Lisboa Summer School Course on Crossed Product C*-Algebras: Lecture 4

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The main theorem

Theorem (with H. Lin; to appear)

Let X be an infinite compact metric space with finite covering dimension, and let $h\colon X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(C^*(\mathbb{Z}, X, h)))$ is dense in $\mathrm{Aff}(T(C^*(\mathbb{Z}, X, h)))$. Then $C^*(\mathbb{Z}, X, h)$ is a simple unital C*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem.

We give some definitions (with explanation) and the main corollary below.

We do not give a full proof here. Instead, we assume X is the Cantor set. The full proof would require three lectures about recursive subhomogeneous algebras and their direct limits, as well as several lectures on technical computations involving KK-theory.

Note: Some of the discussion here is not in the notes, and some of the results are in a slightly different order.

The main corollary

Corollary (with H. Lin; to appear)

Let X be an infinite compact metric space with finite covering dimension, and let $h\colon X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(C^*(\mathbb{Z}, X, h)))$ is dense in $\mathrm{Aff}(T(C^*(\mathbb{Z}, X, h)))$. Then $C^*(\mathbb{Z}, X, h)$ is a simple AH algebra with no dimension growth and with real rank zero.

An AH algebra is a direct limit $\varinjlim A_n$, in which each A_n is a finite direct sum of C*-algebras of the form $C(X, M_r)$, for varying compact metric spaces X and positive integers r. "No dimension growth" means that there is a finite upper bound on the (covering) dimensions of the spaces X which occur throughout in the direct system.

The theorem also implies that $C^*(\mathbb{Z}, X, h)$ is classifiable in the sense of the Elliott program. (This is used in the proof of the corollary.)

Covering dimension

The covering dimension $\dim(X)$ is defined for compact metric spaces X (and more generally). Rather than giving the (somewhat complicated) definition, here are examples:

- The Cantor set has covering dimension zero.
- If X is a compact manifold with dimension d as a manifold, then $\dim(X) = d$.
- If X is a finite CW complex, then $\dim(X)$ is the dimension of the highest dimensional cell.
- $\dim\left([0,1]^{\mathbb{Z}}\right) = \infty$.

The map from K_0 to the affine functions on the tracial state space

For any unital C*-algebra A, we let $T(A)$ be the tracial state space of A. For any compact convex set Δ, we let $\mathrm{Aff}(\Delta)$ be the space of real valued continuous affine functions on Δ, with the supremum norm. Further let $\rho = \rho_A \colon K_0(A) \to \mathrm{Aff}(T(A))$ be the homomorphism determined by $\rho([p])(\tau) = \tau(p)$ for $\tau \in T(A)$ and p a projection in some matrix algebra over A.

In the situation at hand, $T(C^*(\mathbb{Z}, X, h))$ is affinely homeomorphic to the simplex of h-invariant Borel probability measures on X. (See the end of the lecture.)

There is machinery available to compute the range of ρ in the above theorem without computing $C^*(\mathbb{Z}, X, h)$. See, for example, Ruy Exel's Ph.D. thesis. (Reference to the published version in the notes.)

The definition of tracial rank zero

We use the notation $[a, b]$ for the commutator $ab - ba$.

Definition

Let A be a simple unital C*-algebra. Then A has tracial rank zero if for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

1. $\|[a, p]\| < \varepsilon$ for all $a \in F$.
2. $\mathrm{dist}(pap, D) < \varepsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

(This is equivalent to the original definition for simple C*-algebras. See the notes.)

The condition was originally called "tracially AF".

Restricting to the Cantor set

The notes contain a complete proof of the main theorem when X is the Cantor set, using the same methods as in the proof of the full theorem.

The restriction to the Cantor set simplifies the argument by avoiding recursive subhomogeneous C*-algebras and some K-theory computations.

However, for the Cantor set, there is an older and shorter proof, of which the main part is due to Putnam. This proof gives directly the result that $C^*(\mathbb{Z}, X, h)$ is a direct limit of finite direct sums of C*-algebras of the form $C(S^1, M_r)$ (for varying r).

One can get this result for the Cantor set by combining the main theorem with known classification results, so Putnam's argument doesn't give any more in the end.

Other known results: Minimal diffeomorphisms

Let X be a compact manifold, and let $h \colon X \to X$ be a minimal diffeomorphism. Then $C^*(\mathbb{Z}, X, h)$ is a direct limit, with no dimension growth, of recursive subhomogeneous C*-algebras.

There is no condition on $\rho(K_0(C^*(\mathbb{Z}, X, h))) \subset \mathrm{Aff}(T(C^*(\mathbb{Z}, X, h)))$. In particular, $C^*(\mathbb{Z}, X, h)$ sometimes has no nontrivial projections.

