**Ex 1** Reminder: when finding \( \frac{\partial}{\partial y} [f(x, y)] = f_y(x, y) \), the differentiation is with respect to \( y \), and \( x \) is treated as a constant. (Similarly with \( x \) and \( y \) interchanged.)

Review: if \( f(x, y) = y^3 - x(y^2 - 1) \), find \( f_x(x, y) \) and \( f_y(x, y) \).
Ex 2 A particular manufacturer’s productivity happens to be modeled well by a constant elasticity of substitution production function (of which the Cobb-Douglas production function is a special case), and looks like

\[ P(K, L) = 0.3 \left( 0.4K^{-0.5} + 0.6L^{-0.5} \right)^{-2}, \]

for millions of units \( P \) produced at a capital investment of \( K \) million dollars and \( L \) thousand worker-hours every month. Find and interpret the value of the partial derivative \( P_K(10, 6) \), including units.

In the last lecture, we got \( P_K(10, 6) \approx 0.07404 \), but we didn’t specify the units or give the interpretation.
The second order partial derivatives of \( f(x, y) \) are written
\[
\frac{\partial^2}{\partial x^2}[f(x, y)] = f_{xx}(x, y),
\]
\[
\frac{\partial^2}{\partial x \partial y}[f(x, y)] = f_{yx}(x, y),
\]
\[
\frac{\partial^2}{\partial y \partial x}[f(x, y)] = f_{xy}(x, y),
\]
\[
\frac{\partial^2}{\partial y^2}[f(x, y)] = f_{yy}(x, y).
\]

Given a smooth function of several variables \( f \), any “mixed” partial derivatives (i.e. second-order derivatives with a combination of variables in the denominator) are equal. In other words, \( f_{xy} = f_{yx} \).

Find \( f_{xx} \) and \( f_{yy} \) for \( f(x, y) = xe^{2y^2} \)

Verify that the mixed partials for \( g(x, y) = \ln(x^2 - y^2) \) are equal.
Two commodities are: 
\[ \begin{array}{c} \text{substitutes} \\ \text{complementary} \end{array} \] if an increase in demand for one is associated with 
\[ \begin{array}{c} \text{a decrease} \\ \text{an increase} \end{array} \] in demand for the other.

Phrased in terms of calculus, let \( x \) represent the price of product 1 (in dollars per unit) and let \( y \) represent the price of product 2 (in dollars per unit). Let \( f(x, y) \) be the demand (in units) for product 1 at these prices, and let \( g(x, y) \) be the demand (in units) for product 2 at these prices. Then the products are: 
\[ \begin{array}{c} \text{substitutes} \\ \text{complementary} \end{array} \] if \( f_y(x, y) \) and \( g_x(x, y) \) are both 
\[ \begin{array}{c} > 0 \\ < 0 \end{array} \], 
and otherwise are neither complementary nor substitutes.
Ex 5 Local demand for grapefruit is given by \( f(p, n) = 10 + \frac{5}{p + 2} + 3e^{0.4n} \), while demand for oranges is \( g(p, n) = 7 - \frac{4}{p + 6} - 2n \), where each demand is given in thousands of units each month at \( p \) dollars per pound for grapefruit and \( n \) dollars per pound for oranges. Are the products substitutes, complements, or neither?

Thm (Chain Rule for \( f(x, y) \)) Let \( z = f(x, y) \), where \( x \) and \( y \) are functions of \( t \). Then

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}
\]

\( \Delta z \), for changes in \( x \) and \( y \) (\( \Delta x \) and \( \Delta y \), respectively) can be approximated by

\[
\Delta z \approx \frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y
\]

Ex 6 Let \( z = x^2 + \ln(y) \), where \( x = 1 + t^2 \) and \( y = \sqrt{2 + t} \). Find an expression for \( \frac{dz}{dt} \).
Ex 7 (Bonus): Estimate the value of $f_x(2, 5)$ and $f_y(2, 5)$ based on the table of values for function $f$.

<table>
<thead>
<tr>
<th>$f(x, y)$</th>
<th>$x = 1.5$</th>
<th>$x = 2$</th>
<th>$x = 2.5$</th>
<th>$x = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 4$</td>
<td>2</td>
<td>2.6</td>
<td>3</td>
<td>3.2</td>
</tr>
<tr>
<td>$y = 4.5$</td>
<td>1.8</td>
<td>2.8</td>
<td>3.6</td>
<td>4.2</td>
</tr>
<tr>
<td>$y = 5$</td>
<td>1.5</td>
<td>2.7</td>
<td>3.7</td>
<td>4.3</td>
</tr>
<tr>
<td>$y = 5.5$</td>
<td>1.1</td>
<td>2.1</td>
<td>2.9</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Def A point $(a, b)$ is a critical point of function $f(x, y)$ if both $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Ex 8 Find the critical points of $f(x, y) = x^2y - 4y$. 
**Def** Given a twice differentiable function of two variables \( f(x, y) \), we say that \( f \) has a

- relative maximum
- relative minimum

at a point \((a, b)\) if \[ f(a, b) \geq f(x, y) \]

for all points \((x, y)\) near \((a, b)\). We say that \((a, b)\) is a saddle point of \( f \) if it is a critical point, but not a relative extremum (that is, neither a relative maximum nor relative minimum).

**Thm** (The Second Partials Test) To find critical points (values of \( x \) and \( y \) for which relative extrema or saddle points of a twice differentiable function \( f \) may occur) first find all solutions \((a, b)\) to \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \).

To classify these points compute

\[
D(x, y) = f_{xx}(x, y) \cdot f_{yy}(x, y) - [f_{xy}(x, y)]^2
\]

and evaluate it at each critical point \((a, b)\).

If

\[
\begin{bmatrix}
D(a, b) < 0 \\
D(a, b) > 0 \text{ and } f_{xx}(a, b) > 0 \\
D(a, b) > 0 \text{ and } f_{xx}(a, b) < 0 \\
\text{saddle point} \\
\text{relative minimum} \\
\text{relative maximum}
\end{bmatrix}
\]

at a critical point \((a, b)\), then the point is a

- saddle point
- relative minimum
- relative maximum

**Ex 9** For some (twice differentiable) function \( f(x, y) \), we have critical points at \((1, 2)\), and \((-1, 4)\), with \( f_{xx}(1, 2) = 4 \), \( f_{yy}(1, 2) = -3 \), and \( f_{xy}(1, 2) = 1 \), and with \( f_{xx}(-1, 4) = 2 \), \( f_{yy}(-1, 4) = 5 \), and \( f_{xy}(-1, 4) = 3 \). Classify the critical points as relative maxima, relative minima, or saddle points.