moreover, with s_g as above, we have $s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(E_{hk^{-1}}(a))\big)$ for all $h, k \in G$.

Proof.

The first part is immediate from the inequality $\|a\|_\infty \leq \|a\|_{\mathrm{r}}$ above.

The last statement follows by continuity from "picking off coordinates" in the regular representation. □

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$, it makes sense to talk about its coefficients a_g. The first point is that if $C^*(G, A, \alpha) \neq C^*_{\mathrm{r}}(G, A, \alpha)$ (which can happen if G is not amenable, but not if G is amenable), the coefficients $(a_g)_{g \in G}$ do not even uniquely determine the element a. This is why we are only considering reduced crossed products here.

Coefficients in reduced crossed products: Properties

Here are the good things about coefficients.

Proposition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let the maps $E_g\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as in the previous proposition. Then:

1. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_g(a) = 0$ for all $g \in G$, then $a = 0$.
2. If $\pi_0\colon A \to L(H_0)$ is a nondegenerate representation such that $\bigoplus_{g\in G} \pi_0 \circ \alpha_g$ is injective, then the regular representation σ of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to π_0 is injective.
3. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_1(a^*a) = 0$, then $a = 0$.

Proof of the properties of coefficients

(1): Let $\pi_0\colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^*\sigma(a)s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1}g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g\in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^*a = \sum_{g,h\in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^*a) = \sum_{g\in G} u_g^* a_g^* a_g u_g = \sum_{g\in G} \alpha_g^{-1}\big(E_g(a)^*E_g(a)\big).$$

In particular, for each fixed g, we have $E_1(a^*a) \geq \alpha_g^{-1}\big(E_g(a)^*E_g(a)\big)$. By continuity, this inequality holds for all $a \in C^*_{\mathrm{r}}(G, A, \alpha)$. Thus, if $E_1(a^*a) = 0$, then $E_g(a)^*E_g(a) = 0$ for all g, so $a = 0$ by Part (1). This completes the proof.

Injective representations of A always give injective regular representations of the reduced crossed product

It is true for general locally compact groups, not just discrete groups, that the regular representation of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to an injective representation of A is injective. See Theorem 7.7.5 of Pedersen's book.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

Exercise

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as above. Prove that E has the following properties:

1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
2. If $b \geq 0$ then $E(b) \geq 0$.
3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
4. If $a \in A$ and $b \in C^*_{\mathrm{r}}(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$.

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$. In fact, the situation is intractable even for the case of the trivial action of $\mathbb{Z}$ on $\mathbb{C}$. In this case, $l^1(\mathbb{Z}, A, \alpha) = l^1(\mathbb{Z})$. The crossed product is the group C*-algebra $C^*(\mathbb{Z})$, which can be identified with $C(S^1)$. The map $l^1(\mathbb{Z}) \to C(S^1)$ is given by Fourier series: the sequence $a = (a_n)_{n \in \mathbb{Z}_{>0}}$ goes to the function $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$. (This looks more familiar when expressed in terms of 2π-periodic functions on $\mathbb{R}$: it is $t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$.) Failure of convergence of $\sum_{n \in \mathbb{Z}} a_n u_n$ corresponds to the fact that the Fourier series of a continuous function need not converge uniformly. Identifying the coefficient sequences which correspond to elements of the crossed product corresponds to giving a criterion for exactly when a sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$ of complex numbers is the sequence of Fourier coefficients of some continuous function on S^1, a problem for which I know of no satisfactory solution.

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$. Let $\delta_n \in l^2(\mathbb{Z})$ be the function

$$\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

Then the Fourier coefficient a_n is recovered as $a_n = \langle \lambda(a)\delta_0, \delta_n \rangle$. That is, $\lambda(a)\delta_0 \in l^2(\mathbb{Z})$ is given by $\lambda(a)\delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$. Thus, the sequence of Fourier coefficients of a continuous function is always in $l^2(\mathbb{Z})$. (Of course, we already know this, but the calculation here can be applied to more general crossed products.) Unfortunately, this fact is essentially useless for the study of the group C*-algebra. Not only is the Fourier series of a continuous function always in $l^2(\mathbb{Z})$, but the Fourier series of a function in $L^\infty(S^1)$, which is the group von Neumann algebra of $\mathbb{Z}$, is also always in $l^2(\mathbb{Z})$. One will get essentially no useful information from a criterion which can't even exclude any elements of $L^\infty(S^1)$.

The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\mathrm{min}} A$. As far as I know, this problem is also in general intractable.

There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson's book.

The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. Computing the K-theory of a crossed product by $\mathbb{Z}/2\mathbb{Z}$ is harder than computing the K-theory of a crossed product by any of $\mathbb{Z}$, $\mathbb{R}$, or even a (nonabelian) free group!