This is joint with Qing Lin (unpublished). It uses Putnam's methods, but is very long. As a corollary, $C^*(\mathbb{Z}, X, h)$ has stable rank one. Applicable classification results are just now starting to appear.

It should be possible to generalize to minimal homeomorphisms of finite dimensional compact metric spaces.

Other known results: Infinite dimensional spaces

The condition $\dim(X) < \infty$ is needed. Giol and Kerr have an example of a minimal homeomorphism h of an infinite dimensional compact metric space X such that $C^*(\mathbb{Z}, X, h)$ does not have stable rank one.

However, probably h having "mean dimension zero" is enough. Uniquely ergodic implies mean dimension zero, and $\dim(X) < \infty$ implies mean dimension zero.

Other known results: $\mathbb{Z}^d$ acting on the Cantor set, groups which are not discrete, and actions on simple C*-algebras

Let X be the Cantor set, and suppose $\mathbb{Z}^d$ acts freely and minimally on X. Then $C^*(\mathbb{Z}^d, X)$ has stable rank one, real rank zero, and the order on projections over $C^*(\mathbb{Z}^d, X)$ is determined by traces.

All these follow from tracial rank zero, but do not imply tracial rank zero. It should be true that $C^*(\mathbb{Z}^d, X)$ has tracial rank zero, but this is still open.

There is work in progress which will probably give positive results about $C^*(\mathbb{R}, X)$ when $\dim(X) < \infty$ and the action is free and minimal.

There is also a collection of related results on crossed products of simple C*-algebras by actions of $\mathbb{Z}$ and of finite groups which have the tracial Rokhlin property, and generalizations.

Lemma 1

The proof of the main theorem, even for just the Cantor set, requires a number of lemmas. The first essentially says one can use AF algebras in place of finite dimensional algebras in the definition of tracial rank zero.

Lemma

Let A be a simple unital C*-algebra. Suppose that for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$ and a unital AF subalgebra $B \subset A$ with $p \in B$ such that:

1. $\|[a, p]\| < \varepsilon$ for all $a \in F$.
2. $\mathrm{dist}(pap, \, pBp) < \varepsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

Then A has tracial rank zero.

Proof of Lemma 1

Proof.

Let $F \subset A$ be a finite subset, let $\varepsilon > 0$, and let $c \in A$ be a nonzero positive element. Choose p and B as in the hypotheses, with F and c as given and with $\frac{1}{2}\varepsilon$ in place of ε. Let $F_0 \subset pBp$ be a finite set such that $\mathrm{dist}(pap, \, F_0) < \frac{1}{2}\varepsilon$ for all $a \in F$. Since $p \in B$, the algebra pBp is also AF. Choose a unital finite dimensional subalgebra $D \subset pBp$ such that $\mathrm{dist}(b, D) < \frac{1}{2}\varepsilon$ for all $b \in F_0$. Then $\mathrm{dist}(pap, \, D) < \varepsilon$ for all $a \in F$. □

Lemma 2

We need only consider finite subsets of a generating set:

Lemma

Let A be a unital C*-algebra, and let $S \subset A$ be a subset which generates A as a C*-algebra. Assume that the condition of the previous lemma holds for all finite subsets $F \subset S$. Then A has tracial rank zero.

The proof is an exercise.

$C^*(\mathbb{Z}, X, h)_Y$

A key point, in this proof and others, is the construction of a "large" and more tractable subalgebra of the crossed product.

Notation

Let X be a compact metric space, and let $h\colon X \to X$ be a homeomorphism. In the transformation group C*-algebra $C^*(\mathbb{Z}, X, h)$, we write u for the standard unitary representing the generator of $\mathbb{Z}$. For a closed subset $Y \subset X$, we define the C*-subalgebra $C^*(\mathbb{Z}, X, h)_Y$ to be

$$C^*(\mathbb{Z}, X, h)_Y = C^*(C(X),\, uC_0(X \setminus Y)) \subset C^*(\mathbb{Z}, X, h).$$

$C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a groupoid

Although we will not use formally groupoids in these notes, it should be pointed out that $C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a subgroupoid of the transformation group groupoid $\mathbb{Z} \ltimes X$ made from the action of $\mathbb{Z}$ on X generated by h. Informally, we "break" every orbit each time it goes through Y. More formally, for $n < 0$ the pair (n, x) is in the subgroupoid only if all of $h^n(x)$, $h^{n+1}(x)$, $\ldots$, $h^{-1}(x)$ are in $X \setminus Y$, and for $n > 0$ the pair (n, x) is in the subgroupoid only if all of x, $h(x)$, $\ldots$, $h^{n-1}(x)$ are in $X \setminus Y$.

