Ex 1 Reminder: when finding \( \frac{\partial}{\partial y} [f(x, y)] = f_y(x, y) \), the differentiation is with respect to \( y \), and \( x \) is treated as a constant. (Similarly with \( x \) and \( y \) interchanged.)

Review: if \( f(x, y) = y^3 - x(y^2 - 1) \), find \( f_x(x, y) \) and \( f_y(x, y) \).

\[
f_x(x, y) = \frac{\partial}{\partial x} (y^3) - \frac{\partial}{\partial x} (x(y^2 - 1)) = 0 - (y^2 - 1) = -y^2 + 1.
\]

\[
f_y(x, y) = \frac{\partial}{\partial y} (y^3) - \frac{\partial}{\partial y} (x(y^2 - 1)) = 3y^2 - x \frac{\partial}{\partial y} (y^2 - 1) = 3y^2 - x \cdot 2y = 3y^2 - 2xy.
\]

CQ (Section 7.2, #4)

\[
f(x, y) = y^2 + y(x^3 - 5) = y^2 + (x^3 - 5)y
\]

Find \( f_y(2, 1) \)

\[
f_y(x, y) = 2y + x^3 - 5
\]

So \( f_y(2, 1) = 2 \cdot 1 + 2^3 - 5 = 2 + 8 - 5 = 5 \).
Ex 2 A particular manufacturer’s productivity happens to be modeled well by a constant elasticity of substitution production function (of which the Cobb-Douglas production function is a special case), and looks like

\[ P(K, L) = 0.3 \left( 0.4K^{-0.5} + 0.6L^{-0.5} \right)^{-2}, \]

for millions of units \( P \) produced at a capital investment of \( K \) million dollars and \( L \) thousand worker-hours every month. Find and interpret the value of the partial derivative \( P_K(10, 6) \), including units.

In the last lecture, we got \( P_K(10, 6) \approx 0.07404 \), but we didn’t specify the units or give the interpretation.

\[ \frac{\partial P}{\partial K} \text{ has units} \quad \frac{\text{units of } P}{\text{units of } K} \]

Here,

\[ \frac{\text{millions of } P}{\text{units/mth}} = \frac{\text{units/}}{\text{millions of } $/\text{mth}} = \text{Units per } $ \]

\[ \text{units of } \frac{\partial P}{\partial L} = \text{1000s of } \frac{\text{units per month}}{\text{worker-hour}}. \]

Interpretation of \( \frac{\partial P}{\partial K}(10, 6) \approx 0.07404 \):

At 10 million $ of capital and 6 thousand worker-hours/month, each additional $ of capital per month gives about 0.07404 additional units of production per month.

This is the marginal productivity of capital.

\[ \left[ \frac{\partial P}{\partial L} \right] \text{ is the marginal productivity of labor.} \]
Guide for Section 7.2: Partial Derivatives (part 2)

**Def** The second order partial derivatives of \( f(x, y) \) are written

\[
\begin{align*}
\frac{\partial^2}{\partial x^2}[f(x, y)] &= f_{xx}(x, y), \\
\frac{\partial^2}{\partial y^2}[f(x, y)] &= f_{yy}(x, y), \\
\frac{\partial^2}{\partial x \partial y}[f(x, y)] &= f_{yx}(x, y), \\
\frac{\partial^2}{\partial y \partial x}[f(x, y)] &= f_{xy}(x, y),
\end{align*}
\]

and

\[
\frac{\partial^2}{\partial y \partial x}[f(x, y)] = f_{xy}(x, y),
\]

**Thm** Given a smooth function of several variables \( f \), any “mixed” partial derivatives (i.e. second-order derivatives with a combination of variables in the denominator) are equal. In other words, \( f_{xy} = f_{yx} \).

**Ex 3** Find \( f_{xx} \) and \( f_{yy} \) for \( f(x, y) = xe^{2y^2} \).

First:

\[
f_x(x, y) = e^{2y^2}. \quad \text{Comb.}
\]

\[
f_{xx}(x, y) = \frac{\partial}{\partial x} (e^{2y^2}) = 0
\]

Now \( x \) is constant.

\[
f_y(x, y) = xe^{2y^2}. \quad \text{Comb.}
\]

\[
f_{yy}(x, y) = 4x \left[ \frac{\partial}{\partial y} (4y e^{2y^2}) \right] = 4x \left[ 4y e^{2y^2} + y - 4y e^{2y^2} \right]
\]

**Ex 4** Verify that the mixed partials for \( g(x, y) = \ln(x^2 - y^2) \) are equal.

Need \( g_{xy}(x, y) \) and \( g_{yx}(x, y) \)