Preliminaries for computing crossed products

We will shortly do some explicit computations of examples. First, though, we give some useful preliminaries:

- Equivariant maps and functoriality.
- Crossed products of exact sequences.
- Crossed products and direct limits.
- Notation for matrix units.

Equivariant homomorphisms

Let G be a locally compact group. A C*-algebra A equipped with an action $G \to \mathrm{Aut}(A)$ will be called a *G-algebra*. We sometimes refer to (G, A, α) as a G-algebra.

Definition

If (G, A, α) and (G, B, β) are G-algebras, then a homomorphism $\varphi\colon A \to B$ is said to be *equivariant* (or G-equivariant if the group must be specified) if for every $g \in G$, we have $\varphi \circ \alpha_g = \beta_g \circ \varphi$.

For a fixed locally compact group G, the G-algebras and equivariant homomorphisms form a category.

The crossed product construction is functorial for equivariant homomorphisms

Theorem

Let G be a locally compact group. If (G, A, α) and (G, B, β) are G-algebras and $\varphi\colon A \to B$ is an equivariant homomorphism, then there is a homomorphism $\psi\colon C_{\mathrm{c}}(G, A, \alpha) \to C_{\mathrm{c}}(G, B, \beta)$ given by the formula $\psi(a)(g) = \varphi(a(g))$ for $a \in C_{\mathrm{c}}(G, A, \alpha)$ and $g \in G$, and this homomorphism extends by continuity to a homomorphism $L^1(G, A, \alpha) \to L^1(G, B, \beta)$, and then to homomorphisms

$$C^*(G, A, \alpha) \to C^*(G, B, \beta) \quad \text{and} \quad C^*_{\mathrm{r}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, B, \beta).$$

This construction makes the crossed product and reduced crossed product constructions functors from the category of G-algebras to the category of C*-algebras.

This is straightforward. See the notes for details.

Full crossed products preserve exact sequences

Theorem

Let $0 \to J \to A \to B \to 0$ be an exact sequence of G-algebras, with actions γ on J, α on A, and β on B. Then the sequence

$$0 \longrightarrow C^*(G, J, \gamma) \longrightarrow C^*(G, A, \alpha) \longrightarrow C^*(G, B, \beta) \longrightarrow 0$$

is exact.

Proofs can be found in the three places listed in the notes.

The analog for reduced crossed products is in general false.

Full crossed products preserve direct limits

Theorem

Let $\left((G, A_i, \alpha^{(i)})_{i \in I}, (\varphi_{j,i})_{i \le j} \right)$ be a direct system of G-algebras. Let $A = \varinjlim A_i$, with action $\alpha\colon G \to \mathrm{Aut}(A)$ given by $\alpha_g = \varinjlim \alpha_g^{(i)}$. Let

$$\psi_{j,i}\colon C^*\big(G, A_i, \alpha^{(i)}\big) \to C^*\big(G, A_j, \alpha^{(j)}\big)$$

be the map obtained from $\varphi_{j,i}$. Using these maps in the direct system of crossed products, there is a natural isomorphism $C^*(G, A, \alpha) \cong \varinjlim C^*\big(G, A_i, \alpha^{(i)}\big)$.

The proof is done by combining the universal properties of direct limits and crossed products. See the notes.

Notation for matrix units

For any index set S, let $\delta_s \in l^2(S)$ be the standard basis vector, determined by

$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s. \end{cases}$$

For $j, k \in S$, we let the "matrix unit" $e_{j,k}$ be the rank one operator on $l^2(S)$ given by $e_{j,k}\xi = \langle \xi, \delta_k \rangle \delta_j$. This gives the product formula $e_{j,k}e_{l,m} = \delta_{k,l}e_{j,m}$. Conventional matrix units for M_n are obtained by taking $S = \{1, 2, \ldots, n\}$, but we will sometimes want to take S to be a discrete (even finite) group. For $S = \{1, 2\}$, with the obvious choice of matrix representation, we get

$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example: The trivial action

Example

If G acts trivially on the C*-algebra A, then

$$C^*(G, A) \cong C^*(G) \otimes_{\max} A \quad \text{and} \quad C^*_{\mathrm{r}}(G, A) \cong C^*_{\mathrm{r}}(G) \otimes_{\min} A.$$

We describe how to see this when G is discrete and A is unital. Then $C^*(G, A)$ is the universal unital C*-algebra generated by a unital copy of A and a commuting unitary representation of G in the algebra. Since $C^*(G)$ is the universal unital C*-algebra generated by a unitary representation of G in the algebra, this is exactly the universal property of the maximal tensor product.

Example: The trivial action (continued)

For the reduced crossed product, the point is that a regular covariant representation of (G, A) has the form $(\lambda \otimes 1_{H_0}, 1_{L^2(G)} \otimes \pi_0)$ for $\pi_0 \colon A \to L(H_0)$ an arbitrary nondegenerate representation and with $\lambda \colon G \to U(L^2(G))$ being the left regular representation. As we saw above, it suffices to take π_0 to be a single injective representation. Now we are looking at $C^*_{\mathrm{r}}(G)$ on one Hilbert space and A on another, and taking the tensor product of the Hilbert spaces. This is exactly how one gets the minimal tensor product of two C*-algebras.