For actions of $\mathbb{Z}^d$, it appears to be necessary to use subalgebras of the crossed product for which the only nice description is in terms of subgroupoids of the transformation group groupoid.

Lemma 3

The following lemma is due to Putnam.

Lemma

Let X be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a nonempty compact open subset. Then $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra.

Proof of Lemma 3

The proof depends on the construction of Rokhlin towers, which is a crucial element of many structure results for crossed products.

We first claim that there is $N \in \mathbb{Z}_{>0}$ such that $\bigcup_{n=1}^{N} h^{-n}(Y) = X$. Set $U = \bigcup_{n=1}^{\infty} h^{-n}(Y)$, which is a nonempty open subset of X such that $U \subset h(U)$. Then $Z = X \setminus \bigcup_{n=1}^{\infty} h^{-n}(Y)$ is a closed subset of X such that $h(Z) \subset Z$, and $Z \neq X$. Therefore $Z = \varnothing$. So $U = X$, and the claim now follows from compactness of X.

It follows that for each fixed $y \in Y$, the sequence of iterates $h(y), h^2(y), \ldots$ of y under h must return to Y in at most N steps. Define the *first return time* $r(y)$ to be

$$r(y) = \min\{n \geq 1 \colon h^n(y) \in Y\} \leq N.$$

Proof of Lemma 3 (continued)

Let $n(0) < n(1) < \cdots < n(l) \leq N$ be the values of r. Set

$$Y_k = \{y \in Y \colon r(y) = n(k)\}.$$

Then the sets Y_k are compact, open, and partition Y, and the sets $h^j(Y_k)$, for $1 \leq j \leq n(k)$, partition X:

$$Y = \coprod_{k=0}^{l} Y_k \quad \text{and} \quad X = \coprod_{k=0}^{l} \coprod_{j=1}^{n(k)} h^j(Y_k).$$

Each finite sequence $h(Y_k), h^2(Y_k), \ldots, h^{n(k)}(Y_k)$ is a *Rokhlin tower* with base $h(Y_k)$ and height $n(k)$. (It is more common to let the power of h run from 0 to $n(k)-1$. The choice made here, effectively taking the base of the collection of Rokhlin towers to be $h(Y)$ rather than Y, is more convenient for use with our definition of $C^*(\mathbb{Z}, X, h)_Y$.) Further set $X_k = \bigcup_{j=1}^{n(k)} h^j(Y_k)$. The sets X_k then also partition X.

Proof of Lemma 3 (continued)

Define $p_k \in C(X) \subset C^*(\mathbb{Z}, X, h)_Y$ by $p_k = \chi_{X_k}$. Then p_k trivially commutes with every element of $C(X)$. Moreover, suppose $f \in C(X)$ vanishes on Y. Since $Y_k, \ h^{n(k)}(Y_k) \subset Y$, we have

$$\chi_{Y_k} f = 0 \quad \text{and} \quad \chi_{h^{n(k)}(Y_k)} f = 0$$

Use the action of h on the levels of the tower at the first step, and the equations above at the second step, to get

$$p_k u f = u\big(p_k - \chi_{h^{n(k)}(Y_k)} + \chi_{Y_k}\big) f = u p_k f = u f p_k.$$

It follows that p_k commutes with all elements of $C^*(\mathbb{Z}, X, h)_Y$. So it suffices to prove that $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is AF for each k.

Proof of Lemma 3 (continued)

Now $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is the C*-algebra generated by $C(X_k)$ and

$$u(\chi_{X \setminus Y}) p_k = u(\chi_{X_k \setminus h^{n(k)}(Y_k)}) = \sum_{j=1}^{n(k)-1} u(\chi_{h^j(Y_k)})$$

$$= \sum_{j=1}^{n(k)-1} (\chi_{h^{j+1}(Y_k)}) u(\chi_{h^j(Y_k)}).$$

One can now check, although it is a bit tedious to write out the details, that there is an isomorphism $\psi_k \colon p_k C^*(\mathbb{Z}, X, h)_Y p_k \to M_{n(k)} \otimes C(Y_k)$ such that for $f \in C(X_k)$ we have

$$\psi_k(f) = \operatorname{diag}\big(f \circ h|_{Y_k}, \ f \circ h^2|_{Y_k}, \ \ldots, \ f \circ h^{n(k)}|_{Y_k}\big)$$

and for $1 \leq j \leq n(k) - 1$ we have

$$\psi_k\big(\chi_{h^{j+1}(Y_k)} u \chi_{h^j(Y_k)}\big) = e_{j+1,j} \otimes 1.$$

The algebra $M_{n(k)} \otimes C(Y_k)$ is AF because Y_k is totally disconnected. This completes the proof.