Dif. with respect to \( x \) first

\[
g_x(x, y) = \frac{1}{x^2 - y^2} \cdot \frac{\partial}{\partial x} (x^2 - y^2) = \frac{2x}{x^2 - y^2}
\]

\[
g_{xy}(x, y) = \frac{\partial}{\partial y} (g_x(x, y)) = \frac{\partial}{\partial y} \left( \frac{2x}{x^2 - y^2} \right) = \frac{\partial}{\partial y} \left( 2x(x^2 - y^2)^{-1} \right)
\]

\[= 2x \left(-1\right)(x^2 - y^2)^{-2} \frac{\partial}{\partial y} (x^2 - y^2) = 4xy(x^2 - y^2)^{-2}
\]

\[= 4x \left( e^{2y^2} + 4y^2 e^{2y^2} \right)
\]
Guide for Section 7.2: Partial Derivatives (part 2)

\[ g(x,y) = \ln(x^2 - y^2) \]  

New diff. with respect to \( y \) first:

\[ g_y(x,y) = \frac{1}{x^2 - y^2} \cdot (-2y) = -2y (x^2 - y^2)^{-1} \]

\[ g_{yx}(x,y) = \frac{2}{x} (-2y (x^2 - y^2)^{-1}) = -2y \frac{2}{x} \left( (x^2 - y^2)^{-1} \right) \]

\[ = -2y \left( -1 \right) (x^2 - y^2)^{-2} (2x) = 4xy(x^2 - y^2)^{-2} \]

Yes, they are the same.

### CQ (Section 7.2, #4)

\[ \#1 \mbox{ on screen.} \]

\[
\begin{align*}
  g(x,t) &= t x^2 \\
  g_{xt}(x,y) &= \frac{c}{\partial t} \left( \frac{2}{\partial x} \left( t x^2 \right) \right) = \frac{c}{\partial t} (t - 2x) \\
  &= 2x.
\end{align*}
\]

### Def

Two commodities are:

- substitutes
- complementary

if an increase in demand for one is associated with a decrease an increase in demand for the other.

Phrased in terms of calculus, let \( x \) represent the price of product 1 (in dollars per unit) and let \( y \) represent the price of product 2 (in dollars per unit). Let \( f(x,y) \) be the demand (in units) for product 1 at these prices, and let \( g(x,y) \) be the demand (in units) for product 2 at these prices. Then the products are:

- substitutes
- complementary

if \( f_y(x,y) \) and \( g_x(x,y) \) are both \( > 0 \) and \( < 0 \), and otherwise are neither complementary nor substitutes.

### CQ (Section 7.2, #5)

If price of bread increases, then quality of bread sold goes down, so demand for butter goes down. Demand for butter is \( D(v,u) \)

We are talking about \( D_r (v,u) \), and it should be negative.
Ex 5 Local demand for grapefruit is given by \( f(p, n) = 10 + \frac{5}{p+2} + 3e^{0.4n} \), while demand for oranges is \( g(p, n) = 7 - \frac{4}{p+6} - 2n \), where each demand is given in thousands of units each month at \( p \) dollars per pound for grapefruit and \( n \) dollars per pound for oranges. Are the products substitutes, complements, or neither?

\[
\begin{align*}
\text{Need} & \quad f_n(p, n) = \frac{\partial}{\partial n} \left( 10 + \frac{5}{p+2} + 3e^{0.4n} \right) \\
& \quad = 0 + 0 + 3 \cdot (0.4) e^{0.4n} = 1.2e^{0.4n}
\end{align*}
\]

and

\[
\begin{align*}
g_p(p, n) &= \frac{\partial}{\partial p} \left( 7 - \frac{4}{p+6} - 2n \right) \\
& = \frac{-4}{(p+6)^2}
\end{align*}
\]

Both are positive, so substitute.

\[
z = f(x(t), y(t))
\]

Thm (Chain Rule for \( f(x, y) \)) Let \( z = f(x, y) \), where \( x \) and \( y \) are functions of \( t \). Then

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\( \Delta z \), for changes in \( x \) and \( y \) (\( \Delta x \) and \( \Delta y \), respectively) can be approximated by

\[
\Delta z \approx \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y
\]

Ex 6 Let \( z = x^2 + \ln(y) \), where \( x = 1 + t^2 \) and \( y = \sqrt{2+t} \). Find an expression for \( \frac{dz}{dt} \):

\[
\frac{dz}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)
\]

Continued on page 5-1.
\[
\frac{dz}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)
\]
\[
\leq 2t + \frac{1}{\gamma(t)} \cdot y'(t)
\]
\[
= 2(1+t^2) \cdot 2t + \frac{1}{\sqrt{2+t}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2+t}}
\]
\[
= 4t(1+t^2) + \frac{1}{2(2+t)}
\]

**Given problem:**

\[z = f(x, y) = x^2 + \ln(y)\]

\[x(t) = 1+t^2, \quad y(t) = \sqrt{2+t}\]

Find \( \frac{dz}{dt} \) using multivariable chain rule.