Note how full and reduced crossed products parallel maximal and minimal tensor products.

Example: Inner actions

Example

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an inner action of a discrete group G on a unital C*-algebra A. Thus, there is a homomorphism $g \mapsto z_g$ from G to $U(A)$ such that $\alpha_g(a) = z_g a z_g^*$ for all $g \in G$ and $a \in A$. Then $C^*(G, A, \alpha) \cong C^*(G) \otimes_{\max} A$. (This is true even if G is not discrete.)

One shows that the crossed product is the same as for the trivial action. Let $\iota \colon G \to \mathrm{Aut}(A)$ be the trivial action of G on A. As usual, for $g \in G$ let $u_g \in C_{\mathrm{c}}(G, A, \alpha)$ be the standard unitary, but let $v_g \in C_{\mathrm{c}}(G, A, \iota)$ be the standard unitary in the crossed product by the trivial action. Define $\varphi_0 \colon C_{\mathrm{c}}(G, A, \alpha) \to C_{\mathrm{c}}(G, A, \iota)$ by $\varphi_0(a u_g) = a z_g v_g$ for $a \in A$ and $g \in G$, and extend linearly. This map is obviously bijective (the inverse sends $a v_g$ to $a z_g^* u_g$) and isometric for $\| \cdot \|_1$.

Example: Inner actions (continued)

For multiplicativity, it suffices to check the following, for $a, b \in A$ and $g, h \in H$, using the fact that v_g commutes with all elements of A:

$$\begin{aligned}\varphi_0(au_g)\varphi_0(bu_h) &= az_g v_g b z_h v_h = az_g b z_g^* z_{gh} v_g v_h \\ &= a\alpha_g(b) z_{gh} v_{gh} = \varphi_0\big(a\alpha_g(b)u_{gh}\big) = \varphi_0\big((au_g)(bu_h)\big).\end{aligned}$$

For preservation of adjoints:

$$\begin{aligned}\varphi_0(au_g)^* &= (az_g v_g)^* = v_g^* z_g^* a^* = (z_g^* a^* z_g) z_g^* v_g^* \\ &= \alpha_{g^{-1}}(a^*) z_{g^{-1}} v_{g^{-1}} = \varphi_0\big(\alpha_{g^{-1}}(a^*)u_{g^{-1}}\big) = \varphi_0\big((au_g)^*\big).\end{aligned}$$

So φ_0 is an isometric isomorphism of *-algebras, and therefore extends to an isomorphism of the universal C*-algebras.

Example: G acting on itself by translation

Example

If G is discrete and acts on itself by translation, then the crossed product is $K(l^2(G))$. (This is actually true for general G.)

Let $\alpha\colon G \to \mathrm{Aut}(C_0(G))$ denote the action. For $g \in G$, let u_g be the standard unitary, and let $\delta_g \in C_0(G)$ be the function $\chi_{\{g\}}$. Thus $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. Also, $\mathrm{span}\big(\{\delta_g \colon g \in G\}\big)$ is dense in $C_0(G)$. For $g, h \in G$, we have $v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, C_0(G), \alpha)$. Moreover,

$$\begin{aligned}v_{g_1,h_1} v_{g_2,h_2} &= \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}} \\ &= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1}} u_{g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.\end{aligned}$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is v_{g_1,h_2}. Similarly, $v_{g,h}^* = v_{h,g}$. That is, the elements $v_{g,h}$ satisfy the relations for a system of matrix units indexed by G. Also, $\mathrm{span}\big(\{v_{g,h} \colon g, h \in G\}\big)$ is dense in $l^1(G, C_0(G), \alpha)$, and hence in $C^*(G, C_0(G), \alpha)$.

For any finite set $F \subset G$, we thus get a homomorphism

$$\psi_F \colon L(l^2(F)) \to C_{\mathrm{c}}(G, C_0(G), \alpha)$$

sending the matrix unit $e_{g,h} \in L(l^2(F))$ to $v_{g,h}$. If G is finite, we have a surjective homomorphism $L(l^2(G)) \to C^*(G, C_0(G), \alpha)$, necessarily injective since $L(l^2(G))$ is simple.

In general, one assembles the maps ψ_F to get an isomorphism $K(l^2(G)) \to C^*(G, C_0(G), \alpha)$. For details, see the notes.

Since the full crossed product is simple, the map to the reduced crossed product is an isomorphism.

If G acts on $G \times X$ by translation in the first factor and trivially in the second factor, we get the crossed product $C(X, K(l^2(G)))$. (In fact, this is true for an arbitrary action of G on X. See the notes.)