Lemma 4

Notation

Let A be a C*-algebra, and let $p, q \in A$ be projections. We write $p \sim q$ to mean that p and q are Murray-von Neumann equivalent in A, that is, there exists $v \in A$ such that $v^* v = p$ and $vv^* = q$. We write $p \precsim q$ if p is Murray-von Neumann equivalent to a subprojection of q.

Lemma

Let X be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $c \in C^*(\mathbb{Z}, X, h)$ be a nonzero positive element. Then there exists a nonzero projection $p \in C(X)$ such that p is Murray-von Neumann equivalent in $C^*(\mathbb{Z}, X, h)$ to a projection in $\overline{c C^*(\mathbb{Z}, X, h) c}$.

This shows that we can control the size of the leftover in the definition of tracial rank zero using projections in $C(X)$ instead of in the crossed product. This is a big advantage.

What we have so far

Combining our results so far, we have reduced our problem to proving the following:

Let X be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $S = C(X) \cup \{u\}$ (clearly a generating set for $C^*(\mathbb{Z}, X, h)$). Then for every finite subset $F \subset S$, every $\varepsilon > 0$, and every nonempty open set $U \subset X$, there exists a compact open set $Y \subset X$ and a projection $p \in C^*(\mathbb{Z}, X, h)_Y$ such that:

(1) $\| pa - ap \| < \varepsilon$ for all $a \in F \cup \{u\}$.
(2) $pap \in p C^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

The point is that $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra, and (3) says, in view of Lemma 4, that $1 - p$ is "small".

Proof of Lemma 4

Let $E \colon C^*(\mathbb{Z}, X, h) \to C(X)$ be the standard conditional expectation. Then $E(c)$ is a nonzero positive element of $C(X)$. Choose a nonempty compact open subset $K_0 \subset X$ and $\delta > 0$ such that the function $E(c)$ satisfies $E(c)(x) > 4\delta$ for all $x \in K_0$. Choose a finite sum $b = \sum_{n=-N}^{N} b_n u^n \in C^*(\mathbb{Z}, X, h)$ such that $\| b - c \| < \delta$. Since the action of $\mathbb{Z}$ induced by h is free, there is a nonempty compact open subset $K \subset K_0$ such that the sets

$$h^{-N}(K), \, h^{-N+1}(K), \, \ldots, \, h^{N}(K)$$

are disjoint. Set $p = \chi_K \in C(X)$. For $n \in \{-N, -N+1, \ldots, N\} \setminus \{0\}$, the disjointness condition implies that $p u^n p = 0$. Therefore

$$pbp = p b_0 p = pE(b)p.$$

Proof of Lemma 4 (continued)

Using this equation at the first step, we get

$$\| pcp - pE(c)p \| \leq \| pcp - pbp \| + \| pE(b)p - pE(c)p \| \leq 2 \| c - b \| < 2\delta. \quad (1)$$

Since $K \subset K_0$, the function $pE(c)p$ is invertible in $pC(X)p$. In the following, we take inverses in $pC^*(\mathbb{Z}, X, h)p$. Then, in fact, $\| [pE(c)p]^{-1} \| < \frac{1}{4} \delta^{-1}$. The estimate (1) now implies that pcp is invertible in $pC^*(\mathbb{Z}, X, h)p$. Let $a = (pcp)^{-1/2}$, calculated in $pC^*(\mathbb{Z}, X, h)p$. Set $v = apc^{1/2}$. Then

$$vv^* = apcpa = (pcp)^{-1/2} (pcp) (pcp)^{-1/2} = p$$

and

$$v^* v = c^{1/2} p a^2 p c^{1/2} \in \overline{c C^*(\mathbb{Z}, X, h) c}.$$

This completes the proof.

Lemma 5

Lemma

Let X be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a nonempty compact open subset. Let $N \in \mathbb{Z}_{>0}$, and suppose that $Y, \, h(Y), \, \ldots, h^N (Y)$ are disjoint. Then the projections χ_Y and $\chi_{h^N(Y)}$ are Murray-von Neumann equivalent in $C^*(\mathbb{Z}, X, h)_Y$.

Proof.

First, observe that if $Z \subset X$ is a compact open subset such that $Y \cap Z = \varnothing$, then $v = u \chi_Z \in C^*(\mathbb{Z}, X, h)_Y$ and satisfies $v^* v = \chi_Z$ and $v v^* = \chi_{h(Z)}$. Thus $\chi_Z \sim \chi_{h(Z)}$ in $C^*(\mathbb{Z}, X, h)_Y$.

An induction argument now shows that $\chi_{h(Y)} \sim \chi_{h^N(Y)}$ in $C^*(\mathbb{Z}, X, h)_Y$. Also, $\chi_{X \setminus Y} \sim \chi_{X \setminus h(Y)}$ in $C^*(\mathbb{Z}, X, h)_Y$. Since $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra, it follows that $\chi_Y \sim \chi_{h(Y)}$. The result follows by transitivity. □

Lemma 6

Lemma

Let X be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then for any $\varepsilon > 0$, any nonempty open set $U \subset X$, and any finite subset $F \subset C(X)$, there is a compact open set $Y \subset X$ containing y and a projection $p \in C^*(\mathbb{Z}, X, h)_Y$ such that:

(1) $\| p a - a p \| < \varepsilon$ for all $a \in F \cup \{ u \}$.

(2) $p a p \in p C^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{ u \}$.

(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

Lemma 6 finishes the proof

Recall that we reduced our problem to proving the following:

Let X be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $S = C(X) \cup \{ u \}$ Then for every finite subset $F \subset S$, every $\varepsilon > 0$, and every nonempty open set $U \subset X$, there exists compact open set $Y \subset X$ and a projection $p \in C^*(\mathbb{Z}, X, h)_Y$ such that:

(1) $\| p a - a p \| < \varepsilon$ for all $a \in F \cup \{ u \}$.

(2) $p a p \in p C^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{ u \}$.

(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

So Lemma 6 finishes the proof.

Proof of Lemma 6

Let d be the metric on X.

Choose $N_0 \in \mathbb{Z}_{>0}$ so large that $4 \pi / N_0 < \varepsilon$.

Choose $\delta_0 > 0$ with $\delta_0 < \frac{1}{2} \varepsilon$ and so small that $d (x_1, x_2) < 4 \delta_0$ implies $| f (x_1) - f (x_2) | < \frac{1}{4} \varepsilon$ for all $f \in F$.

Choose $\delta > 0$ with $\delta \leq \delta_0$ and such that whenever $d (x_1, x_2) < \delta$ and $0 \leq k \leq N_0$, then $d (h^{-k} (x_1), \, h^{-k} (x_2)) < \delta_0$.

Since h is minimal, there is $N > N_0 + 1$ such that $d (h^N (y), \, y) < \delta$.

Proof of Lemma 6 (continued)

Choose $N + N_0 + 1$ disjoint nonempty open subsets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N \subset U$. Using minimality again, choose $r_{-N_0}, r_{-N_0+1}, \ldots, r_N \in \mathbb{Z}$ such that $h^{r_l}(y) \in U_l$ for $-N_0 \leq l \leq N$. Since h is free, there is a compact open set $Y \subset X$ containing y such that

$$h^{-N_0}(Y),\ h^{-N_0+1}(Y),\ \ldots,\ Y,\ h(Y),\ \ldots,\ h^N(Y)$$

are disjoint and all have diameter less than δ. We may also require that $h^{r_l}(Y) \subset U_l$ for $-N_0 \leq l \leq N$.

Set $q_0 = \chi_Y$. For $-N_0 \leq n \leq N$ set

$$T_n = h^n(Y) \quad \text{and} \quad q_n = u^n q_0 u^{-n} = \chi_{h^n(Y)}.$$

Then the q_n are mutually orthogonal projections in $C(X)$.

Proof of Lemma 6 (continued)

We now have a sequence of projections, in principle going to infinity in both directions:

$$\ldots,\ q_{-N_0},\ \ldots,\ q_{-1},\ q_0,\ q_1,\ \ldots,\ q_{N-N_0},\ \ldots,\ q_{N-1},\ q_N,\ \ldots.$$

The ones shown are orthogonal, and conjugation by u is the shift. The projections q_0 and q_N are the characteristic functions of compact open sets which are disjoint but close to each other, and similarly for the pairs q_{-1} and q_{N-1} down to q_{-N_0} and q_{N-N_0}. We are now going to use Berg's technique to splice this sequence along the pairs of indices $(-N_0,\ N-N_0)$ through $(0, N)$, obtaining a loop of length N on which conjugation by u is approximately the cyclic shift.

Proof of Lemma 6 (continued)

Lemma 5 provides a partial isometry $w \in C^*(\mathbb{Z}, X, h)_Y$ such that $w^* w = q_0$ and $w w^* = q_N$. For $t \in [0, 1]$ define

$$v(t) = \cos(\pi t/2)(q_0 + q_N) + \sin(\pi t/2)(w - w^*).$$

Then $v(t)$ is a unitary in the corner

$$(q_0 + q_N) C^*(\mathbb{Z}, X, h)_Y (q_0 + q_N)$$

whose matrix with respect to the obvious block decomposition is

$$v(t) = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

For $0 \leq k \leq N_0$ define $z_k = u^{-k} v(k/N_0) u^k$.

Proof of Lemma 6 (continued)

We claim that $z_k \in C^*(\mathbb{Z}, X, h)_Y$ for $0 \leq k \leq N_0$. For $k = 0$ this is true by construction. For $1 \leq k \leq N_0$, with

$$a_k = q_0 u^k = (u q_{-1})(u q_{-2}) \cdots (u q_{-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

and

$$b_k = q_N u^k = (u q_{N-1})(u q_{N-2}) \cdots (u q_{N-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

(because $N_0 < N$), and using $T_{-k} \cap T_{N-k} = T_0 \cap T_N = \varnothing$, we can write

$$z_k = (a_k + b_k)^* v(k/N_0)(a_k + b_k) \in C^*(\mathbb{Z}, X, h)_Y.$$

Therefore z_k is a unitary in the corner

$$(q_{-k} + q_{N-k}) C^*(\mathbb{Z}, X, h)_Y (q_{-k} + q_{N-k}).$$

Proof of Lemma 6 (continued)

Moreover, adding estimates on the differences of the matrix entries at the second step,

$$\|uz_{k+1}u^* - z_k\| = \|v(k/N_0) - v((k-1)/N_0)\| \leq 2\pi/N_0 < \tfrac{1}{2}\varepsilon.$$

Now define $e_n = q_n$ for $0 \leq n \leq N - N_0$, and for $N - N_0 \leq n \leq N$ write $k = N - n$ and set $e_n = z_k q_{-k} z_k^*$. The two definitions for $n = N - N_0$ agree because $z_{N_0} q_{-N_0} z_{N_0}^* = q_{N-N_0}$, and moreover $e_N = e_0$. Therefore $ue_{n-1}u^* = e_n$ for $1 \leq n \leq N - N_0$, and also $ue_N u^* = e_1$, while for $N - N_0 < n \leq N$ we have

$$\|ue_{n-1}u^* - e_n\| \leq 2\|uz_{N-n+1}u^* - z_{N-n}\| < \varepsilon.$$

Also, clearly $e_n \in C^*(\mathbb{Z}, X, h)_Y$ for all n.

Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that p satisfies (1) through (3):

1. $\|pa - ap\| < \varepsilon$ for all $a \in F \cup \{u\}$.
2. $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
3. There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than ε, so $\|upu^* - p\| < \varepsilon$. Furthermore, since $p \leq 1 - q_0 = 1 - \chi_Y$, we get $pup \in C^*(\mathbb{Z}, X, h)_Y$. This is (1) and (2) for the element $u \in F \cup \{u\}$.

Proof of Lemma 6 (continued)

Next, let $f \in F$. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than δ. We have $d(h^N(y), y) < \delta$, so the choice of δ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$. Also, $T_{n-N} = h^{n-N}(T_0)$ has diameter less than δ. Therefore $T_{n-N} \cup T_n$ has diameter less than $2\delta + \delta_0 \leq 3\delta_0$. Since f varies by at most $\frac{1}{4}\varepsilon$ on any set with diameter less than $4\delta_0$, and since the sets

$$S_1 = T_1, \ S_2 = T_2, \ \ldots, \ S_{N-N_0-1} = T_{N-N_0-1},$$

$$S_{N-N_0} = T_{N-N_0} \cup T_{-N_0}, \ S_{N-N_0+1} = T_{N-N_0+1} \cup T_{-N_0+1}, \ \ldots,$$

$$\ldots, \ S_N = T_N \cup T_0$$

are disjoint, there is $g \in C(X)$ which is constant on each of these sets and satisfies $\|f - g\| < \frac{1}{2}\varepsilon$.

Proof of Lemma 6 (continued)

Let the values of g on these sets be λ_1 on S_1 through λ_N on S_N. Then $ge_n = e_n g = \lambda_n e_n$ for $0 \leq n \leq N - N_0$. For $N - N_0 < n \leq N$ we use $e_n \in (q_{n-N} + q_n)C^*(\mathbb{Z}, X, h)_Y(q_{n-N} + q_n)$ to get,

$$ge_n = g(q_{n-N} + q_n)e_n = \lambda_n(q_{n-N} + q_n)e_n = e_n(q_{n-N} + q_n)g = e_n g.$$

Since $\|f - g\| < \frac{1}{2}\varepsilon$ and $ge = eg$, it follows that

$$\|pf - fp\| = \|fe - ef\| < \varepsilon.$$

This is (1) for f. That $pfp \in C^*(\mathbb{Z}, X, h)_Y$ follows from the fact that f and p are in this subalgebra. So we also have (2) for f.

Proof of Lemma 6 (continued)

It remains only to verify (3). Using $h^{r_l}(Y) \subset U_l$ for $-N_0 \leq l \leq N$ at the third step, and with $Z = \bigcup_{l=-N_0}^{N} h^{r_l}(Y) \subset U$, we get (with Murray-von Neumann equivalence in $C^*(\mathbb{Z}, X, h)$)

$$1 - p = e \leq \sum_{l=-N_0}^{N} q_l \sim \sum_{l=-N_0}^{N} \chi_{h^{r_l}(Y)} = \chi_Z.$$

This completes the proof.

Some comments on the general case

In the general case, one major complication is that one can't choose the Rokhlin towers to consist of compact open sets. It turns out that one must take Y to be closed with nonempty interior, and replace the sets Y_k by their closures. Then they are no longer disjoint. The algebra $C^*(\mathbb{Z}, X, h)_Y$ is now a very complicated subalgebra of $\bigoplus_{k=0}^{l} M_{n(k)} \otimes C(Y_k)$. It is what is known as a "recursive subhomogeneous algebra". Such algebras are generally not AF, and may have few or no nontrivial projections. The hypothesis on the range of ρ (which was not used above, although it is automatic when X is the Cantor set) must be used to produce sufficiently many nonzero projections and approximating finite dimensional subalgebras.

Some comments on the general case (continued)

Here are a few more details. The algebra $C^*(\mathbb{Z}, X, h)_Y$ is not AF. So we choose closed sets Y_n with

$$Y_0 \supset Y_1 \supset Y_2 \supset \cdots \quad \text{and} \quad \bigcap_{n=0}^{\infty} Y_n = \{y_0\}$$

for a suitable $y_0 \in X$, and such that $\operatorname{int}(Y_n) \neq \varnothing$ for all n. Then

$$C^*(\mathbb{Z}, X, h)_{Y_0} \subset C^*(\mathbb{Z}, X, h)_{Y_1} \subset C^*(\mathbb{Z}, X, h)_{Y_2} \subset \cdots$$

and

$$\overline{\bigcup_{n=0}^{\infty} C^*(\mathbb{Z}, X, h)_{Y_n}} = C^*(\mathbb{Z}, X, h)_{\{y_0\}}.$$

Some comments on the general case (continued)

$C^*(\mathbb{Z}, X, h)_{\{y_0\}}$ is simple and has the same tracial states and K_0-group as $C^*(\mathbb{Z}, X, h)$. (These facts require proof. The one about K_0 is hard, but had already been done by Putnam.) Using this, and some results on dymanics (essentially "mean dimension zero"), the theory of direct limits of recursive subhomogeneous algebras can be used to prove that $C^*(\mathbb{Z}, X, h)_{\{y_0\}}$ has tracial rank zero. This turns out to be a sufficient substitute for $C^*(\mathbb{Z}, X, h)_Y$ being an AF algebra.

The tracial state space of $C^*(\mathbb{Z}, X, h)_{\{y\}}$

We describe the proof of one of the pieces needed.

Lemma

Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then the restriction map $T(C^*(\mathbb{Z}, X, h)) \to T\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ is a bijection.

The proof follows from the following two results. The first is well known, and holds in much greater generality. The proofs are roughly the same, but the proof of the second is more complicated, since the algebra is smaller.

Tracial states and invariant measures

Proposition

Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Then the restriction map $T(C^*(\mathbb{Z}, X, h)) \to T(C(X))$ is a bijection from $T(C^*(\mathbb{Z}, X, h))$ to the set of h-invariant Borel probability measures on X. Moreover, if $E\colon C^*(\mathbb{Z}, X, h) \to C(X)$ is the standard conditional expectation, and μ is an h-invariant Borel probability measure on X, then the tracial state τ_μ on $C^*(\mathbb{Z}, X, h)$ which restricts to μ is given by the formula $\tau_\mu(a) = \int_X E(a)\, d\mu$ for $a \in C^*(\mathbb{Z}, X, h)$.

Proof of the proposition

It is immediate that τ_μ is positive, and that $\tau_\mu(1) = 1$. So τ_μ is a state.

We prove $\tau_\mu(ab) = \tau_\mu(ba)$ for all $a, b \in C^*(\mathbb{Z}, X, h)$. Since τ_μ is continuous, and since $C^*(\mathbb{Z}, X, h)$ is the closed linear span of all elements fu^m with $f \in C(X)$ and $m \in \mathbb{Z}$, it suffices to prove $\tau_\mu((fu^m)(gu^n)) = \tau_\mu((gu^n)(fu^m))$ for all $f, g \in C(X)$ and $m, n \in \mathbb{Z}$. For $n \neq -m$, we have

$$\tau_\mu((fu^m)(gu^n)) = \tau_\mu((gu^n)(fu^m)) = 0,$$

while for $n = -m$, we have, using h-invariance of μ at the second step,

$$\begin{aligned}\tau_\mu((fu^m)(gu^n)) &= \tau_\mu\big(f(u^m g u^{-m})\big) = \int_X f\big(g \circ h^{-m}\big)\, d\mu \\ &= \int_X (f \circ h^m) g\, d\mu = \tau_\mu\big(g(u^{-m} f u^m)\big) = \tau_\mu((gu^n)(fu^m)).\end{aligned}$$

This completes the proof that τ_μ is a tracial state on $C^*(\mathbb{Z}, X, h)$.

Proof of the proposition (continued)

Now let τ be any tracial state on $C^*(\mathbb{Z}, X, h)$. Let μ be the Borel probability measure on X determined by $\tau(f) = \int_X f\, d\mu$ for $f \in C(X)$. We complete the proof by showing that μ is h-invariant, and that $\tau = \tau_\mu$.

For the first, for every $f \in C(X)$, we use the trace property at the second step to get

$$\int_X (f \circ h^{-1})\, d\mu = \tau(ufu^*) = \tau(u^*(uf)) = \tau(f) = \int_X f\, d\mu.$$

Since $f \in C(X)$ is arbitrary, it follows that μ is h-invariant.

Proof of the proposition (continued)

For the second, since $C^*(\mathbb{Z}, X, h)$ is the closed linear span of all elements fu^n with $f \in C(X)$ and $m \in \mathbb{Z}$, it suffices to prove $\tau(fu^n) = 0$ for $f \in C(X)$ and $n \in \mathbb{Z} \setminus \{0\}$. Since h^n has no fixed points, there is an open cover of X consisting of sets U such that $h^n(U) \cap U = \varnothing$. Choose $g_1, g_2, \ldots, g_m \in C(X)$ which form a partition of unity subordinate to this cover. In particular, the supports of g_j and $g_j \circ h^{-n}$ are disjoint for all j. For $1 \leq j \leq m$ we have, using the trace property at the second step, and the relation $u^n g u^{-n} = g \circ h^{-n}$ at the third step,

$$\tau(g_j f u^n) = \tau\big(g_j^{1/2} f u^n g_j^{1/2}\big) = \tau\big(g_j^{1/2} f \big(g_j^{1/2} \circ h^{-n}\big) u^n\big) = \tau(0) = 0.$$

Summing over j gives $\tau(fu^n) = 0$. This completes the proof.

Tracial states on the subalgebra

Lemma

Let X be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then the restriction map $T\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to T(C(X))$ is a bijection from $T\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ to the set of h-invariant Borel probability measures on X.

Proof of the lemma

Applying the previous proposition and restricting from $C^*(\mathbb{Z}, X, h)$ to $C^*(\mathbb{Z}, X, h)_{\{y\}}$, we see that every h-invariant Borel probability measure on X gives a tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$.

Now let τ be any tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$. Let μ be the Borel probability measure on X determined by $\tau(f) = \int_X f \, d\mu$ for $f \in C(X)$. As in the proof of the previous proposition, we complete the proof by showing that μ is h-invariant, and that $\tau = \tau_\mu|_{C^*(\mathbb{Z}, X, h)_{\{y\}}}$.

Proof of the lemma (continued)

For the first, we again show that $\int_X (f \circ h^{-1}) \, d\mu = \int_X f \, d\mu$ for every $f \in C(X)$. This is clearly true for constant functions f. Therefore, it suffices to consider functions f such that $f(y) = 0$. For such a function f, write $f = f_1 f_2^*$ with $f_1 f_2 \in C(X)$ such that $f_1(y) = f_2(y) = 0$. (For example, take $f_1 = f|f|^{-1/2}$ and $f_2 = |f|^{1/2}$.) Then $uf_1, \, uf_2 \in C^*(\mathbb{Z}, X, h)_{\{y\}}$. So

$$f \circ h^{-1} = ufu^* = (uf_1)(uf_2)^* \in C^*(\mathbb{Z}, X, h)_{\{y\}}.$$

We now use the trace property at the second step to get

$$\int_X (f \circ h^{-1}) \, d\mu = \tau\big((uf_1)(uf_2)^*\big) = \tau\big((uf_2)^*(uf_1)\big) = \tau(f) = \int_X f \, d\mu.$$

Thus μ is h-invariant.

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$, which is again in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ because $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is closed under multiplication), and the adjoint of something of this form again has this form (by a similar argument).

It now suffices to prove that if $fu^n \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $n \neq 0$, then $\tau(fu^m) = 0$. The argument is the same as in the proof of the previous Proposition; one merely needs to note that the elements g_j used there are in $C(X) \subset C^*(\mathbb{Z}, X, h)_{\{y\}}$. This completes the proof.